Cornerstones

Emmanuele DiBenedetto
Ugo Gianazza

Partial Differential Equations

Third Edition

Birkhäuser

Birkhäuser

Cornerstones

Series Editor
Steven G. Krantz, Washington University, St. Louis, MO, USA

Editorial Board Member
Robert Lazarsfeld, Stony Brook University, Stony Brook, NY, USA
Peter Petersen, University of California, Los Angeles, CA, USA
Alan Tucker, Stony Brook University, Stony Brook, NY, USA
Scott Wolpert, University of Maryland, College Park, MD, USA

Cornerstones comprises textbooks that focus on what students need to know and what faculty should teach regarding various selected topics in pure and applied mathematics and related subjects. Aimed at aspiring young mathematicians at the advanced undergraduate to the second-year graduate level, books that appear in this series are intended to serve as the definitive advanced texts for the next generation of mathematicians. By enlisting only expert mathematicians and leading researchers in each field who are top-notch expositors with established track records, **Cornerstones** volumes are models of clarity that provide authoritative modern treatments of the essential subjects of pure and applied mathematics while capturing the beauty and excitement of mathematics for the reader.

Emmanuele DiBenedetto • Ugo Gianazza

Partial Differential Equations

Third Edition

Birkhäuser

Emmanuele DiBenedetto (Deceased)

Ugo Gianazza
Department of Mathematics
University of Pavia
Pavia, Italy

ISSN 2197-182X ISSN 2197-1838 (electronic)
Cornerstones
ISBN 978-3-031-46617-5 ISBN 978-3-031-46618-2 (eBook)
https://doi.org/10.1007/978-3-031-46618-2

© Springer Nature Switzerland AG 1995, 2010, 2023
This work is subject to copyright. All rights are reserved by the Publisher, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilms or in any other physical way, and transmission or information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed.
The use of general descriptive names, registered names, trademarks, service marks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.
The publisher, the authors, and the editors are safe to assume that the advice and information in this book are believed to be true and accurate at the date of publication. Neither the publisher nor the authors or the editors give a warranty, expressed or implied, with respect to the material contained herein or for any errors or omissions that may have been made. The publisher remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

This book is published under the imprint Birkhäuser, www.birkhauser-science.com by the registered company Springer Nature Switzerland AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Paper in this product is recyclable.

Contents

0 PRELIMINARIES 1
1 Green's Theorem 1
1.1 Differential Operators and Adjoints 2
2 The Continuity Equation 3
3 The Heat Equation and the Laplace Equation 5
3.1 Variable Coefficients 5
4 A Model for the Vibrating String 6
5 Small Vibrations of a Membrane 8
6 Transmission of Sound Waves 11
7 The Navier–Stokes System 13
8 The Euler Equations 13
9 Isentropic Potential Flows 14
9.1 Steady Potential Isentropic Flows 15
10 Partial Differential Equations 15
Problems and Complements 16
3c The Heat Equation and the Laplace Equation 16
3.1c Basic Physical Assumptions 16
3.2c The Diffusion Equation 17
3.3c Justifying the Postulates (3.3c)–(3.4c) 18
3.4c More on the Postulates (3.3c)–(3.4c) 19

1 QUASI-LINEAR EQUATIONS AND ANALYTIC DATA . 21
1 Quasi-Linear Second-Order Equations in Two Variables 21
2 Characteristics and Singularities 23
2.1 Coefficients Independent of u_x and u_y 24
3 Quasi-Linear Second-Order Equations 25
3.1 Constant Coefficients 27
3.2 Variable Coefficients 27
4 Quasi-Linear Equations of Order $m \geq 1$ 28
4.1 Characteristic Surfaces 29

5 Analytic Data and the Cauchy–Kowalewski Theorem 30
5.1 Reduction to Normal Form ([32]) 30
6 Proof of the Cauchy–Kowalewski Theorem 32
6.1 Estimating the Derivatives of $\mathbf{u}$ at the Origin 33
7 Auxiliary Inequalities 33
8 Auxiliary Estimations at the Origin 35
9 Proof of the Cauchy–Kowalewski Theorem (Concluded) 37
9.1 Proof of Lemma 6.1 37
10 Holmgren's Uniqueness Theorem 38
11 Proof of the Holmgren Uniqueness Theorem 40
11.1 Proof of Lemma 11.1 42

Problems and Complements 43
1c Quasi-Linear Second-Order Equations in Two Variables 43
5c Analytic Data and the Cauchy–Kowalewski Theorem 44
6c Proof of the Cauchy–Kowalewski Theorem 45
8c The Generalized Leibniz Rule 45
9c Proof of the Cauchy–Kowalewski Theorem Concluded 45

2 THE LAPLACE EQUATION 47
1 Preliminaries 47
1.1 The Dirichlet and Neumann Problems 48
1.2 The Cauchy Problem 49
1.3 Well-Posedness and a Counterexample of Hadamard . . . 49
1.4 Radial Solutions 50
2 The Green and Stokes Identities 51
2.1 The Stokes Identities 51
3 Green's Function and the Dirichlet Problem for a Ball 53
3.1 Green's Function for a Ball 55
4 Sub-Harmonic Functions and the Mean Value Property 57
4.1 The Maximum Principle 59
4.2 Structure of Sub-Harmonic Functions 60
5 Estimating Harmonic Functions and Their Derivatives 61
5.1 The Harnack Inequality and the Liouville Theorem 62
5.2 Analyticity of Harmonic Functions 63
6 The Dirichlet Problem 65
7 About the Exterior Sphere Condition 68
7.1 The Case $N = 2$ and ∂E Piecewise Smooth 69
7.2 A Counterexample of Lebesgue for $N = 3$ ([163]) 69
8 The Poisson Integral for the Half Space 70
9 Schauder Estimates of Newtonian Potentials 72
10 Potential Estimates in $L^p(E)$ 75
11 Local Solutions 78
11.1 Local Weak Solutions 79
12 Inhomogeneous Problems 80

- 12.1 On the Notion of Green's Function . . . 80
- 12.2 Inhomogeneous Problems . . . 81
- 12.3 The Case $f \in C_o^\infty(E)$ . . . 82
- 12.4 The Case $f \in C^\eta(\bar{E})$ . . . 83

Problems and Complements . . . 83

- 1c Preliminaries . . . 83
  - 1.1c Newtonian Potentials on Ellipsoids . . . 83
  - 1.2c Invariance Properties . . . 84
- 2c The Green and Stokes Identities . . . 84
- 3c Green's Function and the Dirichlet Problem for the Ball . . . 85
  - 3.1c Separation of Variables . . . 85
- 4c Sub-Harmonic Functions and the Mean Value Property . . . 86
  - 4.1c Reflection and Harmonic Extension . . . 87
  - 4.2c The Weak Maximum Principle . . . 87
  - 4.3c Sub-Harmonic Functions . . . 88
- 5c Estimating Harmonic Functions . . . 89
  - 5.1c Harnack-Type Estimates . . . 90
  - 5.2c Ill Posed Problems. An Example of Hadamard . . . 90
  - 5.3c Removable Singularities . . . 91
- 7c About the Exterior Sphere Condition . . . 92
- 8c Problems in Unbounded Domains . . . 93
  - 8.1c The Dirichlet Problem Exterior to a Ball . . . 93
- 9c Schauder Estimates up to the Boundary ([222, 223]) . . . 94
- 10c Potential Estimates in $L^p(E)$ . . . 95
  - 10.1c Integrability of Riesz Potentials . . . 95
  - 10.2c Second Derivatives of Potentials . . . 96

3 BOUNDARY VALUE PROBLEMS BY DOUBLE LAYER POTENTIALS . . . 97

- 1 The Double-Layer Potential . . . 97
- 2 On the Integral Defining the Double-Layer Potential . . . 99
- 3 The Jump Condition of $W(\partial E, x_o; v)$ Across ∂E . . . 101
- 4 More on the Jump Condition Across ∂E . . . 103
- 5 The Dirichlet Problem by Integral Equations ([192]) . . . 104
- 6 The Neumann Problem by Integral Equations ([192]) . . . 105
- 7 The Green's Function for the Neumann Problem . . . 107
  - 7.1 Finding $\mathcal{G}(\cdot;\cdot)$ . . . 108
- 8 Eigenvalue Problems for the Laplacean . . . 109
  - 8.1 Compact Kernels Generated by Green's Function . . . 110
- 9 Compactness of A_F in $L^p(E)$ for $1 \le p \le \infty$ . . . 110
- 10 Compactness of A_Φ in $L^p(E)$ for $1 \le p < \infty$ . . . 112
- 11 Compactness of A_Φ in $L^\infty(E)$ . . . 112

Problems and Complements . . . 114

- 2c On the Integral Defining the Double-Layer Potential . . . 114

5c The Dirichlet Problem by Integral Equations . . . 115
6c The Neumann Problem by Integral Equations . . . 116
7c The Green's Function for the Neumann Problem . . . 116
  7.1c Constructing $\mathcal{G}(\cdot;\cdot)$ for a Ball in $\mathbb{R}^2$ and $\mathbb{R}^3$ . . . 116
8c Eigenvalue Problems . . . 117

4 INTEGRAL EQUATIONS AND EIGENVALUE PROBLEMS . . . 119
1 Kernels in $L^2(E)$ . . . 119
  1.1 Examples of Kernels in $L^2(E)$ . . . 120
2 Integral Equations in $L^2(E)$ . . . 121
  2.1 Existence of Solutions for Small $|\lambda|$ . . . 121
3 Separable Kernels . . . 122
  3.1 Solving the Homogeneous Equations . . . 123
  3.2 Solving the Inhomogeneous Equation . . . 123
4 Small Perturbations of Separable Kernels . . . 124
  4.1 Existence and Uniqueness of Solutions . . . 125
5 Almost Separable Kernels and Compactness . . . 126
  5.1 Solving Integral Equations for Almost Separable Kernels . . . 127
  5.2 Potential Kernels Are Almost Separable . . . 127
6 Applications to the Neumann Problem . . . 128
7 The Eigenvalue Problem . . . 129
8 Finding a First Eigenvalue and Its Eigenfunctions . . . 131
9 The Sequence of Eigenvalues . . . 132
  9.1 An Alternative Construction Procedure of the Sequence of Eigenvalues . . . 133
10 Questions of Completeness and the Hilbert–Schmidt Theorem . . . 134
  10.1 The Case of $K(x;\cdot) \in L^2(E)$ Uniformly in x . . . 135
11 The Eigenvalue Problem for the Laplacean . . . 136
  11.1 An Expansion of the Green's Function . . . 137

Problems and Complements . . . 138
2c Integral Equations . . . 138
  2.1c Integral Equations of the First Kind . . . 138
  2.2c Abel Equations ([2, 3]) . . . 138
  2.3c Solving Abel Integral Equations . . . 139
  2.4c The Cycloid ([3]) . . . 140
  2.5c Volterra Integral Equations ([266, 267]) . . . 140
3c Separable Kernels . . . 141
  3.1c Hammerstein Integral Equations ([114]) . . . 141
6c Applications to the Neumann Problem . . . 142
9c The Sequence of Eigenvalues . . . 142
10c Questions of Completeness . . . 142
  10.1c Periodic Functions in $\mathbb{R}^N$ . . . 143

10.2c The Poisson Equation with Periodic Boundary Conditions . . . 144
11c The Eigenvalue Problem for the Laplacean . . . 144

5 THE HEAT EQUATION . . . 145

1 Preliminaries . . . 145
1.1 The Dirichlet Problem . . . 145
1.2 The Neumann Problem . . . 146
1.3 The Characteristic Cauchy Problem . . . 146
2 The Cauchy Problem by Similarity Solutions . . . 146
2.1 The Backward Cauchy Problem . . . 150
3 The Maximum Principle and Uniqueness (Bounded Domains) . 150
3.1 A Priori Estimates . . . 151
3.2 Ill Posed Problems . . . 151
3.3 Uniqueness (Bounded Domains) . . . 151
4 The Maximum Principle in $\mathbb{R}^N$ . . . 152
4.1 A Priori Estimates . . . 154
4.2 About the Growth Conditions (4.3) and (4.4) . . . 154
5 Uniqueness of Solutions to the Cauchy Problem . . . 155
5.1 A Counterexample of Tychonov ([263]) . . . 155
6 Initial Data in $L^1_{\text{loc}}(\mathbb{R}^N)$ . . . 157
6.1 Initial Data in the Sense of $L^1_{\text{loc}}(\mathbb{R}^N)$ . . . 158
7 Remarks on the Cauchy Problem . . . 159
7.1 About Regularity . . . 159
7.2 Instability of the Backward Problem . . . 160
8 Estimates Near $t = 0$ . . . 160
9 The Inhomogeneous Cauchy Problem . . . 162
10 Problems in Bounded Domains . . . 163
10.1 The Strong Solution . . . 165
10.2 The Weak Solution and Energy Inequalities . . . 166
11 Energy and Logarithmic Convexity . . . 167
11.1 Uniqueness for Some Ill Posed Problems . . . 168
12 Local Solutions . . . 168
12.1 Variable Cylinders . . . 172
12.2 The Case $|\alpha| = 0$ . . . 172
13 The Harnack Inequality . . . 173
13.1 Compactly Supported Sub-Solutions . . . 174
13.2 Proof of Theorem 13.1 . . . 175
14 Positive Solutions in S_T . . . 177
14.1 Non-Negative Solutions . . . 179
Problems and Complements . . . 181
2c Similarity Methods . . . 181
2.1c The Heat Kernel Has Unit Mass . . . 181
2.2c The Porous Medium Equation . . . 182

2.3c The p-Laplacean Equation . . . 184
2.4c The Error Function . . . 184
2.5c The Appell Transformation ([10]) . . . 184
2.6c The Heat Kernel by Fourier Transform . . . 185
2.7c Rapidly Decreasing Functions . . . 185
2.8c The Fourier Transform of the Heat Kernel . . . 186
2.9c The Inversion Formula . . . 186
3c The Maximum Principle in Bounded Domains . . . 187
3.1c The Blow-Up Phenomenon for Super-Linear Equations . . . 188
3.2c The Maximum Principle for General Parabolic Equations . . . 189
4c The Maximum Principle in $\mathbb{R}^N$ . . . 190
4.1c Counterexamples of the Tychonov Type . . . 191
7c Remarks on the Cauchy Problem . . . 191
12c On the Local Behavior of Solutions . . . 192

6 THE WAVE EQUATION . . . 195
1 The One-Dimensional Wave Equation . . . 195
1.1 A Property of Solutions . . . 196
2 The Cauchy Problem . . . 197
3 Inhomogeneous Problems . . . 198
4 A Boundary Value Problem (Vibrating String) . . . 200
4.1 Separation of Variables . . . 201
4.2 Odd Reflection . . . 202
4.3 Energy and Uniqueness . . . 203
4.4 Inhomogeneous Problems . . . 203
5 The Initial Value Problem in N Dimensions . . . 203
5.1 Spherical Means . . . 204
5.2 The Darboux Formula . . . 204
5.3 An Equivalent Formulation of the Cauchy Problem . . . 205
6 The Cauchy Problem in $\mathbb{R}^3$ . . . 205
7 The Cauchy Problem in $\mathbb{R}^2$ . . . 208
8 The Inhomogeneous Cauchy Problem . . . 210
9 The Cauchy Problem for Inhomogeneous Surfaces . . . 211
9.1 Reduction to Homogeneous Data on $t = \Phi$ . . . 212
9.2 The Problem with Homogeneous Data . . . 212
10 Solutions in Half Space. The Reflection Technique . . . 213
10.1 An Auxiliary Problem . . . 214
10.2 Homogeneous Data on the Hyperplane $x_3 = 0$ . . . 214
11 A Boundary Value Problem . . . 215
12 Hyperbolic Equations in Two Variables . . . 217
13 The Characteristic Goursat Problem . . . 217
13.1 Proof of Theorem 13.1: Existence . . . 217
13.2 Proof of Theorem 13.1: Uniqueness . . . 219
13.3 Goursat Problems in Rectangles . . . 220

14 The Noncharacteristic Cauchy Problem and the Riemann Function . . . 220
15 Symmetry of the Riemann Function . . . 222

Problems and Complements . . . 223
2c The d'Alembert Formula . . . 223
3c Inhomogeneous Problems . . . 223
  3.1c The Duhamel Principle ([61]) . . . 223
4c Solutions for the Vibrating String . . . 224
6c Cauchy Problems in $\mathbb{R}^3$ . . . 227
  6.1c Asymptotic Behavior . . . 227
  6.2c Radial Solutions . . . 227
  6.3c Solving the Cauchy Problem by Fourier Transform . . . 229
7c Cauchy Problems in $\mathbb{R}^2$ and the Method of Descent . . . 230
  7.1c The Cauchy Problem for $N = 4, 5$ . . . 231
8c Inhomogeneous Cauchy Problems . . . 231
  8.1c The Wave Equation for the N and $(N + 1)$-Laplacean . . . 231
  8.2c Miscellaneous Problems . . . 232
10c The Reflection Technique . . . 236
11c Problems in Bounded Domains . . . 237
  11.1c Uniqueness . . . 237
  11.2c Separation of Variables . . . 238
12c Hyperbolic Equations in Two Variables . . . 239
  12.1c The General Telegraph Equation . . . 239
14c Goursat Problems . . . 239
  14.1c The Riemann Function and the Fundamental Solution of the Heat Equation . . . 240

7 QUASI-LINEAR EQUATIONS OF FIRST ORDER . . . 241
1 Quasi-Linear Equations . . . 241
2 The Cauchy Problem . . . 242
  2.1 The Case of Two Independent Variables . . . 242
  2.2 The Case of N Independent Variables . . . 243
3 Solving the Cauchy Problem . . . 243
  3.1 Constant Coefficients . . . 244
  3.2 Solutions in Implicit Form . . . 245
4 Equations in Divergence Form and Weak Solutions . . . 246
  4.1 Surfaces of Discontinuity . . . 247
  4.2 The Shock Line . . . 247
5 The Initial Value Problem . . . 248
  5.1 Conservation Laws . . . 249
6 Conservation Laws in One Space Dimension . . . 249
  6.1 Weak Solutions and Shocks . . . 251
  6.2 Lack of Uniqueness . . . 252
7 Hopf Solution of The Burgers Equation . . . 253

8 Weak Solutions to (6.4) When $a(\cdot)$ is Strictly Increasing 254
 8.1 Lax Variational Solution 255
9 Constructing Variational Solutions I 256
 9.1 Proof of Lemma 9.1 257
10 Constructing Variational Solutions II 258
11 The Theorems of Existence and Stability 261
 11.1 Existence of Variational Solutions 261
 11.2 Stability of Variational Solutions 261
12 Proof of Theorem 11.1 262
 12.1 The Representation Formula (11.4) 262
 12.2 Initial Datum in the Sense of $L^1_{\text{loc}}(\mathbb{R})$ 263
 12.3 Weak Forms of the PDE 264
13 The Entropy Condition 265
 13.1 Entropy Solutions 265
 13.2 Variational Solutions of (6.4) Are Entropy Solutions . . . 266
 13.3 Remarks on the Shock and the Entropy Conditions 267
14 The Kruzhkov Uniqueness Theorem 269
 14.1 Proof of the Uniqueness Theorem I 269
 14.2 Proof of the Uniqueness Theorem II 271
 14.3 Stability in $L^1(\mathbb{R}^N)$ 272
15 The Maximum Principle for Entropy Solutions 272

Problems and Complements 274
3c Solving the Cauchy Problem 274
6c Explicit Solutions to the Burgers Equation 275
 6.2c Invariance of Burgers Equations by Some Transformation of Variables 276
 6.3c The Generalized Riemann Problem 276
13c The Entropy Condition 277
14c The Kruzhkov Uniqueness Theorem 279

8 NONLINEAR EQUATIONS OF FIRST ORDER 281
1 Integral Surfaces and Monge's Cones 281
 1.1 Constructing Monge's Cones 282
 1.2 The Symmetric Equation of Monge's Cones 282
2 Characteristic Curves and Characteristic Strips 283
 2.1 Characteristic Strips 284
3 The Cauchy Problem 285
 3.1 Identifying the Initial Data $p(0, s)$ 285
 3.2 Constructing the Characteristic Strips 286
4 Solving the Cauchy Problem 286
 4.1 Verifying (4.3) 287
 4.2 A Quasi-Linear Example in $\mathbb{R}^2$ 288
5 The Cauchy Problem for the Equation of Geometrical Optics . 289
 5.1 Wave Fronts, Light Rays, Local Solutions and Caustics . 290

6 The Initial Value Problem for Hamilton–Jacobi Equations 290
7 The Cauchy Problem in Terms of the Lagrangian 292
8 The Hopf Variational Solution 293
8.1 The First Hopf Variational Formula 294
8.2 The Second Hopf Variational Formula 294
9 Semigroup Property of Hopf Variational Solutions 295
10 Regularity of Hopf Variational Solutions 296
11 Hopf Variational Solutions (8.3) Are Weak Solutions of the Cauchy Problem (6.4) 297
12 Some Examples 299
12.1 Example I 299
12.2 Example II 300
12.3 Example III 300
13 Uniqueness 301
14 More on Uniqueness and Stability 303
14.1 Stability in $L^p(\mathbb{R}^N)$ for All $p \geq 1$ 303
14.2 Comparison Principle 304
15 Semi-Concave Solutions of the Cauchy Problem 304
15.1 Uniqueness of Semi-Concave Solutions 304
16 A Weak Notion of Semi-Concavity 305
17 Semi-Concavity of Hopf Variational Solutions 306
17.1 Weak Semi-Concavity of Hopf Variational Solutions Induced by the Initial Datum u_o 306
17.2 Strictly Convex Hamiltonian 307
18 Uniqueness of Weakly Semi-Concave Variational Hopf Solutions 309

9 LINEAR ELLIPTIC EQUATIONS WITH MEASURABLE COEFFICIENTS 313

1 Weak Formulations and Weak Derivatives 313
1.1 Weak Derivatives 314
2 Embeddings of $W^{1,p}(E)$ 315
2.1 Compact Embeddings of $W^{1,p}(E)$ 316
3 Multiplicative Embeddings of $W_o^{1,p}(E)$ and $\tilde{W}^{1,p}(E)$ 316
3.1 Some Consequences of the Multiplicative Embedding Inequalities 317
4 The Homogeneous Dirichlet Problem 318
5 Solving the Homogeneous Dirichlet Problem (4.1) by the Riesz Representation Theorem 318
6 Solving the Homogeneous Dirichlet Problem (4.1) by Variational Methods 319
6.1 The Case $N = 2$ 320
6.2 Gâteaux Derivative and The Euler Equation of $J(\cdot)$ 321
7 Solving the Homogeneous Dirichlet Problem (4.1) by Galerkin Approximations 321

7.1 On the Selection of an Orthonormal System in $W_o^{1,2}(E)$ 322
7.2 Conditions on $\mathbf{f}$ and f for the Solvability of the Dirichlet Problem (4.1) 323
8 Traces on ∂E of Functions in $W^{1,p}(E)$ 323
8.1 The Segment Property 323
8.2 Defining Traces 324
8.3 Characterizing the Traces on ∂E of Functions in $W^{1,p}(E)$ 325
9 The Inhomogeneous Dirichlet Problem 325
10 The Neumann Problem 326
10.1 A Variant of (10.1) 327
11 The Eigenvalue Problem 328
12 Constructing The Eigenvalues of (11.1) 329
13 The Sequence of Eigenvalues and Eigenfunctions 331
14 A Priori $L^\infty(E)$ Estimates for Solutions of the Dirichlet Problem (9.1) 333
15 Proof of Propositions 14.1–14.2 334
15.1 An Auxiliary Lemma on Fast Geometric Convergence . . 335
15.2 Proof of Proposition 14.1 for $N > 2$ 335
15.3 Proof of Proposition 14.1 for $N = 2$ 336
16 A Priori $L^\infty(E)$ Estimates for Solutions of the Neumann Problem (10.1) 336
17 Proof of Propositions 16.1–16.2 338
17.1 Proof of Proposition 16.1 for $N > 2$ 340
17.2 Proof of Proposition 16.1 for $N = 2$ 341
18 Miscellaneous Remarks on Further Regularity 341

Problems and Complements 342
1c Weak Formulations and Weak Derivatives 342
1.1c The Chain Rule in $W^{1,p}(E)$ 342
2c Embeddings of $W^{1,p}(E)$ 343
2.1c Proof of (2.4) 344
2.2c Compact Embeddings of $W^{1,p}(E)$ 344
3c Multiplicative Embeddings of $W_o^{1,p}(E)$ and $\tilde{W}^{1,p}(E)$ 345
3.1c Proof of Theorem 3.1 for $1 \le p < N$ 345
3.2c Proof of Theorem 3.1 for $p \ge N > 1$ 347
3.3c Proof of Theorem 3.2 for $1 \le p < N$ and E Convex 348
5c Solving the Homogeneous Dirichlet Problem (4.1) by the Riesz Representation Theorem 349
6c Solving the Homogeneous Dirichlet Problem (4.1) by Variational Methods 350
6.1c More General Variational Problems 350
6.8c Gâteaux Derivatives, Euler Equations and Quasi-Linear Elliptic Equations 352

8c Traces on ∂E of Functions in $W^{1,p}(E)$ . . . 353
  8.1c Extending Functions in $W^{1,p}(E)$ . . . 353
  8.2c The Trace Inequality . . . 354
  8.3c Characterizing the Traces on ∂E of Functions in $W^{1,p}(E)$ . . . 355
9c The Inhomogeneous Dirichlet Problem . . . 357
  9.1c The Lebesgue Spike . . . 357
  9.2c Variational Integrals and Quasi-Linear Equations . . . 357
10c The Neumann Problem . . . 358
11c The Eigenvalue Problem . . . 359
12c Constructing the Eigenvalues . . . 359
13c The Sequence of Eigenvalues and Eigenfunctions . . . 359
14c A Priori $L^\infty(E)$ Estimates for Solutions of the Dirichlet Problem (9.1) . . . 359
15c A Priori $L^\infty(E)$ Estimates for Solutions of the Neumann Problem (10.1) . . . 360
  15.1c Back to the Quasi-Linear Dirichlet Problem (9.1c) . . . 360

10 DEGIORGI CLASSES . . . 363
1 Quasi-Linear Equations and DeGiorgi Classes . . . 363
  1.1 DeGiorgi Classes . . . 365
2 Local Boundedness of Functions in the DeGiorgi Classes . . . 366
  2.1 Proof of Theorem 2.1 for $1 < p < N$ . . . 367
  2.2 Proof of Theorem 2.1 for $p = N$ . . . 368
3 Hölder Continuity of Functions in the DG Classes . . . 369
  3.1 On the Proof of Theorem 3.1 . . . 370
4 Estimating the Values of u by the Measure of the Set Where u Is Either Near μ^+ or Near μ^- . . . 370
5 Reducing the Measure of the Set Where u is Either Near μ^+ or Near μ^- . . . 371
  5.1 The Discrete Isoperimetric Inequality . . . 372
  5.2 Proof of Proposition 5.1 . . . 373
6 Proof of Theorem 3.1 . . . 374
7 Boundary DeGiorgi Classes: Dirichlet Data . . . 375
  7.1 Continuity up to ∂E of Functions in the Boundary DG Classes (Dirichlet Data) . . . 376
8 Boundary DeGiorgi Classes: Neumann Data . . . 377
  8.1 Continuity up to ∂E of Functions in the Boundary DG Classes (Neumann Data) . . . 379
9 The Harnack Inequality . . . 380
  9.1 Proof of Theorem 9.1. Preliminaries . . . 380
  9.2 Proof of Theorem 9.1. Expansion of Positivity . . . 381
  9.3 Proof of Theorem 9.1 . . . 381
10 Harnack Inequality and Hölder Continuity . . . 383

11 Local Clustering of the Positivity Set of Functions in $W^{1,1}(E)$ 384
12 A Proof of the Harnack Inequality Independent of Hölder Continuity 386

11 LINEAR PARABOLIC EQUATIONS IN DIVERGENCE FORM WITH MEASURABLE COEFFICIENTS 389
1 Parabolic Spaces and Embeddings 389
  1.1 Steklov Averages 393
2 Weak Formulations 393
3 The Homogeneous Dirichlet Problem 394
4 The Energy Inequality 396
5 Existence of Solutions of the Homogeneous Cauchy–Dirichlet Problem (3.1) by Galerkin Approximations 397
6 Uniqueness of Solutions of the Homogeneous Cauchy–Dirichlet Problem (3.1) 403
7 Traces of Functions on $\Sigma \overset{\text{def}}{=} \partial E \times (0,T]$ 403
8 The Inhomogeneous Dirichlet Problem 405
9 The Neumann Problem 406
  9.1 The Energy Inequality for the Neumann Problem 408
  9.2 A Variant of Problems (3.1) and (9.1) 410
10 A Priori $L^\infty(E_T)$ Estimates for Solutions of the Cauchy–Dirichlet Problem (8.1) 410
11 Proof of Propositions 10.1–10.2 412
12 A Priori $L^\infty(E_T)$ Estimates for Solutions of the Neumann Problem (9.1) 416
13 Proof of Propositions 12.1–12.2 418
14 Miscellaneous Remarks on Further Regularity 422
15 Gaussian Bounds on the Fundamental Solution 423
  15.1 The Gaussian Upper Bound 425
  15.2 The Gaussian Lower Bound 436

Problems and Complements 441
3c The Homogeneous Dirichlet Problem 441
5c Existence of Solutions of the Homogeneous Dirichlet Problem (3.1) by Galerkin Approximations 442
7c Traces of Functions on $\Sigma \overset{\text{def}}{=} \partial E \times (0,T]$ 445
8c The Inhomogeneous Dirichlet Problem 445
  8.1c Parabolic Quasi-Minima 446
9c The Neumann Problem 447
10c A Priori $L^\infty(E_T)$ Estimates for Solutions of the Dirichlet Problem (8.1) 448
12c A Priori $L^\infty(E_T)$ Estimates for Solutions of the Neumann Problem (9.1) 449
15c Gaussian Bounds on the Fundamental Solution 449

12 PARABOLIC DEGIORGI CLASSES 451
1 Quasi-Linear Equations and DeGiorgi Classes 451
  1.1 Parabolic DeGiorgi Classes 456
2 Local Boundedness of Functions in the PDG Classes 456
3 Hölder Continuity of Functions in the PDG Classes 459
  3.1 On the Proof of Theorem 3.1 460
4 Estimating the Values of u by the Measure of the Set Where u is Either Near μ^+ or Near μ^- 461
5 Reducing the Measure of the Set Where u is Either Near μ^+ or Near μ^- 464
  5.1 Proof of Proposition 5.1 464
6 Propagating in Time the Measure-Theoretical Information 467
  6.1 Proof of Proposition 6.1 468
7 Proof of Theorem 3.1 470
8 Boundary Parabolic DeGiorgi Classes: Dirichlet Data 472
  8.1 Lateral Conditions 474
  8.2 Initial Conditions 475
  8.3 Definition of Boundary Parabolic DeGiorgi Classes 476
  8.4 Continuity up to $\partial_p E_T$ of Functions in the Boundary PDG Classes (Dirichlet Data) 476
9 Boundary Parabolic DeGiorgi Classes: Neumann Data 479
  9.1 Lateral Boundary 481
  9.2 Definition of Boundary Parabolic DeGiorgi Classes 484
  9.3 Continuity up to S_T of Functions in the Boundary PDG Classes (Neumann Data) 484
10 The Harnack Inequality 485
  10.1 Proof of Theorem 10.1. Preliminaries 486
  10.2 Proof of Theorem 10.1. Expansion of Positivity 487
  10.3 Proof of Theorem 10.1 488
  10.4 The Mean Value Harnack Inequality 492
11 The Harnack Inequality Implies the Hölder Continuity 494
12 A Consequence of the Harnack Inequality 495
13 A More Straightforward Proof of the Hölder Continuity 498

Problems and Complements 499
2c Local Boundedness of Functions in the PDG Classes 499
3c Hölder Continuity of Solutions of Linear Parabolic Equations . 500
6c Propagating in Time the Measure-Theoretical Information 501
  6.1c Proof of Proposition 6.1c 501
7c Proof of Theorem 3.1 505
11c The Harnack Inequality 507

13 PARABOLIC EQUATIONS IN NONDIVERGENCE FORM 509
1 Introductory Material 509
1.1 Introduction 509
1.2 The Pucci Equation 513
1.3 The Bellman–Dirichlet Equation 516
1.4 Remarks on the Concept of Ellipticity 517
1.5 Equations of Mini-Max Type 518
2 Maximum Principles 519
2.1 Linear Equations 519
2.2 Quasi-Linear Equations 523
3 The Aleksandrov Maximum Principle 529
3.1 Basic Geometric Notions 529
3.2 Increasing Concave Hull of u 531
3.3 Auxiliary Lemmas 537
3.4 Embedding by Normal Mapping 539
3.5 Estimates of the Supremum of a Function 544
3.6 Maximum Principle for Nonlinear Operators 551
4 Local Estimates and the Harnack Inequality 552
4.1 A Local Maximum Principle 553
4.2 A Covering Lemma 558
4.3 Two Technical Lemmas 562
4.4 The Harnack Inequality for Linear Equations 569
4.5 The Harnack Inequality for Quasi-Linear Equations 576
4.6 Local Hölder Continuity of Solutions 580
4.7 Hölder Continuity of Solutions of Quasi-Linear Equations 583

Problems and Complements 584
1c Introductory Material 584
1.1c Introduction 584
1.3c The Bellman–Dirichlet Equation 585
3c The Aleksandrov Maximum Principle 589
3.5c Estimates of the Supremum of a Function 589

14 NAVIER–STOKES EQUATIONS 591
1 Navier–Stokes Equations in Dimensionless Form 591
2 Steady-State Flow with Homogeneous Boundary Data 593
2.1 Uniqueness of Solutions to (2.1) 594
3 Existence of Solutions to (2.1) 595
4 Nonhomogeneous Boundary Data 597
4.1 Uniqueness of Solutions to (4.1) 599
4.2 Existence of Solutions to (4.1) 599
5 Recovering the Pressure 600
6 Steady-State Flows in Unbounded Domains 601

6.1 Assumptions on **a** and **f** . . . 602
6.2 Toward a Notion of a Solution to (6.1) . . . 603
7 Existence of Solutions to (6.1) . . . 603
7.1 Approximating Solutions and A Priori Estimates . . . 603
7.2 The Limiting Process . . . 606
8 Time-Dependent Navier–Stokes Equations in Bounded Domains . . . 606
9 The Galerkin Approximations . . . 609
10 Selecting Subsequences Strongly Convergent in $L^2(E_T;\mathbb{R}^3)$ . . . 611
11 The Limiting Process and Proof of Theorem 8.1 . . . 613
12 Higher Integrability and Some Consequences . . . 615
12.1 The $L^{p,q}(E_T;\mathbb{R}^N)$ Spaces . . . 615
12.2 The Case $N=2$ . . . 617
13 Energy Identity for the Homogeneous Boundary Value Problem with Higher Integrability . . . 617
14 Stability and Uniqueness for the Homogeneous Boundary Value Problem with Higher Integrability . . . 620
15 Local Regularity of Solutions with Higher Integrability . . . 622
16 Proof of Theorem 15.1 – Introductory Results . . . 625
17 Proof of Theorem 15.1 Continued . . . 627
18 Proof of Theorem 15.1 Concluded . . . 631
19 Regularity of the Initial-Boundary Value Problem . . . 632
20 Recovering the Pressure in the Time-Dependent Equations . . . 633

Problems and Complements . . . 635
1c Navier–Stokes Equations in Dimensionless Form . . . 635
4c Nonhomogeneous Boundary Data . . . 636
4.1c Solving (4.1) by Galerkin Approximations . . . 637
4.2c Extending Fields $\mathbf{a}\in W^{\frac{1}{2},2}(\partial E;\mathbb{R}^3)$, Satisfying (4.2) into Solenoidal Fields $\mathbf{b}\in W^{1,2}(E;\mathbb{R}^3)$ . . . 638
4.3c Proof of Proposition 4.3c . . . 644
4.4c The Case of a General Domain E . . . 645
5c Recovering the Pressure . . . 646
5.1c Proof of Proposition 5.1 for $\mathbf{u}\in H^\perp\cap C^\infty(E;\mathbb{R}^3)$ . . . 646
5.2c Proof of Proposition 5.1 for $\mathbf{u}\in H^\perp$ . . . 647
5.3c More General Versions of Proposition 5.1 . . . 649
8c Time-Dependent Navier–Stokes Equations in Bounded Domains . . . 649
10c Selecting Subsequences Strongly Convergent in $L^2(E_T)$ . . . 649
10.1c Proof of Friedrichs' Lemma . . . 650
10.2c Compact Embedding of $W^{1,p}$ into $L^q(Q)$ for $1\le q<p_*$ . . . 650
10.3c Solutions Global in Time . . . 651
11c The Limiting Process and Proof of Theorem 8.1 . . . 651
12c Higher Integrability and Some Consequences . . . 652

13c Energy Identity for the Homogeneous Boundary Value Problem with Higher Integrability . . . 652
15c Local Regularity of Solutions with Higher Integrability . . . 653
16c Proof of Theorem 15.1 – Introductory Results . . . 653
20c Recovering the Pressure in the Time-Dependent Equations . . . 656

15 QUASI-LINEAR FIRST-ORDER SYSTEMS . . . 657
1 Hyperbolic Systems . . . 657
2 Some Examples . . . 658
  2.1 Incompressible Euler Equations . . . 659
  2.2 Reacting Gas Flow in 1–Space Dimension . . . 660
  2.3 A Weakly Hyperbolic System Arising in Magnetohydrodynamics . . . 661
3 Uniqueness of Smooth Solutions . . . 663
4 Existence of Solutions: The Linear Theory . . . 665
  4.1 A Family of Approximating Problems . . . 666
  4.2 Estimate of H_i, $i = 1, 2, 3$ . . . 668
  4.3 Proof of Theorem 4.1 . . . 671
5 Existence of Solutions: The Nonlinear Theory . . . 672
6 Counterexamples to Uniqueness . . . 674
7 Back to Quasi-Linear First-Order Strictly Hyperbolic Systems . . . 679
  7.1 A First Example . . . 681
  7.2 A Second Example . . . 682
8 Lax Shock Conditions . . . 682
9 Shocks . . . 684
  9.1 An Example . . . 687
10 Centered Rarefaction Waves . . . 688
  10.1 An Example . . . 690
11 Contact Discontinuities . . . 691
  11.1 An Example . . . 693
12 The Riemann Problem . . . 693
13 Convex Entropies . . . 696
  13.1 Examples of Entropies for 2 × 2 Systems . . . 703
14 The Glimm Existence Result . . . 704
15 Some Final Comments . . . 706

Problems and Complements . . . 706
2c Some Examples . . . 707
5c Existence of Solutions: The Nonlinear Theory . . . 708
6c Proof of Theorem 6.1 . . . 708
7c Back to Quasi-Linear First-Order Strictly Hyperbolic Systems . . . 715
12c The Riemann Problem . . . 716
13c Convex Entropies . . . 720

References . . . 723

Index . . . 737

Preface

Preface to the Third Edition

This is a revised and extended version of the 2010 second edition of the introduction to partial differential equations authored by Emmanuele DiBenedetto. Even though the material is essentially the same for a large part, nevertheless there is a relevant new portion, which covers different topics.

The guiding principle that led us during the preparation of this monograph is twofold, and indeed it was already at the heart of the second edition.

On one hand, there is the close and existential connection between the theory of PDEs and the modelling of physical phenomena: quite a number of well-known physical facts are described in terms of partial differential equations; the nature of the phenomenon suggests what kind of results mathematicians should look for, and at the same time, it is a common fact that, once established, analytical tools prove to be powerful even largely beyond the problem they were originally developed for.

On the other hand, in the mathematical theory of PDEs there are topics, which can by now be considered as classical, but at the same time serve as a natural introduction to very active research fields. Therefore, it seems quite natural to provide a self-contained introduction to these results, specifically aimed at people who approach the field for the first time.

With these two main ideas as a sort of roadmap, we set out to work, and revised the 2010 second edition. Some changes are perhaps limited in terms of number of pages, but they are not less important than other topics, to which full new chapters are devoted.

In the Complements of Chapter 0, we offered a detailed description of Einstein's approach to the Brownian motion; in Chapter 1 we added Holmgren's uniqueness theorem, which in our opinion represents a natural complement to the Cauchy–Kowalewski theorem; in Chapter 5, Section 2.2c in the Complements about the porous medium equation was largely expanded, following suggestions by students who used the monograph as textbook in their PDE class in the past; as for Chapter 6 we briefly revised the sections devoted to Problems and Complements, adding further exercises, examples, and even some explicit hints of solutions.

The first substantial novelty is represented by Chapters 11 and 12, where linear parabolic equations in divergence form with bounded and measurable coefficients, and parabolic DeGiorgi classes are respectively discussed. These two chapters represent a sort of parabolic analogue of Chapters 9 and 10, and indeed a certain degree of similarity in their structures is apparent and explicitly sought for. Even though the notion of parabolic DeGiorgi class might not be as widely accepted as the corresponding elliptic one, nevertheless they are being used more and more, and they are of considerable help in highlighting the deep structural nature of many results. Chapter 11 ends with the Gaussian bounds on the fundamental solution, which lie at the heart of the celebrated Nash's result, but are not so easy to find in the available literature.

Chapter 13 is devoted to parabolic equations in nondivergence form, modelled on the prototype class of linear parabolic equations with bounded and measurable coefficients. Here the focus is on solutions in Sobolev spaces, and not on solutions in the viscosity sense. The strict analogy with the corresponding results of Chapter 12 is clear, as is the deep difference in terms of the analytical tools needed to achieve them. We discussed at length whether it was the case to add also an analogous chapter about elliptic equations in nondivergence form, but we decided in the negative, due to the enormous amount of literature already available on this topic.

The Navier–Stokes equations are discussed in Chapter 14. We tried and concentrated on the main results, at the same time giving a brief introduction to the most important and interesting open problems.

Chapter 15 is the last one and deals with quasi-linear first order systems. We first consider smooth solutions, and then treat the main classical results due to the pioneering work of Lax, Glimm, and collaborators. We obviously do not dwell on the recent accomplishments in this field, where new and important results are still regularly achieved.

Vanderbilt University, *Emmanuele DiBenedetto*,
University of Pavia, *Ugo Gianazza*,
May 2021

Besides the sheer scientific aspect described above, there is another difference with respect to the previous edition. When Emmanuele DiBenedetto decided to prepare a new edition of his 2010 manuscript, he asked me to help him; I was very surprised by this request, but at the same time very glad to accept it. The plan was discussed together, as well as the distribution of who should take care of what. The pandemic made things a bit more difficult than what we had originally expected, because a lot of work had to be done at distance, but eventually it turned out to be not so invasive. What really impacted the

Emmanuele DiBenedetto, 1947–2021

work was Emmanuele's illness, which cut too short an extraordinary life and a great friendship. Up to his last days, Emmanuele kept on suggesting, advising, correcting his part, and mine as well.

When Emmanuele passed away, Chris Eder and Springer were very supportive and confident that I could finish the work by myself, simply following the original plan. I am very grateful for that. The work you now have in your hands is precisely the result of such a roadmap. Emmanuele's understanding and vision of PDEs theory in general, and of its classical aspects in particular were impressive, and always freely shared, without pretending to keep things for himself in any way: although this monograph is mostly a fruit of his work, at the same time it is dedicated to him.

When Emmanuele completed a monograph, he used to send a copy to people active in the field, in order to collect opinions, suggestions, criticisms, etc. It seemed to me that it is a very good way to proceed and that it was worth doing the same in this case as well. Therefore, I am indebted to many friends, collaborators, colleagues and students, who read large parts of this revised version, with particular care for the new chapters, and suggested changes, improvements, corrections, or simply used the previous editions in their classes and reported about the comments by students.

In particular, I am grateful with Naian Liao, who read the parabolic chapters and suggested various improvements, Luc Tartar, who provided very useful and interesting literature about quasi-linear first-order systems, Giuseppe Savaré and Sandro Salsa for their kind remarks, Enrico Vitali for a lot of invaluable bibliographic references, Giorgio Metafune, who provided the example of Section 10c of the Complements of Chapter 11. As for students, special thanks go to G. Cavalleri, M. Ferrari, B. Minniti, and E. Tolotti.

University of Pavia, *Ugo Gianazza*,
July 2023

Preface to the Second Edition

This is a revised and extended version of my 1995 elementary introduction to partial differential equations. The material is essentially the same except for three new chapters. The first (Chapter 8) is about nonlinear equations of first order and in particular Hamilton–Jacobi equations. It builds on the continuing idea that PDEs, although a branch of mathematical analysis, are closely related to models of physical phenomena. Such underlying physics in turn provides ideas of solvability. The Hopf variational approach to the Cauchy problem for Hamilton–Jacobi equations is one of the clearest and most incisive examples of such an interplay. The method is a perfect blend of classical mechanics, through the role and properties of the Lagrangian and Hamiltonian, and calculus of variations. A delicate issue is that of identifying "uniqueness classes." An effort has been made to extract the geometrical conditions on the graph of solutions, such as quasi-concavity, for uniqueness to hold.

Chapter 9 is an introduction to weak formulations, Sobolev spaces, and direct variational methods for linear and quasi-linear elliptic equations. While terse, the material on Sobolev spaces is reasonably complete, at least for a PDE user. It includes all the basic embedding theorems, including their proofs, and the theory of traces. Weak formulations of the Dirichlet and Neumann problems build on this material. Related variational and Galerkin methods, as well as eigenvalue problems, are presented within their weak framework. The Neumann problem is not as frequently treated in the literature as the Dirichlet problem; an effort has been made to present the underlying theory as completely as possible. Some attention has been paid to the local behavior of these weak solutions, both for the Dirichlet and Neumann problems. While efficient in terms of existence theory, weak solutions provide limited information on their local behavior. The starting point is a sup bound for the solutions and weak forms of the maximum principle. A further step is their local Hölder continuity.

An introduction to these local methods is in Chapter 10 in the framework of DeGiorgi classes. While originating from quasi-linear elliptic equations, these classes have a life of their own. The investigation of the local and boundary behavior of functions in these classes, involves a combination of methods from PDEs, measure theory, and harmonic analysis. We start by tracing them back to quasi-linear elliptic equations, and then present in detail some of these methods. In particular, we establish that functions in these classes are locally bounded and locally Hölder continuous, and we give conditions for the regularity to extend up to the boundary. Finally, we prove that non-negative functions on the DeGiorgi classes satisfy the Harnack inequality. This, on the one hand, is a surprising fact, since these classes require only some sort of Caccioppoli-type energy bounds. On the other hand, this raises the question of understanding their structure, which to date is still not fully understood. While some facts about these classes are scattered in the literature, this is perhaps the first systematic presentation of DeGiorgi classes in their own right. Some of the material is as recent as last year. In this respect, these last two chapters provide a background on a spectrum of techniques in local behavior of solutions of elliptic PDEs, and build toward research topics of current active investigation.

The presentation is more terse and streamlined, than in the first edition. Some elementary background material (Weierstrass Theorem, mollifiers, Theorem of Ascoli–Arzelá, Jensen's inequality, etc..) has been removed.

I am indebted to many colleagues and students who, over the past twelve years, have offered critical suggestions and pointed out misprints, imprecise statements, and points that were not clear on a first reading. Among these Giovanni Caruso, Xu Guoyi, Hanna Callender, David Petersen, Mike O'Leary, Changyong Zhong, Justin Fitzpatrick, Abey Lopez, Haichao Wang. Special thanks go to Matt Calef for reading carefully a large portion of the manuscript and providing suggestions and some simplifying arguments. The help of U. Gianazza, has been greatly appreciated. He has read the entire manuscript with extreme care and dedication, picking up points that needed to be clarified. I am very much indebted to Ugo.

I would like to thank Avner Friedman, James Serrin, Constantine Dafermos, Bob Glassey, Giorgio Talenti, Luigi Ambrosio, Juan Manfredi, John Lewis, Vincenzo Vespri, and Gui Qiang Chen for examining the manuscript in detail and for providing valuable comments. Special thanks to David Kinderlehrer for his suggestion to include material on weak formulations and direct methods. Without his input and critical reading, the last two chapters probably would not have been written. Finally, I would like to thank Ann Kostant and the entire team at Birkhäuser for their patience in coping with my delays.

Vanderbilt University, *Emmanuele DiBenedetto*,
June 2009

Preface to the First Edition

These notes are meant to be a self contained, elementary introduction to partial differential equations (PDEs). They assume only advanced differential calculus and some basic L^p theory. Although the basic equations treated in this book, given its scope, are linear, I have made an attempt to approach then from a nonlinear perspective.

Chapter 1 is focused on the Cauchy-Kowalewski theorem. We discuss the notion of characteristic surfaces and use it to classify partial differential equations. The discussion grows from equations of second order in two variables to equations of second order in N variables to PDEs of any order in N variables.

In Chapters 2 and 3 we study the Laplace equation and connected elliptic theory. The existence of solutions for the Dirichlet problem is proven by the Perron method. This method clarifies the structure of the sub(super)-harmonic functions, and it is closely related to the modern notion of *viscosity solution*. The elliptic theory is complemented by the Harnack and Liouville theorems, the simplest version of Schauder's estimates, and basic L^p-potential estimates. Then, in Chapter 3 the Dirichlet and Neumann problems, as well as eigenvalue problems for the Laplacean, are cast in terms of integral equations. This requires some basic facts concerning double layer potentials and the notion of compact subsets of L^p, which we present.

In Chapter 4 we present the Fredholm theory of integral equations and derive necessary and sufficient conditions for solving the Neumann problem. We solve eigenvalue problems for the Laplacean, generate orthonormal systems in L^2, and discuss questions of completeness of such systems in L^2. This provides a theoretical basis for the method of separation of variables.

Chapter 5 treats the heat equation and related parabolic theory. We introduce the representation formulas, and discuss various comparison principles. Some focus has been placed on the uniqueness of solutions to the Cauchy problem and their behavior as $|x| \to \infty$. We discuss Widder's theorem and the structure of the non-negative solutions. To prove the parabolic Harnack estimate we have used an idea introduced by Krylov and Safonov in the context of fully nonlinear equations.

The wave equation is treated in Chapter 6 in its basic aspects. We derive representation formulas and discuss the role of the characteristics, propagation of signals, and questions of regularity. For general linear second-order hyperbolic equations in two variables, we introduce the Riemann function and prove its symmetry properties. The sections on Goursat problems represent a concrete application of integral equations of Volterra type.

Chapter 7 is an introduction to conservation laws. The main points of the theory are taken from the original papers of Hopf and Lax from the 1950s. Space is given to the minimization process and the meaning of taking the initial data in the sense of L^1. The uniqueness theorem we present is due to Kruzhkov (1970). We discuss the meaning of *viscosity solution* vis-à-vis the notion of sub-solutions and maximum principle for parabolic equations.

The theory is complemented by an analysis of the asymptotic behavior, again following Hopf and Lax.

Even though the layout is theoretical, I have indicated some of the physical origins of PDEs. Reference is made to potential theory, similarity solutions for the porous medium equation, generalized Riemann problems, etc.

I have also attempted to convey the notion of *ill posed* problems, mainly via some examples of Hadamard.

Most of the background material, arising along the presentation, has been stated and proved in the complements. Examples include the theorem of Ascoli–Arzelà, Jensen's inequality, the characterization of compactness in L^p, mollifiers, basic facts on convex functions, and the Weierstrass theorem. A book of this kind is bound to leave out a number of topics, and this book is no exception. Perhaps the most noticeable omission here is some treatment of numerical methods.

These notes have grown out of courses in PDEs I taught over the years at Indiana University, Northwestern University and the University of Rome II, Italy. My thanks go to the numerous students who have pointed out misprints and imprecise statements. Of these, special thanks go to M. O'Leary, D. Diller, R. Czech, and A. Grillo. I am indebted to A. Devinatz for reading a large portion of the manuscript and for providing valuable critical comments. I have also benefited from the critical input of M. Herrero, V. Vespri, and J. Manfredi, who have examined parts of the manuscript. I am grateful to E. Giusti for his help with some of the historical notes. The input of L. Chierchia has been crucial. He has read a large part of the manuscript and made critical remarks and suggestions. He has also worked out in detail a large number of the problems and supplied some of his own. In particular, he wrote the first draft of problems **2.7**–**2.13** of Chapter 5 and **6.10**–**6.11** of Chapter 6. Finally I like to thank M. Cangelli and H. Howard for their help with the graphics.

0

PRELIMINARIES

1 Green's Theorem

Let E be an open set in $\mathbb{R}^N$, and let k be a non-negative integer. Denote by $C^k(E)$ the collection of all real-valued, k-times continuously differentiable functions in E. A function f is in $C^k_o(E)$ if $f \in C^k(E)$, and its support is contained in E. A function $f : \bar{E} \to \mathbb{R}$ is in $C^k(\bar{E})$, if $f \in C^k(E)$ and all partial derivatives $\partial^\ell f / \partial x_i^\ell$ for all $i = 1, \dots, N$ and $\ell = 0, \dots, k$, admit continuous extensions up to ∂E. The boundary ∂E is of class C^1 if for all $y \in \partial\Omega$, there exists $\varepsilon > 0$ such that within the ball $B_\varepsilon(y)$ centered at y and radius ε, ∂E can be implicitly represented, in a local system of coordinates, as a level set of a function $\varPhi \in C^1(B_\varepsilon(y))$ such that $|\nabla\varPhi| \neq 0$ in $B_\varepsilon(y)$.

If ∂E is of class C^1, let $\mathbf{n}(x) = \big(n_1(x), \dots, n_N(x)\big)$ denote the unit normal exterior to E at $x \in \partial E$. Each of the components $n_j(\cdot)$ is well defined as a continuous function on ∂E. A real vector-valued function

$$\bar{E} \ni x \to \mathbf{f}(x) = \big(f_1(x), \dots, f_N(x)\big) \in \mathbb{R}^N$$

is of class $C^k(E)$, $C^k(\bar{E})$, or $C^k_o(E)$ if all components f_j belong to these classes.

Theorem 1.1. *Let E be a bounded domain of $\mathbb{R}^N$ with boundary ∂E of class C^1. Then for every $\mathbf{f} \in C^1(\bar{E})$*

$$\int_E \operatorname{div} \mathbf{f}\, dx = \int_{\partial E} \mathbf{f} \cdot \mathbf{n}\, d\sigma$$

where dx is the Lebesgue measure in E and $d\sigma$ denotes the Lebesgue surface measure on ∂E.

This is also referred to as the *divergence theorem*, or as the formula of *integration by parts*. It continues to hold if $\mathbf{n}$ is only $d\sigma$-a.e. defined in ∂E. For example, ∂E could be a cube in $\mathbb{R}^N$. More generally, ∂E could be the finite

© Springer Nature Switzerland AG 2023

E. DiBenedetto, U. Gianazza, *Partial Differential Equations*,
Cornerstones, https://doi.org/10.1007/978-3-031-46618-2_1

union of portions of surfaces of class C^1. The domain E need not be bounded, provided $|\mathbf{f}|$ and $|\nabla\mathbf{f}|$ decay sufficiently fast as $|x| \to \infty$.[1]

1.1 Differential Operators and Adjoints

Given a symmetric matrix $(a_{ij}) \in \mathbb{R}^N \times \mathbb{R}^N$, a vector $\mathbf{b} \in \mathbb{R}^N$, and $c \in \mathbb{R}$, consider the formal expression

$$\mathcal{L}(\cdot) = a_{ij}\frac{\partial^2}{\partial x_i \partial x_j} + b_i\frac{\partial}{\partial x_i} + c \tag{1.1}$$

where we have adopted the Einstein summation convention, i.e., repeated indices in a monomial expression mean summation over those indices. The formal adjoint of $\mathcal{L}(\cdot)$ is

$$\mathcal{L}^*(\cdot) = a_{ij}\frac{\partial^2}{\partial x_i \partial x_j} - b_i\frac{\partial}{\partial x_i} + c.$$

Thus $\mathcal{L} = \mathcal{L}^*$ if $\mathbf{b} = 0$. If $u, v \in C^2(\bar{E})$ for a bounded open set $E \subset \mathbb{R}^N$ with boundary ∂E of class C^1, the divergence theorem yields the Green's formula

$$\int_E [v\mathcal{L}(u) - u\mathcal{L}^*(v)]dx = \int_{\partial E} [(va_{ij}u_{x_j}n_i - ua_{ij}v_{x_i}n_j) + uv\mathbf{b}\cdot\mathbf{n}]d\sigma. \tag{1.2}$$

If $u, v \in C_o^2(E)$, then

$$\int_E [v\mathcal{L}(u) - u\mathcal{L}^*(v)]dx = 0. \tag*{$(1.2)_o$}$$

More generally, the entries of the matrix (a_{ij}) as well as $\mathbf{b}$ and c might be smooth functions of x. In such a case, for $v \in C^2(\bar{E})$, define

$$\mathcal{L}^*(v) = \frac{\partial^2 (a_{ij}v)}{\partial x_i \partial x_j} - \frac{\partial (b_i v)}{\partial x_i} + cv.$$

The Green's formula $(1.2)_o$ continues to hold for every pair of functions $u, v \in C_o^2(E)$. If u and v do not vanish near ∂E, a version of (1.2) continues to hold, where the right-hand side contains the extra boundary integral

$$\int_{\partial E} uv\frac{\partial a_{ij}}{\partial x_i}n_j\, d\sigma.$$

---

[1] Identifying precise conditions on ∂E and $\mathbf{f}$ for which one can *integrate by parts* is part of *geometric measure theory* ([102]).

2 The Continuity Equation

Let $t \to E(t)$ be a set-valued function that associates to each t in some open interval $I \subset \mathbb{R}$ a bounded open set $E(t) \subset \mathbb{R}^N$, for some $N \geq 2$. Assume that the boundaries $\partial E(t)$ are uniformly of class C^1, and that there exists a bounded open set $E \subset \mathbb{R}^N$ such that $E(t) \subset E$, for all $t \in I$. Our aim is to compute the derivative

$$\frac{d}{dt}\int_{E(t)} \rho(x,t)dx \qquad \text{for a given} \quad \rho \in C^1(E \times I).$$

Regard points $x \in E(t)$ as moving along the trajectories $t \to x(t)$ with velocities $\dot{x} = \mathbf{v}(x,t)$. Assume that the motion, or deformation, of $E(\cdot)$ is smooth in the sense that $(x,t) \to \mathbf{v}(x,t)$ is continuous in a neighborhood of $E \times I$. Forming the difference quotient gives

$$\begin{aligned}
&\frac{d}{dt}\int_{E(t)} \rho(x,t)dx \\
&\quad = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\Big(\int_{E(t+\Delta t)} \rho(x,t+\Delta t)dx - \int_{E(t)} \rho(x,t)dx\Big) \\
&\quad = \lim_{\Delta t \to 0} \int_{E(t)} \frac{\rho(x,t+\Delta t) - \rho(x,t)}{\Delta t} dx \\
&\qquad + \lim_{\Delta t \to 0} \frac{1}{\Delta t}\Big(\int_{E(t+\Delta t)-E(t)} \rho(t)dx - \int_{E(t)-E(t+\Delta t)} \rho(t)dx\Big).
\end{aligned} \tag{2.1}$$

The first limit is computed by carrying the limit under the integral, yielding

$$\lim_{\Delta t \to 0} \int_{E(t)} \frac{\rho(x,t+\Delta t) - \rho(x,t)}{\Delta t} dx = \int_{E(t)} \rho_t \, dx.$$

As for the second, first compute the difference of the last two volume integrals by means of Riemann sums as follows. Fix a number $0 < \Delta\sigma \ll 1$, and approximate $\partial E(t)$ by means of a polyhedron with faces of area not exceeding $\Delta\sigma$ and tangent to $\partial E(t)$ at some of their interior points. Let $\{F_1, \dots, F_n\}$ for some $n \in \mathbb{N}$ be a finite collection of faces making up the approximating polyhedron, and let x_i for $i = 1, \dots, n$, be a selection of their tangency points with $\partial E(t)$. Then approximate the set

$$\big[E(t+\Delta t) - E(t)\big] \bigcup \big[E(t) - E(t+\Delta t)\big]$$

by the union of the cylinders of basis F_i and height $\mathbf{v}(x_i,t) \cdot \mathbf{n}\Delta t$, built with their axes parallel to the outward normal to $\partial E(t)$ at x_i. Therefore, for Δt fixed

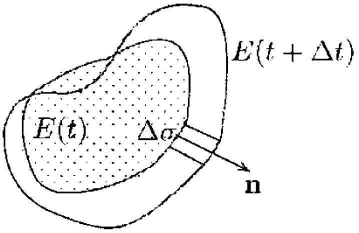

Fig. 2.1

$$\frac{1}{\Delta t}\left(\int_{E(t+\Delta t)-E(t)}\rho(t)dx-\int_{E(t)-E(t+\Delta t)}\rho(t)dx\right)$$
$$=\lim_{\Delta\sigma\to 0}\sum_{i=1}^{n}\rho(x_i,t)\mathbf{v}(x_i,t)\cdot\mathbf{n}\Delta\sigma+O(\Delta t)=\int_{\partial E(t)}\rho\mathbf{v}\cdot\mathbf{n}d\sigma+O(\Delta t).$$

Letting now $\Delta t\to 0$ in (2.1) yields

$$\frac{d}{dt}\int_{E(t)}\rho\,dx=\int_{E(t)}\rho_t\,dx+\int_{\partial E(t)}\rho\mathbf{v}\cdot\mathbf{n}\,d\sigma. \tag{2.2}$$

By the Green's theorem

$$\int_{\partial E(t)}\rho\mathbf{v}\cdot\mathbf{n}\,d\sigma=\int_{E(t)}\operatorname{div}(\rho\mathbf{v})\,dx.$$

Therefore (2.2) can be equivalently written as

$$\frac{d}{dt}\int_{E(t)}\rho\,dx=\int_{E(t)}[\rho_t+\operatorname{div}(\rho\mathbf{v})]\,dx. \tag{2.3}$$

Consider now an ideal fluid filling a region $E\subset\mathbb{R}^3$. Assume that the fluid is compressible (say a gas) and let $(x,t)\to\rho(x,t)$ denote its density. At some instant t, cut a region $E(t)$ out of E and follow the motion of $E(t)$ as if each of its points were identified with the moving particles. Whatever the subregion $E(t)$, during the motion the mass is conserved. Therefore

$$\frac{d}{dt}\int_{E(t)}\rho\,dx=0.$$

By the previous calculations and the arbitrariness of $E(t)\subset E$

$$\rho_t+\operatorname{div}(\rho\mathbf{v})=0\qquad\text{in}\quad E\times\mathbb{R}. \tag{2.4}$$

This is referred to as the equation of continuity or conservation of mass.

3 The Heat Equation and the Laplace Equation

Any quantity that is conserved as it *moves* within an open set E with velocity $\mathbf{v}$ satisfies the *conservation law* (2.4). Let u be the temperature of a material homogeneous body occupying the region E. If c is the heat capacity, the thermal energy stored at $x \in E$ at time t is $cu(x,t)$. By Fourier's law the energy "moves" following gradients of temperature, i.e.,

$$cu\mathbf{v} = -k\nabla u \tag{3.1}$$

where k is the conductivity ([78, 30]). Putting this in (2.4) yields the *heat equation*

$$u_t - \frac{k}{c}\Delta u = 0. \tag{3.2}$$

Now let u be the pressure of a fluid moving with velocity $\mathbf{v}$ through a region E of $\mathbb{R}^N$ occupied by a porous medium. The porosity p_o of the medium is the relative infinitesimal fraction of space occupied by the pores and available to the fluid. Let μ, k, and ρ denote respectively kinematic viscosity, permeability, and density. By Darcy's law ([226])

$$\mathbf{v} = -\frac{kp_o}{\mu}\nabla u. \tag{3.3}$$

Assume that k and μ are constant. If the fluid is incompressible, then $\rho =$ const, and it follows from (2.4) that $\operatorname{div}\mathbf{v} = 0$. Therefore the pressure u satisfies

$$\operatorname{div}\nabla u = \Delta u = u_{x_i x_i} = 0 \quad \text{in } E. \tag{3.4}$$

The latter is the Laplace equation for the function u. A fluid whose velocity is given as the gradient of a scalar function is a *potential fluid* ([268]).

3.1 Variable Coefficients

Consider now the same physical phenomena taking place in nonhomogeneous, anisotropic media. For heat conduction in such media, temperature gradients might generate heat propagation in preferred directions, which themselves might depend on $x \in E$. As an example one might consider the heat diffusion in a solid of given conductivity, in which is embedded a bundle of curvilinear material fibers of different conductivity. Thus in general, the conductivity of the composite medium is a tensor dependent on the location $x \in E$ and time t, represented formally by an $N \times N$ matrix $k = \big(k_{ij}(x,t)\big)$. For such a tensor, the product on the right-hand side of (3.1) is the row-by-column product of the matrix (k_{ij}) and the column vector ∇u. Enforcing the same conservation of energy (2.4) yields a nonhomogeneous, anisotropic version of the heat equation (3.2), in the form

$$u_t - \big(a_{ij}(x,t)u_{x_i}\big)_{x_j} = 0 \quad \text{in } E, \text{ where } a_{ij} = \frac{k_{ij}}{c}. \tag{3.5}$$

Similarly, the permeability of a nonhomogeneous, anisotropic porous medium is a position-dependent tensor $\big(k_{ij}(x)\big)$. Then, analogous considerations applied to (3.3), imply that the velocity potential u of the flow of a fluid in a heterogeneous, anisotropic porous medium satisfies the partial differential equation

$$\big(a_{ij}(x)u_{x_i}\big)_{x_j} = 0 \quad \text{in } E, \text{ where } a_{ij} = \frac{p_o k_{ij}}{\mu}. \tag{3.6}$$

The physical, tensorial nature of either heat conductivity or permeability of a medium implies that (a_{ij}) is symmetric, bounded, and positive definite in E. However, no further information is available on these coefficients, since they reflect interior properties of physical domains, not accessible without altering the physical phenomenon we are modeling. This raises the question of the meaning of (3.5)–(3.6). Indeed, even if $u \in C^2(E)$, the indicated operations are not well defined for $a_{ij} \in L^\infty(E)$. A notion of solution will be given in Chapter 10, along with solvability methods.

Equations (3.5)–(3.6) are said to be in divergence form. Equations in non-divergence form are of the type

$$a_{ij}(x)u_{x_i x_j} = 0 \quad \text{in } E \tag{3.7}$$

and arise in the theory of stochastic control ([146]).

4 A Model for the Vibrating String

Consider a material string of constant linear density ρ whose end points are fixed, say at 0 and 1. Assume that the string is vibrating in the plane (x, y), set the interval $(0, 1)$ on the x-axis, and let $(x, t) \to u(x, t)$ be the y-coordinate of the string at the point $x \in (0, 1)$ at the instant $t \in \mathbb{R}$. The basic physical assumptions are:

(i) The dimensions of the cross sections are negligible with respect to the length, so that the string can be identified, for all t, with the graph of $x \to u(x, t)$.
(ii) Let $(x, t) \to \mathbf{T}(x, t)$ denote the *tension*, i.e., the sum of the internal forces per unit length, generated by the displacement of the string. Assume that $\mathbf{T}$ at each point $(x, u(x, t))$ is tangent to the string. Letting $T = |\mathbf{T}|$, assume that $(x, t) \to T(x, t)$ is t-independent.
(iii) Resistance of the material to flexure is negligible with respect to the tension.
(iv) Vibrations are small in the sense that $|u|^\theta$ and $|u_x|^\theta$ for $\theta > 1$ are negligible when compared with $|u|$ and $|u_x|$.

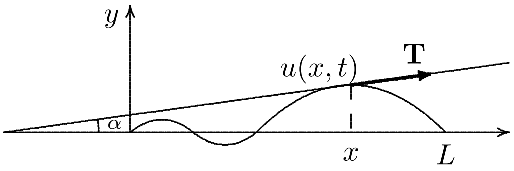

Fig. 4.1

The tangent line to the graph of $u(\cdot, t)$ at $(x, u(x, t))$ forms with the x-axis an angle $\alpha \in (0, \pi/2)$ given by

$$\sin\alpha = \frac{u_x}{\sqrt{1+u_x^2}}.$$

Therefore the vertical component of the tension $\mathbf{T}$ at $(x, u(x, t))$ is

$$T\sin\alpha = T\frac{u_x}{\sqrt{1+u_x^2}}.$$

Consider next, for t fixed, a small interval $(x_1, x_2) \subset (0, 1)$ and the corresponding portion of the string of extremities $\big(x_1, u(x_1, t)\big)$ and $\big(x_2, u(x_2, t)\big)$. Such a portion is in instantaneous equilibrium if both the x and y components of the sum of all forces acting on it are zero. The components in the y-direction are:

1. The difference of the y-components of $\mathbf{T}$ at the two extremities, i.e.,
$$\left(T\frac{u_x}{\sqrt{1+u_x^2}}\right)(x_2, t) - \left(T\frac{u_x}{\sqrt{1+u_x^2}}\right)(x_1, t) = \int_{x_1}^{x_2} \frac{\partial}{\partial x}\left(T\frac{u_x}{\sqrt{1+u_x^2}}\right) dx.$$
2. The total load acting on the portion, i.e.,
$$-\int_{x_1}^{x_2} p(x, t)dx, \qquad \text{where } \ p(\cdot, t) = \{\text{load per unit length}\}.$$
3. The inertial forces due to the vertical acceleration $u_{tt}(x, t)$, i.e.,
$$\int_{x_1}^{x_2} \rho\frac{\partial^2}{\partial t^2}u(x, t)dx.$$

Therefore the portion of the string is instantaneously in equilibrium if

$$\int_{x_1}^{x_2} \rho\frac{\partial^2}{\partial t^2}u(x, t)dx = \int_{x_1}^{x_2} \left[\frac{\partial}{\partial x}\left(T\frac{u_x}{\sqrt{1+u_x^2}}\right)(x, t) + p(x, t)\right] dx.$$

Dividing by $\Delta x = x_2 - x_1$ and letting $\Delta x \to 0$ gives

$$\rho\frac{\partial^2}{\partial t^2}u - \frac{\partial}{\partial x}\left(T\frac{u_x}{\sqrt{1+u_x^2}}\right) = p \quad \text{ in } \ (0, 1)\times\mathbb{R}. \tag{4.1}$$

The balance of forces along the x-direction involves only the tension and gives

$$(T\cos\alpha)(x_1,t) = (T\cos\alpha)(x_2,t)$$

or equivalently

$$\left(\frac{T}{\sqrt{1+u_x^2}}\right)(x_1,t) = \left(\frac{T}{\sqrt{1+u_x^2}}\right)(x_2,t).$$

From this

$$\int_{x_1}^{x_2} \frac{\partial}{\partial x}\left(\frac{T}{\sqrt{1+u_x^2}}\right) dx = 0 \quad \text{for all } (x_1,x_2) \subset (0,1).$$

Therefore

$$\frac{\partial}{\partial x}\left(\frac{T}{\sqrt{1+u_x^2}}\right) = 0 \quad \text{and} \quad x \to \left(\frac{T}{\sqrt{1+u_x^2}}\right)(x,t) = T_o$$

for some $T_o > 0$ independent of x. In view of the physical assumptions (ii) and (iv), may take T_o also independent of t. These remarks in (4.1) yield the partial differential equation

$$\frac{\partial^2}{\partial t^2}u - c^2\frac{\partial^2 u}{\partial x^2} = f \quad \text{in } (0,1)\times\mathbb{R} \tag{4.2}$$

where

$$c^2 = \frac{T_o}{\rho} \quad \text{and} \quad f(x,t) = \frac{p(x,t)}{\rho}.$$

This is the *wave equation* in one space variable.

Remark 4.1 The assumption that ρ is constant is a “linear” assumption in the sense that leads to the linear wave equation (4.2). Nonlinear effects due to variable density were already observed by D. Bernoulli ([17]), and by S.D. Poisson ([206]).

5 Small Vibrations of a Membrane

A *membrane* is a rigid thin body of constant density ρ, whose thickness is negligible with respect to its extension. Assume that, at rest, the membrane occupies a bounded open set $E \subset \mathbb{R}^2$, and that it begins to vibrate under the action of a vertical load, say $(x,t) \to p(x,t)$. Identify the membrane with the graph of a smooth function $(x,t) \to u(x,t)$ defined in $E \times \mathbb{R}$ and denote by $\nabla u = (u_{x_1}, u_{x_2})$ the spatial gradient of u. The relevant physical assumptions are:

(i) Forces due to flexure are negligible.

(ii) Vibrations occur only in the direction $\mathbf{u}$ normal to the position of rest of the membrane. Moreover , vibrations are small, in the sense that $u_{x_i}u_{x_j}$ and uu_{x_i} for $i, j = 1, 2$ are negligible when compared to u and $|\nabla u|$.

(iii) The tension $\mathbf{T}$ has constant modulus, say $|\mathbf{T}| = T_o > 0$.

Cut a small ideal region $G_o \subset E$ with boundary ∂G_o of class C^1, and let G be the corresponding portion of the membrane. Thus G is the graph of $u(\cdot, t)$ restricted to G_o, or equivalently, G_o is the projection on the plane $u = 0$ of the portion G of the membrane. Analogously, introduce the curve Γ limiting G and its projection $\Gamma_o = \partial G_o$. The tension $\mathbf{T}$ acts at points $P \in \Gamma$ and is tangent to G at P and normal to Γ. If $\boldsymbol{\tau}$ is the unit vector of $\mathbf{T}$ and $\mathbf{n}$ is the exterior unit normal to G at P, let $\mathbf{e}$ be the unit tangent to Γ at P oriented so that the triple $\{\boldsymbol{\tau}, \mathbf{e}, \mathbf{n}\}$ is positive and $\boldsymbol{\tau} = \mathbf{e} \wedge \mathbf{n}$. Our aim is to compute the vertical component of $\mathbf{T}$ at $P \in \Gamma$. If $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is the positive unit triple along the coordinate axes, we will compute the quantity $\mathbf{T} \cdot \mathbf{k} = T_o \boldsymbol{\tau} \cdot \mathbf{k}$.

Consider a parametrization of Γ_o, say

$$s \to P_o(s) = \big(x_1(s), x_2(s)\big) \quad \text{for } s \in \{\text{some interval of } \mathbb{R}\}.$$

The unit exterior normal to ∂G_o is given by

$$\boldsymbol{\nu} = \frac{(x_2', -x_1')}{\sqrt{x_1'^2 + x_2'^2}}.$$

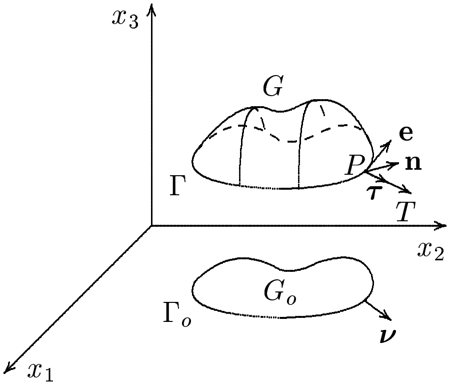

Fig. 5.1

Consider also the corresponding parametrization of Γ

$$s \to P(s) = \big(P_o(s), u(x_1(s), x_2(s), t)\big).$$

The unit tangent $\mathbf{e}$ to Γ is

$$\mathbf{e} = \frac{(x_1', x_2', x_3')}{\sqrt{x_1'^2 + x_2'^2 + x_3'^2}} = \frac{(\dot{P}_o, \dot{P}_o \cdot \nabla u)}{\sqrt{|\dot{P}_o|^2 + |\dot{P}_o \cdot \nabla u|^2}}$$

and the exterior unit normal to G at Γ is

$$\mathbf{n} = \frac{(-\nabla u, 1)}{\sqrt{1 + |\nabla u|^2}}.$$

Therefore

$$\boldsymbol{\tau} = \mathbf{e} \wedge \mathbf{n} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1' & x_2' & x_3' \\ -u_{x_1} & -u_{x_2} & 1 \end{pmatrix} \frac{1}{J}$$

where

$$J = \sqrt{1 + |\nabla u|^2}\sqrt{|\dot{P}_o|^2 + |\dot{P}_o \cdot \nabla u|^2}.$$

From this

$$J\boldsymbol{\tau} \cdot \mathbf{k} = (x_2', -x_1') \cdot \nabla u = |\dot{P}_o|\, \nabla u \cdot \boldsymbol{\nu}.$$

If β is the cosine of the angle between the vectors ∇u and $\dot{P}_o$

$$\boldsymbol{\tau} \cdot \mathbf{k} = \frac{\nabla u \cdot \boldsymbol{\nu}}{J_\beta}, \qquad J_\beta = \sqrt{1 + |\nabla u|^2}\sqrt{1 + \beta^2 |\nabla u|^2}.$$

Since

$$(1 + \beta^2 |\nabla u|^2) \le J_\beta \le (1 + |\nabla u|^2)$$

by virtue of the physical assumption (ii)

$$\mathbf{T} \cdot \mathbf{k} \approx T_o \nabla u \cdot \boldsymbol{\nu}.$$

Next, write down the equation of instantaneous equilibrium of the portion G of the membrane. The vertical loads on G, the vertical contribution of the tension $\mathbf{T}$, and the inertial force due to acceleration u_{tt} are respectively

$$\int_{G_o} p(x,t)\, dx, \qquad \int_{\partial G_o} T_o \nabla u \cdot \boldsymbol{\nu}\, d\sigma, \qquad \int_{G_o} \rho u_{tt}\, dx$$

where $d\sigma$ is the measure along the curve ∂G_o. Instantaneous equilibrium of every portion of the membrane implies that

$$\int_{G_o} \rho u_{tt}\, dx = \int_{\partial G_o} T_o \nabla u \cdot \boldsymbol{\nu}\, d\sigma + \int_{G_o} p(x,t)\, dx.$$

By Green's theorem

$$\int_{\partial G_o} T_o \nabla u \cdot \boldsymbol{\nu}\, d\sigma = \int_{G_o} T_o \operatorname{div}(\nabla u)\, dx.$$

Therefore

$$\int_{G_o} [\rho u_{tt} - T_o \operatorname{div} \nabla u - p]\, dx = 0$$

for all $t \in \mathbb{R}$ and all $G_o \subset E$. Thus

$$u_{tt} - c^2 \Delta u = f \quad \text{ in } E \times \mathbb{R} \tag{5.1}$$

where

$$c^2 = \frac{T_o}{\rho}, \qquad f = \frac{p}{\rho}, \qquad \Delta u = \operatorname{div}(\nabla u).$$

Equation (5.1), modeling small vibrations of a stretched membrane, is the two-dimensional wave equation.

6 Transmission of Sound Waves

An ideal *compressible* fluid is moving within a region $E \subset \mathbb{R}^3$. Let $\rho(x,t)$ and $\mathbf{v}(x,t)$ denote its density and velocity at $x \in E$ at the instant t. Each x can be regarded as being in motion along the trajectory $t \to x(t)$ with velocity $x'(t)$. Therefore, denoting by $v_i(x,t)$ the components of $\mathbf{v}$ along the x_i-axes, then

$$\dot{x}_i(t) = v_i(x(t), t), \qquad i = 1, 2, 3.$$

The acceleration has components

$$\ddot{x}_i = \frac{\partial v_i}{\partial t} + \frac{\partial v_i}{\partial x_j}\dot{x}_j = \frac{\partial v_i}{\partial t} + (\mathbf{v} \cdot \nabla) v_i$$

where ∇ denotes the gradient with respect to the space variables only. Cut any region $G_o \subset E$ with boundary ∂G_o of class C^1. Since G_o is instantaneously in equilibrium, the balance of forces acting on G_o must be zero. These are:

(i) The inertial forces due to acceleration

$$\int_{G_o} \rho \big[\mathbf{v}_t + (\mathbf{v} \cdot \nabla)\mathbf{v}\big]\, dx.$$

(ii) The Kelvin forces due to pressure. Let $p(x,t)$ be the pressure at $x \in E$ at time t. The forces due to pressure on G are

$$\int_{\partial G_0} p\boldsymbol{\nu}\, d\sigma, \qquad \boldsymbol{\nu} = \{\text{outward unit normal to } \partial G_o\}.$$

(iii) The sum of the external forces, and the internal forces due to friction

$$-\int_{G_o} \mathbf{f}\, dx.$$

Therefore

$$\int_{G_o} \rho[\mathbf{v}_t + (\mathbf{v}\cdot\nabla)\mathbf{v}]\,dx = -\int_{\partial G_o} p\boldsymbol{\nu}\,d\sigma + \int_{G_o} \mathbf{f}\,dx.$$

By Green's theorem

$$\int_{\partial G_o} p\boldsymbol{\nu}\,d\sigma = \int_{G_o} \nabla p\,dx.$$

Therefore by the arbitrariness of $G_o \subset E$

$$\rho\,[\mathbf{v}_t + (\mathbf{v}\cdot\nabla)\mathbf{v}] = -\nabla p + \mathbf{f} \quad \text{in } E\times\mathbb{R}. \tag{6.1}$$

Assume the following physical, modeling assumptions:

(a) The fluid moves with small relative velocity and small time variations of density. Therefore second-order terms of the type $v_i v_{j,x_h}$ and $\rho_t v_i$ are negligible with respect to first order terms.
(b) Heat transfer is slower than pressure drops, i.e., the process is adiabatic and $\rho = h(p)$ for some $h \in C^2(\mathbb{R})$.

Expanding $h(\cdot)$ about the equilibrium pressure p_o, renormalized to be zero, gives

$$\rho = a_o p + a_1 p^2 + \cdots .$$

Assume further that the pressure is close to the equilibrium pressure, so that all terms of order higher than one are negligible when compared to $a_o p$. These assumptions in (6.1) yield

$$\frac{\partial}{\partial t}(\rho\mathbf{v}) = -\nabla p + \mathbf{f} \quad \text{in } E\times\mathbb{R}.$$

Now take the divergence of both sides to obtain

$$\frac{\partial}{\partial t}\operatorname{div}(\rho\mathbf{v}) = -\Delta p + \operatorname{div}\mathbf{f} \quad \text{in } E\times\mathbb{R}.$$

From the continuity equation

$$\operatorname{div}(\rho\mathbf{v}) = -\rho_t = -a_o p_t.$$

Combining these remarks gives the equation of the pressure in the propagation of sound waves in a fluid, in the form

$$\frac{\partial^2 p}{\partial t^2} - c^2\Delta p = f \quad \text{in } E\times\mathbb{R} \tag{6.2}$$

where

$$c^2 = \frac{1}{a_o}, \qquad f = -\frac{\operatorname{div}\mathbf{f}}{a_o}.$$

Equation (6.2) is the wave equation in three space dimensions ([205]).

7 The Navier–Stokes System

The system (6.1) is rather general and holds for any ideal fluid. If the fluid is incompressible, then $\rho = \text{const}$, and the continuity equation (2.4) gives

$$\operatorname{div} \mathbf{v} = 0. \tag{7.1}$$

If in addition the fluid is viscous, the internal forces due to friction can be represented by $\mu\rho\Delta\mathbf{v}$, where $\mu > 0$ is the *kinematic viscosity* ([268]). Therefore (6.1) yields the Navier–Stokes system

$$\frac{\partial}{\partial t}\mathbf{v} - \mu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \frac{1}{\rho}\nabla p = \mathbf{f}_e \tag{7.2}$$

where $\mathbf{f}_e = \mathbf{f}/\rho$ are the external forces acting on the system. The unknowns are the three components of the velocity and the pressure p, to be determined from the system of four equations (7.1) and (7.2).

8 The Euler Equations

Let S denote the entropy function of a gas undergoing an adiabatic process. The pressure p and the density ρ are linked by the equation of state

$$p = f(S)\rho^{1+\alpha}, \qquad \alpha > 0 \tag{8.1}$$

for some smooth function $f(\cdot)$. The entropy $S\big(x(t), t\big)$ of an infinitesimal portion of the gas moving along the Lagrangian path $t \to x(t)$ is conserved. Therefore ([268])

$$\frac{d}{dt}S = 0$$

where formally

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla$$

is the *total derivative.* The system of the Euler equations of the process is

$$\rho\big[\mathbf{v}_t + (\mathbf{v}\cdot\nabla)\mathbf{v}\big] = -\nabla p + \mathbf{f} \tag{8.2}$$

$$\rho_t + \operatorname{div}(\mathbf{v}\rho) = 0 \tag{8.3}$$

$$\frac{\partial}{\partial t}\frac{p}{\rho^{1+\alpha}} + \mathbf{v}\cdot\nabla\frac{p}{\rho^{1+\alpha}} = 0. \tag{8.4}$$

The first is the pointwise balance of forces following Newton's law along the Lagrangian paths of the motion. The second is the conservation of mass, and the last is the conservation of entropy.

9 Isentropic Potential Flows

A flow is isentropic if $S = \text{const}$. In this case, the equation of state (8.1) permits one to define the pressure as a function of the density alone. Let $u \in C^2(\mathbb{R}^3 \times \mathbb{R})$ be the velocity potential, so that $\mathbf{v} = \nabla u$. Assume that $\mathbf{f} = 0$, and rewrite (8.2) as

$$\frac{\partial}{\partial t} u_{x_i} + u_{x_j} u_{x_i x_j} = -\frac{p_{x_i}}{\rho}, \qquad i = 1,2,3. \tag{9.1}$$

From this

$$\frac{\partial}{\partial x_i}\left(u_t + \frac{1}{2}|\nabla u|^2 + \int_0^p \frac{ds}{\rho(s)} \right) = 0, \qquad i = 1,2,3.$$

From the equation of state

$$\int_0^p \frac{ds}{\rho(s)} = \frac{1+\alpha}{\alpha}\frac{p}{\rho}.$$

Combining these calculations, gives the Bernoulli law for an isentropic potential flow[2]

$$u_t + \frac{1}{2}|\nabla u|^2 + \frac{1+\alpha}{\alpha}\frac{p}{\rho} = g \tag{9.2}$$

where $g(\cdot)$ is a function of t only. The positive quantity

$$c^2 = \frac{dp}{d\rho} = (1+\alpha)\frac{p}{\rho}$$

has the dimension of the square of a velocity, and c represents the local speed of sound. Notice that c need not be constant. Next multiply the ith equation in (9.1) by u_{x_i} and add for $i = 1,2,3$ to obtain

$$\frac{1}{2}\frac{\partial}{\partial t}|\nabla u|^2 + \frac{1}{2}\nabla u \cdot \nabla |\nabla u|^2 = -\frac{1}{\rho}\nabla p \cdot \nabla u. \tag{9.3}$$

Using the continuity equation

$$\begin{aligned} -\frac{1}{\rho}\nabla p \cdot \nabla u &= -\nabla \frac{p}{\rho} \nabla u - \frac{p}{\rho}\frac{1}{\rho}\nabla \rho \cdot \nabla u \\ &= -\nabla \frac{p}{\rho} \cdot \nabla u + \frac{p}{\rho}\frac{1}{\rho}\rho_t + \frac{p}{\rho}\Delta u. \end{aligned}$$

From the equation of state

---

[2] Daniel Bernoulli, 1700–1782, botanist and physiologist, made relevant discoveries in hydrodynamics. His father, Johann B. 1667–1748, and his uncle Jakob 1654–1705, brother of Johann, were both mathematicians. Jakob and Johann are known for their contributions to the calculus of variations.

$$\frac{d}{dt}f(S) = \frac{d}{dt}\frac{p}{\rho^{1+\alpha}} = \frac{1}{\rho^\alpha}\frac{d}{dt}\frac{p}{\rho} + \frac{p}{\rho}\frac{d}{dt}\frac{1}{\rho^\alpha} = 0.$$

From this, expanding the total derivative

$$\frac{\partial}{\partial t}\frac{p}{\rho} + \nabla u \cdot \nabla \frac{p}{\rho} - \alpha \frac{p}{\rho^2}[\rho_t + \nabla u \cdot \nabla \rho] = 0.$$

Again, by the equation of continuity

$$\rho_t + \nabla u \cdot \nabla \rho = \rho \Delta u.$$

Therefore

$$-\nabla \frac{p}{\rho} \cdot \nabla u = \frac{\partial}{\partial t}\frac{p}{\rho} + \alpha \frac{p}{\rho}\Delta u.$$

Combining these calculations in (9.3) gives

$$c^2 \Delta u - \frac{1}{2}\nabla u \cdot \nabla |\nabla u|^2 = \frac{1}{2}\frac{\partial}{\partial t}|\nabla u|^2 - \frac{1}{\rho}\frac{\partial}{\partial t}p. \tag{9.4}$$

9.1 Steady Potential Isentropic Flows

For steady flows, rewrite (9.4) in the form

$$(c^2\delta_{ij} - u_{x_i}u_{x_j})u_{x_i x_j} = 0. \tag{9.5}$$

The matrix of the coefficients of the second derivatives $u_{x_i x_j}$ is

$$\left(\delta_{ij} - \frac{u_{x_i}u_{x_j}}{c^2}\right)$$

and its eigenvalues are

$$\lambda_1 = 1 - \frac{|\nabla u|^2}{c^2} \qquad \text{and} \qquad \lambda_2 = 1.$$

Using the steady-state version of the Bernoulli law (9.2) gives the first eigenvalue in terms only of the pressure p and the density ρ. The ratio $M = |\nabla u|/c$ of the speed of a body to the speed of sound in the surrounding medium is called the Mach number.[3]

10 Partial Differential Equations

The equations and systems of the previous sections are examples of PDEs. Let $u \in C^m(E)$ for some $m \in \mathbb{N}$, and for $j = 1, 2, \ldots, m$, let $D^j u$ denote the vector of all the derivatives of u of order j. For example, if $N = m = 2$, denoting (x, y) the coordinates in $\mathbb{R}^2$

---

[3] Ernst Mach, 1838–1916. Mach one is the speed of sound; Mach two is twice the speed of sound; ...

$$D^1 u = (u_x, u_y) \quad \text{and} \quad D^2 u = (u_{xx}, u_{xy}, u_{yy}).$$

A partial differential equation is a functional link among the variables

$$x,\ u,\ D^1 u,\ D^2 u,\ \ldots,\ D^m u$$

that is

$$F(x, u, D^1 u, D^2 u, \ldots, D^m u) = 0.$$

The PDE is of order m if the gradient of F with respect to $D^m u$ is not identically zero. It is linear if for all $u, v \in C^m(E)$ and all $\alpha, \beta \in \mathbb{R}$

$$\begin{aligned} &F\big(x, (\alpha u + \beta v), D^1(\alpha u + \beta v), D^2(\alpha u + \beta v), \ldots, D^m(\alpha u + \beta v)\big) \\ &= \alpha F\big(x, u, D^1 u, D^2 u, \ldots, D^m u\big) + \beta F\big(x, v, D^1 v, D^2 v, \ldots, D^m v\big). \end{aligned}$$

It is *quasi-linear* if it is linear with respect to the highest order derivatives. Typically a quasi-linear PDE takes the form

$$\sum_{m_1 + \cdots + m_N = m} a_{m_1, \ldots, m_N} \frac{\partial^m u}{\partial^{m_1} x_1 \cdots \partial^{m_N} x_N} + F_o = 0$$

where m_j are non-negative integers and the coefficients $a_{m_1, \ldots, m_N}$, and the forcing term F_o, are given smooth functions of $(x, u, D^1 u, D^2 u, \ldots, D^{m-1} u)$. If the PDE is quasi-linear, the sum of the terms involving the derivatives of highest order, is the principal part of the PDE.

Problems and Complements

3c The Heat Equation and the Laplace Equation

It is worth devoting some space to Einstein's description of the Brownian Motion [64, 63, 240] and to its surprising connection with the Heat Equation.

3.1c Basic Physical Assumptions

Particles suspended in a fluid undergo random motions owing to disordered collisions generated by local thermal gradients. The trajectories are irregular and intertwined, and cannot be efficiently representable by the classical concepts of piecewise differentiable curves. Instead, the motion will be described by the density or distribution of particles, $u(x, t)$ providing the number of particles, per unit volume, present at x at time t. The basic assumptions are:

- **H1** *The motion of each particle is independent of the motion of the others.* This is in general false if the motion is caused by mutual collisions, which by their own nature mutually affect the motion of each particle. It can only be justified as an "average" property of the particles, in the sense that the motion, computed and averaged, in some sense, over a large number of particles, is not affected by the addition of a further particle, however it is positioned within the system.
- **H2** *For each time t, the motion of a particle at times later than t depends only on its position at time t and not on its kinematic at times preceding and up to t.* This is also false in general, in the classical formulation of Mechanics, where the motion after a time t depends on the kinematic at time t. Einstein justifies this assumption provided we look at times $t+\tau$ where $0<\tau\ll 1$ is so small that the only mechanics affecting the particle is a possible collision at time t and not its velocity at t. In the words of Einstein ... τ *is negligible with respect to observable times...*

The multitude of particles suggests looking at the particles only as a statistical average. Moreover, the numerous collisions, and consequent lack of differentiability of the intertwined paths, suggest renouncing to the classical notions of *velocity* and *acceleration*.

For $y\in\mathbb{R}^N$ let $p(y,\varDelta t)dy$ be the probability of each particle undergoing a displacement y in the time interval $(t,t+\varDelta t)$. In view of **H1**–**H2**, such a probability is the same for each particle. Moreover,

$$\int_{\mathbb{R}^N} p(y,\varDelta t)dy = 1 \qquad \text{for all } \ \varDelta t>0. \tag{3.1c}$$

It is also assumed that the medium is *isotropic*, that is, the probability density is independent of the direction of y so that $p(y,\varDelta t)=p(|y|,\varDelta t)$.

3.2c The Diffusion Equation

The particles that are in x at time $t+\varDelta t$ are those that were in $x-y$ at time t and that have been displaced by y in the time interval $(t,t+\varDelta t)$. Thus,

$$u(x,t+\varDelta t)=\int_{\mathbb{R}^N} u(x-y,t)p(y,\varDelta t)dy.$$

By Taylor's expansion (formal at this stage)

$$u(x-y,t)=u(x,t)-\sum_{j=1}^{N} u_{x_j}(x,t)y_j+\frac{1}{2}\sum_{i,j=1}^{N} u_{x_ix_j}(x,t)y_iy_j+\mathcal{R}(x,y,t).$$

Therefore,

$$\begin{aligned}u(x,t+\Delta t) =&u(x,t)\int_{\mathbb{R}^N} p(y,\Delta t)dy\\&-\sum_{j=1}^{N} u_{x_j}(x,t)\int_{\mathbb{R}^N} y_j p(y,\Delta t)dy\\&+\frac{1}{2}\sum_{i,j=1}^{N} u_{x_i x_j}\int_{\mathbb{R}^N} y_i y_j p(y,\Delta t)dy\\&+\int_{\mathbb{R}^N} \mathcal{R}(x,y,t)p(y,\Delta t)dy.\end{aligned}$$

Using that $p(y,\Delta t) = p(|y|,\Delta t)$ one verifies that

$$\begin{aligned}&\textstyle\int_{\mathbb{R}^N} y_j p(y,\Delta t)dy = 0 &&\text{for } i = 1,\dots,N;\\&\textstyle\int_{\mathbb{R}^N} y_i y_j p(y,\Delta t)dy = 0 &&\text{for all } i \neq j;\\&\textstyle\int_{\mathbb{R}^N} y_j^2 p(y,\Delta t)dy = 2k(\Delta t) &&\text{for } j = 1,\dots,N\end{aligned}$$

for a function $k(\cdot)$. Combining these calculations

$$\frac{u(x,t+\Delta t)-u(x,t)}{\Delta t} - \frac{k(\Delta t)}{\Delta t}\Delta u = \frac{1}{\Delta t}\int_{\mathbb{R}^N} \mathcal{R}(x,y,t)p(y,\Delta t)dy. \tag{3.2c}$$

Einstein then postulates that

$$\lim_{\Delta t\to 0}\frac{k(\Delta t)}{\Delta t} = k \quad \text{for a given positive constant } k\text{, and} \tag{3.3c}$$

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_{\mathbb{R}^N} \mathcal{R}(x,y,t)p(y,\Delta t)dy = 0. \tag{3.4c}$$

Assuming these two postulates for the moment, and letting $\Delta t \to 0$ in (3.2c), imply that the density u of particles in Brownian motion suspended in a fluid satisfies the heat equation

$$u_t - k\Delta u = 0 \quad \text{within the fluid being observed.} \tag{3.5c}$$

3.3c Justifying the Postulates (3.3c)–(3.4c)

The first postulate is simply accepted by Einstein. The second is essentially a consequence of the first. By the definition of the remainder $\mathcal{R}(x,y,t)$

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_{\mathbb{R}^N} |\mathcal{R}(x,y,t)|p(y,\Delta t)dy \le \lim_{\Delta t\to 0}\frac{A(x,t)}{\Delta t}\int_{\mathbb{R}^N} |y|^3|p(y,\Delta t)dy$$

for a constant $A(x,t)$ depending on (x,t), which, however, in this process, are fixed. Now it is reasonable to postulate that if $0 < \Delta t \ll 1$, those displacements y that have an appreciable probability of actually occurring are small in length, proportionally to Δt. Thus, it is postulated that

$$\int_{\mathbb{R}^N} |y|^3 |p(y, \Delta t) dy \le B \int_{|y| \le C\Delta t} |y|^3 p(y, \Delta t) dy$$

for given constants B and C. This implies (3.4c).

3.4c More on the Postulates (3.3c)–(3.4c)

These postulates are verified for the probability density $p(\cdot, \cdot)$ given by

$$\Gamma(y, t) = \frac{1}{(4\pi k t)^{N/2}} e^{-\frac{|y|^2}{4kt}}. \tag{3.6c}$$

Verify that such a $p(\cdot, \cdot)$ satisfies (3.1c) for all $t > 0$, and (3.3c)–(3.4c) for all $\Delta t > 0$. The probability that a particle in Brownian motion, is at y at time t, starting from $y = 0$ at time $t = 0$, could be statistically computed by effecting a large number of experiments, and then by taking their arithmetic average. It can be shown that as the number of experiments tends to infinity the corresponding probability tends to the function Γ in (3.6c), for a physical constant k that depends on the medium. Such a constant can be given a precise physical meaning by the following considerations. The mean distance $\ell(t)$ traveled by the particle in time t, originating at the origin is the square-root of the expected value of $|y|^2$, that is

$$\ell^2(t) = \int_{\mathbb{R}^N} |y|^2 \Gamma(y, t) dy = \frac{4kt}{\pi^{N/2}} \int_{\mathbb{R}^N} |y|^2 e^{-|y|^2} dy = \text{const}(N) kt. \tag{3.7c}$$

3.5 Verify that $\mathbb{R}^N \times \mathbb{R}^+ \ni (y, t) \to \Gamma(y, t)$ satisfies the heat equation (3.5c).

3.6 Justify the calculations in (3.7c) and compute explicitly the constant const(N).

3.7 Verify that the postulates (3.4c)–(3.4c) are verified for such a Γ.

3.8 Let $u_o(\cdot)$ be a bounded and continuous function in $\mathbb{R}^N$. Verify that the function $u = \Gamma * u_o$ satisfies the heat equation (3.5c) and that

$$\lim_{t \to 0} u(\cdot, t) = u_o \quad \text{over compact subsets of } \mathbb{R}^N.$$

3.9 Let

$$K_N(x, y) = \begin{cases} \frac{1}{(N-2)\omega_N} \frac{1}{|x-y|^{N-2}} & \text{for } N > 2; \\ \\ \frac{1}{2\pi} \ln |x - y| & \text{for } N = 2. \end{cases}$$

$$K'_{N+1}(x, y; t) = \frac{2}{\omega_{N+1}} \frac{1}{[|x - y|^2 + t^2]^{\frac{N+1}{2}}}.$$

Verify that

$$\begin{aligned} &\Delta_x K_N = \Delta_y K_n = 0 \quad \text{in } x \ne y \\ &\Delta_{x;t} K'_{N+1} = \Delta_{y;t} K'_{N+1} = 0 \quad \text{for } t > 0. \end{aligned}$$

Verify that

$$K'_{N+1} = \frac{\partial}{\partial \tau} K_{N+1}(x, y; t, \tau)\bigg|_{\tau=0}.$$

1

QUASI-LINEAR EQUATIONS AND ANALYTIC DATA

1 Quasi-Linear Second-Order Equations in Two Variables

Let (x, y) denote the variables in $\mathbb{R}^2$, and consider the quasi-linear equation

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} = D \tag{1.1}$$

where $(x, y, u_x, u_y) \to A, B, C, D(x, y, u_x, u_y)$ are given smooth functions of their arguments. The equation is of order two if at least one of the coefficients A, B, C is not identically zero. Let $\varGamma$ be a curve in $\mathbb{R}^2$ of parametric representation

$$\varGamma = \begin{cases} x = \xi(t) \\ y = \eta(t) \end{cases} \in C^2(-\delta, \delta) \text{ for some } \ \delta > 0.$$

On $\varGamma$, prescribe the Cauchy data

$$u\big|_{\varGamma} = v, \quad u_x\big|_{\varGamma} = \varphi, \quad u_y\big|_{\varGamma} = \psi \tag{1.2}$$

where $t \to v(t), \varphi(t), \psi(t)$ are given functions in $C^2(-\delta, \delta)$. Since

$$\frac{d}{dt} u\big(\xi(t), \eta(t)\big) = u_x \xi' + u_y \eta' = \varphi \xi' + \psi \eta' = v'$$

of the three functions v, φ, ψ, only two can be assigned independently. The Cauchy problem for (1.1) and $\varGamma$, consists in finding $u \in C^2(\mathbb{R}^2)$ satisfying the PDE and the Cauchy data (1.2). Let u be a solution of the Cauchy problem (1.1)–(1.2), and compute its second derivatives on $\varGamma$. By (1.1) and the Cauchy data

$$\begin{aligned} Au_{xx} + 2Bu_{xy} + Cu_{yy} &= D \\ \xi' u_{xx} + \eta' u_{xy} \qquad\qquad &= \varphi' \\ \xi' u_{xy} + \eta' u_{yy} &= \psi'. \end{aligned}$$

Here A, B, C are known, since they are computed on $\varGamma$. Precisely

© Springer Nature Switzerland AG 2023

E. DiBenedetto, U. Gianazza, *Partial Differential Equations*,
Cornerstones, https://doi.org/10.1007/978-3-031-46618-2_2

$$A, B, C\big|_{\Gamma} = A, B, C(\xi, \eta, v, \varphi, \psi).$$

Therefore, u_{xx}, u_{xy}, and u_{yy} can be computed on Γ, provided

$$\det \begin{pmatrix} A & 2B & C \\ \xi' & \eta' & 0 \\ 0 & \xi' & \eta' \end{pmatrix} \neq 0. \tag{1.3}$$

We say that Γ is a characteristic curve if (1.3) does not hold, i.e., if

$$A\eta'^2 - 2B\xi'\eta' + C\xi'^2 = 0. \tag{1.4}$$

In general, the property of Γ being a characteristic depends on the Cauchy data assigned on it. If Γ admits a local representation of the type

$$y = y(x) \quad \text{in a neighborhood of some } x_o \in \mathbb{R} \tag{1.5}$$

the characteristics are the graphs of the possible solutions of the differential equation

$$y' = \frac{B \pm \sqrt{B^2 - AC}}{A}.$$

Associate with (1.1) the matrix of the coefficients

$$M = \begin{pmatrix} A & B \\ B & C \end{pmatrix}.$$

Using (1.4) and the matrix M, we classify, locally, the family of quasi-linear equations (1.1) as **elliptic** if $\det M > 0$, i.e., if there exists no real characteristic; **parabolic** if $\det M = 0$, i.e., if there exists one family of real characteristics; **hyperbolic** if $\det M < 0$, i.e., if there exist two families of real characteristics. The elliptic, parabolic, or hyperbolic nature of (1.1) may be different in different regions of $\mathbb{R}^2$. For example, the Tricomi equation ([259])

$$yu_{xx} - u_{yy} = 0$$

is elliptic in the region $[y < 0]$, parabolic on the x-axis and hyperbolic in the upper half-plane $[y > 0]$. The characteristics are solutions of $\sqrt{y}y' = \pm 1$ in the upper half-plane $[y > 0]$.

The elliptic, parabolic, or hyperbolic nature of the PDE may also depend upon the solution itself. As an example, consider the equation of steady compressible fluid flow of a gas of density u and velocity $\nabla u = (u_x, u_y)$ in $\mathbb{R}^2$, introduced in (9.5) of the Preliminaries

$$(c^2 - u_x^2)u_{xx} - 2u_x u_y u_{xy} + (c^2 - u_y^2)u_{yy} = 0$$

where $c > 0$ is the speed of sound. Compute

$$\det M = \det \begin{pmatrix} c^2 - u_x^2 & -u_x u_y \\ -u_x u_y & c^2 - u_y^2 \end{pmatrix} = c^2(c^2 - |\nabla u|^2).$$

Therefore the equation is elliptic for sub-sonic flow ($|\nabla u| < c$), parabolic for sonic flow ($|\nabla u| = c$), and hyperbolic for super-sonic flow ($|\nabla u| > c$). The Laplace equation

$$\Delta u = u_{xx} + u_{yy} = 0$$

is elliptic. The heat equation

$$H(u) = u_y - u_{xx} = 0$$

is parabolic with characteristic lines $y = \text{const}$. The wave equation

$$\Box u = u_{yy} - c^2 u_{xx} = 0 \quad c \in \mathbb{R}$$

is hyperbolic with characteristic lines $x \pm cy = \text{const}$.

2 Characteristics and Singularities

If Γ is a characteristic, the Cauchy problem (1.1)–(1.2) is in general not solvable, since the second derivatives of u cannot be computed on Γ. We may attempt to solve the PDE (1.1) on each side of Γ and then piece together the functions so obtained. Assume that Γ divides $\mathbb{R}^2$ into two regions E_1 and E_2 and let $u_i \in C^2(\bar{E}_i)$, for $i = 1, 2$, be possible solutions of (1.1) in E_i satisfying the Cauchy data (1.2). These are taken in the sense of

$$\lim_{\substack{(x,y)\to(\xi(t),\eta(t))\\(x,y)\in E_i}} u_i(x,y),\, u_{i,x}(x,y),\, u_{i,y}(x,y) = v(t), \varphi(t), \psi(t).$$

Setting

$$u = \begin{cases} u_1 & \text{in } E_1 \\ u_2 & \text{in } E_2 \end{cases}$$

the function u is of class C^1 across Γ. If $f_i \in C(\bar{E}_i)$, for $i = 1, 2$, and

$$f = \begin{cases} f_1 & \text{in } E_1 \\ f_2 & \text{in } E_2 \end{cases}$$

let $[f]$ denote the jump of f across Γ, i.e.,

$$[f](t) = \lim_{\substack{(x,y)\to(\xi(t),\eta(t))\\(x,y)\in E_1}} f_1(x,y) - \lim_{\substack{(x,y)\to(\xi(t),\eta(t))\\(x,y)\in E_2}} f_2(x,y).$$

From the assumptions on u,

$$[u] = [u_x] = [u_y] = 0. \tag{2.1}$$

From (1.1),

$$A[u_{xx}] + 2B[u_{xy}] + C[u_{yy}] = 0. \tag{2.2}$$

Assume that Γ has the local representation (1.5). Then using (2.1), compute

$$[u_{xx}] + [u_{xy}]y' = 0, \qquad [u_{xy}] + [u_{yy}]y' = 0.$$

Therefore

$$[u_{xx}] = [u_{yy}]y'^2, \qquad [u_{xy}] = -[u_{yy}]y'. \tag{2.3}$$

Let $J = [u_{yy}]$ denote the jump across Γ of the second y-derivative of u. From (2.2) and (2.3)

$$J(Ay'^2 - 2By' + C) = 0.$$

If Γ is not a characteristic, then $(Ay'^2 - 2By' + C) \neq 0$. Therefore $J = 0$, and u is of class C^2 across Γ. If $J \neq 0$, then Γ must be a characteristic. Thus if a solution of (1.1) in a region $E \subset \mathbb{R}^2$ suffers discontinuities in the second derivatives across a smooth curve, these must occur across a characteristic.

2.1 Coefficients Independent of u_x and u_y

Assume that the coefficients A, B, C and the term D are independent of u_x and u_y, and that $u \in C^3(\bar{E}_i)$, $i = 1, 2$. Differentiate (1.1) with respect to y in E_i, form differences, and take the limit as $(x, y) \to \Gamma$ to obtain

$$A[u_{xxy}] + 2B[u_{xyy}] + C[u_{yyy}] = 0. \tag{2.4}$$

Differentiating the jump J of u_{yy} across Γ gives

$$J' = [u_{xyy}] + [u_{yyy}]y'. \tag{2.5}$$

From the second jump condition in (2.3), by differentiation

$$-y'J' - y''J = [u_{xxy}] + [u_{xyy}]y'. \tag{2.6}$$

We eliminate $[u_{xxy}]$ from (2.4) and (2.6) and use (2.5) to obtain

$$\begin{aligned} A(y'J' + y''J) &= (2B - Ay')[u_{xyy}] + C[u_{yyy}] \\ &= (2B - Ay')J' + (Ay'^2 - 2By' + C)[u_{yyy}]. \end{aligned}$$

Therefore, if Γ is a characteristic

$$2(B - Ay')J' = Ay''J.$$

This equation describes how the jump J of u_{yy} at some point $P \in \Gamma$ propagates along Γ. In particular, either J vanishes identically, or it is never zero on Γ.

3 Quasi-Linear Second-Order Equations

Let E be a region in $\mathbb{R}^N$, and let $u \in C^2(E)$. A quasi-linear equation in E takes the form

$$A_{ij} u_{x_i x_j} = F \tag{3.1}$$

where we have adopted the summation notation and

$$(x, u, \nabla u) \to A_{ij}, F(x, u, \nabla u) \quad \text{for} \quad i, j = 1, 2, \dots, N$$

are given smooth functions of their arguments. The equation is of order two if not all the coefficients A_{ij} are identically zero. By possibly replacing A_{ij} with

$$\frac{A_{ij} + A_{ji}}{2}$$

we may assume that the matrix (A_{ij}) of the coefficients is symmetric. Let Γ be a hypersurface of class C^2 in $\mathbb{R}^N$, given as a level set of $\Phi \in C^2(E)$; say for example, $\Gamma = [\Phi = 0]$. Assume that $\nabla\Phi \neq 0$ and let $\boldsymbol{\nu} = \nabla\Phi/|\nabla\Phi|$ be the unit normal to Γ oriented in the direction of increasing Φ. For $x \in \Gamma$, introduce a local system of $N-1$ mutually orthogonal unit vectors $\{\boldsymbol{\tau}_1, \dots, \boldsymbol{\tau}_{N-1}\}$ chosen so that the n-tuple $\{\boldsymbol{\tau}_1, \dots, \boldsymbol{\tau}_{N-1}, \boldsymbol{\nu}\}$ is congruent to the orthonormal Cartesian system $\{\mathbf{e}_1, \dots, \mathbf{e}_N\}$. Given $f \in C^1(E)$, compute the derivatives of f, normal and tangential to Γ from

$$D_{\boldsymbol{\nu}} f = \nabla f \cdot \boldsymbol{\nu}, \quad D_{\boldsymbol{\tau}_j} f = \nabla f \cdot \boldsymbol{\tau}_j, \quad j = 1, \dots, N-1.$$

If $\tau_{i,j} = \boldsymbol{\tau}_i \cdot \mathbf{e}_j$, and $\nu_j = \boldsymbol{\nu} \cdot \mathbf{e}_j$, are the projections of $\boldsymbol{\tau}_i$ and $\boldsymbol{\nu}$ on the coordinate axes

$$D_{\boldsymbol{\tau}_j} f = (\tau_{j,1}, \dots, \tau_{j,N}) \cdot \nabla f \qquad \text{and} \qquad D_{\boldsymbol{\nu}} f = (\nu_1, \dots, \nu_N) \cdot \nabla f.$$

Introduce the unitary matrix

$$T = \begin{pmatrix} \tau_{1,1} & \tau_{1,2} & \cdots & \tau_{1,N} \\ \tau_{2,1} & \tau_{2,2} & \cdots & \tau_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ \tau_{N-1,1} & \tau_{N-1,2} & \cdots & \tau_{N-1,N} \\ \nu_1 & \nu_2 & \cdots & \nu_N \end{pmatrix} \tag{3.2}$$

and write

$$\nabla f = T^{-1} \begin{pmatrix} D_{\boldsymbol{\tau}} f \\ D_{\boldsymbol{\nu}} f \end{pmatrix} \qquad \text{where } T^{-1} = T^t. \tag{3.3}$$

The Cauchy data of u on Γ are

$$u\big|_\Gamma = v, \quad D_{\boldsymbol{\tau}_j} u\big|_\Gamma = \varphi_j,\ j = 1, \dots, N-1, \quad D_{\boldsymbol{\nu}} u\big|_\Gamma = \psi \tag{3.4}$$

regarded as restrictions to Γ of smooth functions defined on the whole of E. These must satisfy the compatibility conditions

$$D_{\boldsymbol{\tau}_j} v = \varphi_j, \qquad j = 1, \dots, N-1. \tag{3.5}$$

Therefore, only v and ψ can be given independently. The Cauchy problem for (3.1) and Γ consists in finding a function $u \in C^2(E)$ satisfying the PDE and the Cauchy data (3.4). If u is a solution to the Cauchy problem (3.1)–(3.4), compute the second derivatives of u on Γ

$$\begin{aligned} u_{x_i} &= \tau_{k,i} u_{\tau_k} + \nu_i u_\nu \\ u_{x_i x_j} &= \tau_{k,i} \frac{\partial}{\partial x_j} u_{\tau_k} + \nu_i \frac{\partial}{\partial x_j} u_\nu \\ &= \tau_{k,i}\tau_{l,j} u_{\tau_k \tau_l} + \tau_{k,i}\nu_j u_{\tau_k \nu} + \tau_{k,j}\nu_i u_{\tau_k \nu} + \nu_i \nu_j u_{\nu\nu}. \end{aligned} \tag{3.6}$$

From the compatibility conditions (3.4) and (3.5)

$$D_{\boldsymbol{\tau}_i}(D_{\boldsymbol{\tau}_j} u) = D_{\boldsymbol{\tau}_i}\varphi_j, \qquad D_{\boldsymbol{\tau}_i}(D_{\boldsymbol{\nu}} u) = D_{\boldsymbol{\tau}_i}\psi.$$

Therefore, of the terms on the right-hand side of (3.6), all but the last are known on Γ. Using the PDE, one computes

$$A_{ij}\nu_i\nu_j u_{\nu\nu} = \tilde{F} \quad \text{on } \Gamma \tag{3.7}$$

where $\tilde{F}$ is a known function of Γ and the Cauchy data on it. We conclude that $u_{\nu\nu}$, and hence all the derivatives $u_{x_i x_j}$, can be computed on Γ provided

$$A_{ij}\Phi_{x_i}\Phi_{x_j} \neq 0. \tag{3.8}$$

Both (3.7) and (3.8) are computed at fixed points $P \in \Gamma$. We say that Γ is a characteristic at P if (3.8) is violated, i.e., if

$$(\nabla\Phi)^t (A_{ij})(\nabla\Phi) = 0 \qquad \text{at } P.$$

Since (A_{ij}) is symmetric, its eigenvalues are real and there is a unitary matrix U such that

$$U^{-1}(A_{ij})U = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_N \end{pmatrix}.$$

Let $\xi = Ux^t$ denote the coordinates obtained from x by the rotation induced by U. Then

$$(\nabla\Phi)^t (A_{ij})(\nabla\Phi) = [U^{-1}(\nabla\Phi)]^t U^{-1}(A_{ij})U[U^{-1}(\nabla\Phi)] = \lambda_i \Phi^2_{\xi_i}.$$

Therefore Γ is a characteristic at P if

$$\lambda_i \Phi^2_{\xi_i} = 0. \tag{3.9}$$

Writing this for all $P \in E$ gives a first-order PDE in Φ. Its solutions permit us to find the characteristic surfaces as the level sets $[\Phi = \text{const}]$.

3.1 Constant Coefficients

If the coefficients A_{ij} are constant, (3.9) is a first-order PDE with constant coefficients. The PDE in (3.1) is classified according to the number of positive and negative eigenvalues of (A_{ij}). Let p and n denote the number of positive and negative eigenvalues of (A_{ij}), and consider the pair (p, n). The equation (3.1) is classified as **elliptic** if either $(p, n) = (N, 0)$ or $(p, n) = (0, N)$. In either of these cases, it follows from (3.9) that

$$0 = |\lambda_i \varPhi_{\xi_i}^2| \ge \min_{1 \le i \le N} |\lambda_i| |\nabla \varPhi|^2.$$

Therefore, there exist no characteristic surfaces.

Equation (3.1) is classified as **hyperbolic** if $p + n = N$ and $p, n \ge 1$. Without loss of generality, we may assume that eigenvalues are ordered so that $\lambda_1, \dots, \lambda_p$ are positive and that $\lambda_{p+1}, \dots, \lambda_N$ are negative. In such a case, (3.9) takes the form

$$\sum_{i=1}^{p} \lambda_i \varPhi_{\xi_i}^2 = \sum_{j=p+1}^{N} |\lambda_j| \varPhi_{\xi_j}^2.$$

This is solved by

$$\varPhi^{\pm}(\xi) = \sum_{i=1}^{p} \sqrt{\lambda_i} \xi_i \pm k \sum_{j=p+1}^{N} \sqrt{|\lambda_j|} \xi_j$$

where

$$k^2 = \Big(\sum_{i=1}^{p} \lambda_i^2 \Big) \Big/ \Big(\sum_{j=p+1}^{N} \lambda_j^2 \Big).$$

The hyperplanes $[\varPhi^{\pm} = \text{const}]$ are two families of characteristic surfaces for (3.1). In the literature these PDE are further classified according to the values of p and n. Namely they are called *hyperbolic* if either $p = 1$ or $n = 1$. Otherwise they are called *ultra-hyperbolic.*

Equation (3.1) is classified as **parabolic** if $p + n < N$. Then at least one of the eigenvalues is zero. If, say, $\lambda_1 = 0$, then (3.9) is solved by any function of ξ_1 only, and the hyperplane $\xi_1 = \text{const}$ is a characteristic surface.

3.2 Variable Coefficients

In analogy with the case of constant coefficients we classify the PDE in (3.1) at each point $P \in E$ as elliptic, hyperbolic, or parabolic according to the number of positive and negative eigenvalues of (A_{ij}) at P. The classification is local, and for coefficients depending on the solution and its gradient, the nature of the equation may depend on its own solutions.

4 Quasi-Linear Equations of Order $m \ge 1$

An N-dimensional multi-index α, of size $|\alpha|$, is an N-tuple of non-negative integers whose sum is $|\alpha|$, i.e.,

$$\alpha = (\alpha_1, \dots, \alpha_N), \quad \alpha_i \in \mathbb{N} \cup \{0\},\ i = 1, \dots, N, \quad |\alpha| = \sum_{i=1}^{N} \alpha_i.$$

If $f \in C^m(E)$ for some $m \in \mathbb{N}$, and α is a multi-index of size m, let

$$D^\alpha f = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_N}}{\partial x_N^{\alpha_N}} f.$$

If $|\alpha| = 0$ let $D^\alpha f = f$. By $D^{m-1} f$ denote the vector of all the derivatives $D^\alpha f$ for $0 \le |\alpha| \le m-1$. Consider the quasi-linear equation

$$\sum_{|\alpha|=m} A_\alpha D^\alpha u = F \tag{4.1}$$

where $(x, D^{m-1}u) \to A_\alpha, F(x, D^{m-1}u)$ are given smooth functions of their arguments. The equation is of order m if not all the coefficients A_α are identically zero. If $\mathbf{v} = (v_1, \dots, v_N)$ is a vector in $\mathbb{R}^N$ and α is an N-dimensional multi-index, let

$$\mathbf{v}^\alpha = v_1^{\alpha_1} v_2^{\alpha_2} \cdots v_N^{\alpha_N}.$$

Prescribe a surface Γ as in the previous section and introduce the matrix T as in (3.2), so that the differentiation formula (3.3) holds. Denoting by $\beta = (\beta_1, \dots, \beta_{N-1})$ an $(N-1)$-dimensional multi-index of size $|\beta| \le m$, set

$$D^\beta_{\boldsymbol{T}} f = D^{\beta_1}_{\boldsymbol{T}_1} D^{\beta_2}_{\boldsymbol{T}_2} \cdots D^{\beta_{N-1}}_{\boldsymbol{T}_{N-1}} f.$$

Write N-dimensional multi-indices as $\alpha = (\beta, s)$, where s is a non-negative integer, and for $|\alpha| \le m$, set

$$D^\alpha_{\boldsymbol{T},\boldsymbol{\nu}} f = D^\beta_{\boldsymbol{T}} D^s_{\boldsymbol{\nu}} f.$$

The Cauchy data of u on Γ are

$$D^\alpha_{\boldsymbol{T},\boldsymbol{\nu}} u\big|_\Gamma = f_\alpha \in C^m(E) \quad \text{for all } |\alpha| < m. \tag{4.2}$$

Among these we single out the Dirichlet data

$$u\big|_\Gamma = f_o, \tag*{(4.2)$_{\rm D}$}$$

the normal derivatives

$$D^s_{\boldsymbol{\nu}} u\big|_\Gamma = f_s, \qquad |\alpha| = s \le m-1, \tag*{(4.2)$_{\boldsymbol{\nu}}$}$$

and the tangential derivatives

$$D^{\beta}_{\boldsymbol{T}} u\big|_{\Gamma} = f_{\beta}, \qquad |\beta| < m. \tag{4.2$_{\boldsymbol{T}}$}$$

Of these, only $(4.2)_{\mathrm{D}}$ and $(4.2)_{\boldsymbol{\nu}}$ can be given independently. The remaining ones must be assigned to satisfy the compatibility conditions

$$D^{s}_{\boldsymbol{\nu}} f_{\beta} = D^{\beta}_{\boldsymbol{T}} f_{s} \quad \text{for all } |\beta| \ge 0, \quad |\beta| + s \le m-1. \tag{4.3}$$

The Cauchy problem for (4.1) consists in finding a function $u \in C^m(E)$ satisfying (4.1) in E and the Cauchy data (4.2) on Γ.

4.1 Characteristic Surfaces

If u is a solution of the Cauchy problem, compute its derivatives of order m on Γ, by using (4.1), the Cauchy data (4.2) and the compatibility conditions (4.3). Proceeding as in formula (3.6), for a multi-index α of size $|\alpha| = m$

$$D^{\alpha} u = \nu_1^{\alpha_1} \cdots \nu_N^{\alpha_N} D^{m}_{\boldsymbol{\nu}} u + g \qquad \text{on } \Gamma$$

where g is a known function that can be computed a priori in terms of Γ, the Cauchy data (4.2), and the compatibility conditions (4.3). Putting this in (4.1) gives

$$\sum_{|\alpha|=m} A_{\alpha} \boldsymbol{\nu}^{\alpha} D^{m}_{\boldsymbol{\nu}} u = \tilde{F}, \qquad \boldsymbol{\nu}^{\alpha} = \nu_1^{\alpha_1} \cdots \nu_N^{\alpha_N}$$

where $\tilde{F}$ is known in terms of Γ and the data. Therefore all the derivatives, normal and tangential, up to order m can be computed on Γ, provided

$$\sum_{|\alpha|=m} A_{\alpha} \boldsymbol{\nu}^{\alpha} \ne 0. \tag{4.4}$$

We say that Γ is a characteristic surface if (4.4) is violated, i.e.,

$$\sum_{|\alpha|=m} A_{\alpha} (D\Phi)^{\alpha} = 0. \tag{4.5}$$

In general, the property of Γ being a characteristic depends on the Cauchy data assigned on it, unless the coefficients A_{α} are independent of $D^{m-1}u$.

Condition (4.5) was derived at a fixed point of Γ. Writing it at all points of E gives a first-order nonlinear PDE in Φ whose solutions permit one to find the characteristics associated with (4.1) as the level sets of Φ. To (4.1) associate the *characteristic form*

$$\mathcal{L}(\xi) = A_{\alpha} \xi^{\alpha}.$$

If $\mathcal{L}(\xi) \ne 0$ for all $\xi \in \mathbb{R}^N - \{0\}$, then there are no characteristic hypersurfaces, and (4.1) is said to be *elliptic*.

5 Analytic Data and the Cauchy–Kowalewski Theorem

A real-valued function f defined in an open set $G \subset \mathbb{R}^k$, for some $k \in \mathbb{N}$, is analytic at $\eta \in G$, if in a neighborhood of η, $f(y)$ can be represented as a convergent power series of $y - \eta$. The function f is analytic in G if it is analytic at every $\eta \in G$.

Consider the Cauchy problem for (4.1) with analytic data. Precisely, assume that Γ is noncharacteristic and analytic about one of its points x_o; the Cauchy data (4.2) satisfy the compatibility conditions (4.3) and are analytic at x_o. Finally, the coefficients A_α and the free term F are analytic about the point $(x_o, u(x_o), D^{m-1}u(x_o))$.

The Cauchy–Kowalewski Theorem asserts that under these circumstances, the Cauchy problem (4.1)–(4.2) has a solution u, analytic at x_o. Moreover, the solution is unique within the class of analytic solutions at x_o.

5.1 Reduction to Normal Form ([32])

Up to an affine transformation of the coordinates, we may assume that x_o coincides with the origin and that Γ is represented by the graph of $x_N = \Phi(\bar{x})$, with $\bar{x} = (x_1, \dots, x_{N-1})$, where $\bar{x} \to \Phi(\bar{x})$ is analytic at the origin of $\mathbb{R}^{N-1}$. Flatten Γ about the origin by introducing new coordinates $(\bar{x}, t)$ where $t = x_N - \Phi(\bar{x})$. In this way Γ becomes a $(N-1)$-dimensional open neighborhood of the origin lying on the hyperplane $t = 0$. Continue to denote by u, A_α, and F the transformed functions and rewrite (4.1) as

$$A_{(0,\dots,m)} \frac{\partial^m}{\partial t^m} u = \sum_{\substack{|\beta|+s=m \\ 0<s<m}} A_{(\beta,s)} \frac{\partial^s}{\partial t^s} D^\beta u + \sum_{|\beta|=m} A_{(\beta,0)} D^\beta u + F. \tag{5.1}$$

Here D^β operates only on the variables $\bar{x}$. Since $[t = 0]$ is not a characteristic surface, (4.4) implies

$$A_{(0,\dots,m)} \big(\bar{x}, 0, D^{m-1} u(\bar{x}, 0)\big) \neq 0$$

and this continues to hold in a neighborhood of $\big(0, 0, D^{m-1}u(0,0)\big)$, since the functions A_α are analytic near such a point. Divide (5.1) by the coefficient of $D_t^m u$ and continue to denote by the same letters the transformed terms on the right-hand side. Next, introduce the vector $\mathbf{u} = D^{m-1}u$, and let $u_\alpha = D^\alpha u$ for $|\alpha| \le m-1$, be one component of this vector. If $\alpha = (0, \dots, m-1)$, then the derivative

$$\frac{\partial}{\partial t} u_\alpha = \frac{\partial^m}{\partial t^m} u$$

satisfies (5.1). If $\alpha = (\beta, s)$ and $s \le (m-2)$, then

$$\frac{\partial}{\partial t} u_\alpha = \frac{\partial}{\partial x_i} \tilde{u}_\alpha$$

for some $i = 1, \dots, N-1$ and some component $\tilde{u}_\alpha$ of the vector $\mathbf{u}$. Therefore, the PDE of order m in (5.1) can be rewritten as a first-order system in the normal form

$$\frac{\partial}{\partial t} u_i = A_{ijk} \frac{\partial}{\partial x_j} u_k + F_i(\bar{x}, t, \mathbf{u}).$$

The Cauchy data on Γ reduce to $\mathbf{u}(\bar{x}, 0) = \mathbf{f}(\bar{x})$. Therefore setting

$$\mathbf{v}(\bar{x}, t) = \mathbf{u}(\bar{x}, t) - \mathbf{f}(0)$$

and transforming the coefficients A_{ijk} and the function F accordingly, reduces the problem to one with Cauchy data vanishing at the origin.

The coefficients A_{ijk} as well as the free term F, can be considered independent of the variables $(\bar{x}, t)$. Indeed these can be introduced as auxiliary dependent variables by setting

$$u_j = x_j, \qquad \text{satisfying} \quad \frac{\partial}{\partial t} u_j = 0 \text{ and } u_j(\bar{x}, 0) = x_j$$

for $j = 1, \dots, N-1$, and

$$u_N = t, \qquad \text{satisfying } \frac{\partial}{\partial t} u_N = 1 \text{ and } u_N(\bar{x}, 0) = 0.$$

These remarks permit one to recast the Cauchy problem (4.1)–(4.2), in the *normal form*

$$\begin{gathered} \mathbf{u} = (u_1, u_2, \dots, u_\ell), \quad \ell \in \mathbb{N} \\ \frac{\partial}{\partial t} \mathbf{u} = \mathbf{A}_j(\mathbf{u}) \frac{\partial}{\partial x_j} \mathbf{u} + \mathbf{F}(\mathbf{u}) \\ \mathbf{u}(x, 0) = \mathbf{u}_o(x), \qquad \mathbf{u}_o(0) = 0 \end{gathered} \tag{5.2}$$

where $\mathbf{A}_j = (A_{ik})_j$ are $\ell \times \ell$ matrices and $\mathbf{F} = (F_1, \dots, F_\ell)$ are known functions of their arguments. We have also renamed and indexed the *space* variables, on the hyperplane $t = 0$, as $x = (x_1, \dots, x_N)$.

Theorem 5.1 (Cauchy–Kowalewski). *Assume that* $\mathbf{u} \to A_{ikj}(\mathbf{u}), F_i(\mathbf{u})$ *and* $x \to u_{o,i}(x)$ *for* $i, j, k = 1, \dots, \ell$ *are analytic in a neighborhood of the origin. Then there exists a unique analytic solution of the Cauchy problem (5.2) in a neighborhood of the origin.*

For linear systems, the theorem was first proved by Cauchy ([33]). It was generalized to nonlinear systems by Sonja Kowalewskaja[1] ([140]). A generalization is also due to G. Darboux ([41]).

---

[1] Sonja Kowalewskaja, 1850–1891, was born in Russia into an upper middle class family. Karl Weierstrass was the advisor of her PhD thesis, which was devoted to the analyticity of solutions of the Cauchy Problem. She obtained a chair of mathematics in Stockholm in 1884, where she died at 41 of pneumonia.

Theorem 5.1 is a fundamental achievement, which lies at the heart of a lot of different results. However, as Lars Gårding wrote in [95] about a general linear partial differential equation of the kind

$$P(x, D_x u) = f, \tag{5.3}$$

it was pointed out very emphatically by Hadamard that it is not natural to consider only analytic solutions and source functions f even P has analytic coefficients. This reduces the interest of the Cauchy-Kowalewski theorem which says that (5.3) *has locally analytic solutions if P and f are analytic. The Cauchy-Kowalewski theorem does not distinguish between classes of differential operators which have, in fact, very different properties such as the Laplace operator and the wave operator.*

6 Proof of the Cauchy–Kowalewski Theorem

First use the system in (5.2) to compute all the derivatives, at the origin, of a possible solution. Then using these numbers, construct the formal Taylor's series of an anticipated solution, say

$$\mathbf{u}(x,t) = \sum_{|\beta|+s\ge 0} \frac{D^\beta D_t^s \mathbf{u}(0,0)}{\beta! s!} x^\beta t^s \tag{6.1}$$

where $x^\beta = x_1^{\beta_1} \cdots x_N^{\beta_N}$, and $\beta! = \beta_1! \cdots \beta_N!$. If this series converges in a neighborhood of the origin, then (6.1) defines a function $\mathbf{u}$, analytic near $(0,0)$. Such a $\mathbf{u}$ is a solution to (5.2). Indeed, substituting the power series (6.1) on the left- and right-hand sides of the system in (5.2), gives two analytic functions whose derivatives of any order coincide at $(0,0)$. Thus they must coincide in a neighborhood of the origin. Uniqueness within the class of analytic solutions follows by the same unique continuation principle. Therefore the proof of the theorem reduces to showing that the series in (6.1), with all the coefficients $D^\beta D^s \mathbf{u}(0,0)$ computed from (5.2), converges about the origin.

The convergence of the series could be established, indirectly, by the method of the majorant ([131], 73–78). This was the original approach of A. Cauchy, followed also by S. Kowalewskaja and G. Darboux. The convergence of the series, can also be established by a direct estimation of all the derivatives of $\mathbf{u}$. This is the method we present here. This approach, originally due to Lax ([157]), has been further elaborated and extended by A. Friedman [82]. It has also been generalized to an infinite dimensional setting by M. Shimbrot and R.E. Welland ([238]).

Let α and β be N-dimensional multi-indices, denote by m a non-negative integer and set $\alpha^m = \alpha_1^m \cdots \alpha_N^m$, and

$$\alpha + \beta = (\alpha_1 + \beta_1, \dots, \alpha_N + \beta_N), \qquad \binom{\alpha+\beta}{\alpha} = \frac{(\alpha+\beta)!}{\alpha!\beta!}.$$

Denote by ι the multi-index $\iota = (1, \dots, 1)$, so that

$$\beta + \iota = (\beta_1 + 1, \beta_2 + 1, \dots, \beta_N + 1).$$

The convergence of the series in (6.1) is a consequence of the following

Lemma 6.1 *There exist constants C_o and C such that,*

$$|D^\beta D_t^s \mathbf{u}(0,0)| \le C_o C^{|\beta|+s-1} \frac{\beta! s!}{(\beta+\iota)^2 (s+1)^2} \tag{6.2}$$

for all N-dimensional multi-indices β, and all non-negative integers s.

6.1 Estimating the Derivatives of u at the Origin

We will establish first the weaker inequality

Lemma 6.2 *There exist constants C_o and C such that*

$$|D^\beta D_t^s \mathbf{u}(0,0)| \le C_o C^{|\beta|+s-1} \frac{(|\beta|+s)!}{(\beta+\iota)^2 (s+1)^2} \tag{6.3}$$

for all N-dimensional multi-indices β and all non-negative integers s.

This inequality holds for $s = 0$, since $D^\beta \mathbf{u}(0,0) = D^\beta \mathbf{u}_o(0)$, and $\mathbf{u}_o$ is analytic. We will show that if it does hold for s it continues to hold for $s+1$.

Let $\gamma = (\gamma_1, \dots, \gamma_\ell)$ be an ℓ-dimensional multi-index. Since $\mathbf{A}_j(\cdot)$ and $\mathbf{F}(\cdot)$ are analytic at the origin

$$A_{ikj}(\mathbf{u}) = \sum_{|\gamma| \ge 0} A_\gamma^{ikj} \mathbf{u}^\gamma, \qquad F_i(\mathbf{u}) = \sum_{|\gamma|\ge 0} F_\gamma^i \mathbf{u}^\gamma \tag{6.4}$$

where $\mathbf{u}^\gamma = u_1^{\gamma_1} \cdots u_\ell^{\gamma_\ell}$, and A_γ^{ikj} and F_γ^i are constants satisfying

$$|F_\gamma^i| + |A_\gamma^{ikj}| \le M_o M^{|\gamma|} \tag{6.5}$$

for all $i, k = 1, \dots, \ell$ and all $j = 1, \dots, N$, for two given positive constants M_o and M. From (5.2) and (6.4), compute

$$D^\beta D_t^{s+1} u_i = \sum_{|\gamma|\ge 1} A_\gamma^{ikj} D^\beta D_t^s \left(\mathbf{u}^\gamma \frac{\partial u_k}{\partial x_j} \right) + \sum_{|\gamma| \ge 1} F_\gamma^i D^\beta D_t^s \mathbf{u}^\gamma. \tag{6.6}$$

The induction argument requires some preliminary estimates.

7 Auxiliary Inequalities

Lemma 7.1 *Let m, n, i, j be non-negative integers such that $n \ge i$ and $m \ge j$. Then*

$$\sum_{i=0}^{n} \frac{1}{(n-i+1)^2 (i+1)^2} \le \frac{16}{(n+1)^2} \tag{7.1}$$

and

$$\text{and} \qquad \binom{n}{i} \binom{m}{j} \le \binom{n+m}{i+j}. \tag{7.2}$$

Proof. If $1 \le i \le \frac{1}{2}n$, the argument of the sum is majorized by

$$\frac{4}{(n+1)^2}\frac{1}{(i+1)^2} \quad \text{for all integers} \quad 1 \le i \le \tfrac{1}{2}n.$$

If $\frac{1}{2}n < i \le n$, it is majorized by

$$\frac{4}{(n+1)^2}\frac{1}{(n-i+1)^2} \quad \text{for all integers} \quad \tfrac{1}{2}n < i \le n.$$

Thus in either case

$$\sum_{i=0}^{n} \frac{1}{(n-i+1)^2(i+1)^2} \le \frac{8}{(n+1)^2} \sum_{i=0}^{\infty} \frac{1}{(i+1)^2}.$$

Inequality (7.2) is proved by induction on m, by making use of the identity

$$\binom{m+1}{j} = \binom{m}{j} + \binom{m}{j-1}. \qquad \blacksquare$$

Next, we establish a multi-index version of Lemma 7.1. If β and σ are N-dimensional multi-indices, we say that $\sigma \le \beta$, if and only if $\sigma_i \le \beta_i$ for all $i = 1, \dots, N$.

Lemma 7.2 *Let α and β be N-dimensional multi-indices. Then*

$$\sum_{\sigma \le \beta} \frac{1}{(\beta - \sigma + \iota)^2(\sigma + \iota)^2} \le \frac{16^N}{(\beta + \iota)^2}. \tag{7.3}$$

and

$$\binom{\alpha+\beta}{\alpha} \le \binom{|\alpha+\beta|}{|\alpha|}. \tag{7.4}$$

Proof (of (7.3)). For 1-dimensional multi-indices, (7.3) is precisely (7.1). Assuming that (7.3) holds true for multi-indices of dimension $k \in \mathbb{N}$, will show that it continues to hold for multi-indices of dimension $k+1$. Let β and σ be k-dimensional multi-indices and let

$$\tilde{\beta} = (\beta_1, \dots, \beta_k, \beta_{k+1}), \qquad \tilde{\sigma} = (\sigma_1, \dots, \sigma_k, \sigma_{k+1})$$

denote multi-indices of dimension $k+1$. Then by the induction hypothesis and (7.1)

$$\begin{aligned} \sum_{\tilde{\sigma} \le \tilde{\beta}} \frac{1}{(\tilde{\beta} - \tilde{\sigma} + \iota)^2(\tilde{\sigma} + \iota)^2} &= \sum_{\sigma \le \beta} \frac{1}{(\beta - \sigma + \iota)^2(\sigma + \iota)^2} \\ &\quad \times \sum_{\sigma_{k+1}=0}^{\beta_{k+1}} \frac{1}{(\beta_{k+1} - \sigma_{k+1} + 1)^2(\sigma_{k+1} + 1)^2} \\ &\le \frac{16^k}{(\beta + \iota)^2} \frac{16}{(\beta_{k+1} + 1)^2} = \frac{16^{k+1}}{(\tilde{\beta} + \iota)^2}. \end{aligned} \qquad \blacksquare$$

Proof (of (7.4)). By (7.2), the inequality holds for 2-dimensional multi-indices. Assuming that it holds for k-dimensional multi-indices, we show that it continues to hold for multi-indices of dimension $k+1$. Let α and β be $(k+1)$-dimensional multi-indices. Then by the induction hypothesis

$$\begin{aligned}\binom{\alpha+\beta}{\alpha} &= \frac{(\alpha+\beta)!}{\alpha!\beta!} = \frac{\prod_{j=1}^{k}(\alpha_j+\beta_j)}{\prod_{j=1}^{k}\alpha_j!\beta_j!}\,\frac{(\alpha_{k+1}+\beta_{k+1})!}{\alpha_{k+1}!\beta_{k+1}!}\\ &\le \frac{\left[\sum\limits_{j=1}^{k}(\alpha_j+\beta_j)\right]!}{\left[\sum\limits_{j=1}^{k}\alpha_j\right]!\left[\sum\limits_{j=0}^{k}\beta_j\right]!}\,\frac{(\alpha_{k+1}+\beta_{k+1})!}{\alpha_{k+1}!\beta_{k+1}!}\\ &\le \binom{\sum\limits_{j=1}^{k}(\alpha_j+\beta_j)}{\sum\limits_{j=1}^{k}\alpha_j}\binom{\alpha_{k+1}+\beta_{k+1}}{\alpha_{k+1}} \le \binom{\sum\limits_{j=1}^{k+1}(\alpha_j+\beta_j)}{\sum\limits_{j=1}^{k+1}\alpha_j} = \binom{|\alpha+\beta|}{|\alpha|}. \quad \blacksquare\end{aligned}$$

8 Auxiliary Estimations at the Origin

Lemma 8.1 *Let* $\mathbf{u}=(u_1,\dots,b_\ell)$ *satisfy (6.3). Then for every* $1\le p,q\le\ell$ *and every* N*-dimensional multi-index* β

$$|D^\beta D_t^s(u_pu_q)(0,0)| \le \left(\frac{cC_o}{C}\right)^2 C^{|\beta|+s}\frac{(|\beta|+s)!}{(\beta+\iota)^2(s+1)^2} \tag{8.1}$$

where $c=16^{N+1}$. *For every* ℓ*-dimensional multi-index* γ

$$|D^\beta D_t^s\mathbf{u}^\gamma(0,0)| \le \left(\frac{cC_o}{C}\right)^{|\gamma|} C^{|\beta|+s}\frac{(|\beta|+s)!}{(\beta+\iota)^2(s+1)^2}. \tag{8.2}$$

Moreover, for all indices $h=1,\dots,N$ *and* $k=1,\dots,\ell$

$$|D^\beta D_t^s(\mathbf{u}^\gamma D_{x_h}u_k)(0,0)| \le C_o\left(\frac{cC_o}{C}\right)^{|\gamma|} C^{|\beta|+s}\frac{(|\beta|+s+1)!}{(\beta+\iota)^2(s+1)^2}. \tag{8.3}$$

Proof (of (8.1)). By the generalized Leibniz rule

$$D^\beta D_t^s u_pu_q = \sum_{j=0}^{s}\binom{s}{j}\sum_{\sigma\le\beta}\binom{\beta}{\sigma}(D^{\beta-\sigma}D_t^{s-j}u_p)(D^\sigma D_t^j u_q)$$

where σ is an N-dimensional multi-index of size $|\sigma|\le|\beta|$. Compute this at the origin and estimate the right-hand side by using (6.3), which is assumed to hold for all the t-derivatives of order up to s. By (7.4)

$$\binom{\beta}{\sigma}\binom{s}{j} \le \binom{|\beta|}{|\sigma|}\binom{s}{j} \le \binom{|\beta|+s}{|\sigma|+j}.$$

Therefore

$$\begin{aligned}|D^\beta D_t^s(u_p u_q)(0,0)| \le & C_o^2 C^{|\beta|+s-2}(|\beta|+s)! \\ & \times \sum_{j=0}^{s} \frac{1}{(s-j+1)^2(j+1)^2} \sum_{\sigma\le\beta} \frac{1}{(\beta-\sigma+\iota)^2(\sigma+\iota)^2}.\end{aligned}$$

To prove (8.1) estimate the two sums on the right-hand side with the aid of (7.1) of Lemma 7.1 and (7.3) of Lemma 7.2. ■

Proof (of (8.2)). The proof is by induction. If γ is a ℓ-dimensional multi-index of either form

$$(\dots,1,\dots,1,\dots), \qquad (\dots,2,\dots)$$

then (8.2) is precisely (8.1) for such a multi-index. Therefore (8.2) holds for multi-indices of size $|\gamma|=2$. Assuming it does hold for multi-indices of size $|\gamma|$, we will show that it continues to hold for all ℓ-dimensional multi-indices

$$\tilde\gamma = (\gamma_1,\dots,\gamma_p+1,\dots,\gamma_\ell), \qquad 1\le p\le \ell$$

of size $|\gamma|+1$. By the Leibniz rule

$$D^\beta D_t^s \mathbf{u}^{\tilde\gamma} = \sum_{j=0}^{s}\binom{s}{j}\sum_{\sigma\le\beta}\binom{\beta}{\sigma}(D^{\beta-\sigma}D_t^{s-j}\mathbf{u}^\gamma)(D^\sigma D_t^j u_p).$$

First compute this at the origin. Then estimate $D^{\beta-\sigma}D_t^{s-j}\mathbf{u}^\gamma(0,0)$ by the induction hypothesis, and the terms $D^\sigma D_t^j u_p(0,0)$ by (6.3).

The estimation is concluded by proceeding as in the proof of (8.1). ■

Proof (of (8.3)). By the generalized Leibniz rule

$$D^\beta D_t^s(\mathbf{u}^\gamma D_{x_h}u_k) = \sum_{j=0}^{s}\binom{s}{j}\sum_{\sigma\le\beta}\binom{\beta}{\sigma}(D^{\beta-\sigma}D_t^{s-j}\mathbf{u}^\gamma)(D^\sigma D_t^j D_{x_h}u_k).$$

First compute this at the origin. Then majorize the terms involving $\mathbf{u}^\gamma$, by means of (8.2), and the terms $D^\sigma D_{x_h} D_t^j u_k(0,0)$ by (6.3). This gives

$$\begin{aligned}|D^\beta D_t^s(\mathbf{u}^\gamma D_{x_h}u_k)(0,0)| \le & C_o\left(\frac{cC_o}{C}\right)^{|\gamma|} C^{|\beta|+s} \\ \times \sum_{j=0}^{s}\sum_{\sigma\le\beta} & \binom{|\beta|+s}{|\sigma|+j}\frac{(|\beta-\sigma|+s-j)!(|\sigma|+j+1)!}{(\beta-\sigma+\iota)^2(\tilde\sigma+\iota)^2(s-j+1)^2(j+1)^2}\end{aligned} \tag{8.4}$$

where $\tilde\sigma$ is the N-dimensional multi-index $\tilde\sigma=(\sigma_1,\dots,\sigma_h+1,\dots,\sigma_N)$. Estimate

$$(|\sigma|+j+1)! \le (|\sigma|+j)!(|\beta|+s+1) \quad\text{ and }\quad (\tilde\sigma+\iota)^2 \ge (\sigma+\iota)^2.$$

These estimates in (8.4) yield

$$\begin{aligned}|D^\beta D_t^s(\mathbf{u}^\gamma D_{x_h}u_k)(0,0)| &\le C_o\left(\frac{cC_o}{C}\right)^{|\gamma|}C^{|\beta|+s}(|\beta|+s)!(|\beta|+s+1)\\ &\quad\times\sum_{j=0}^{s}\frac{1}{(s-j+1)^2(j+1)^2}\sum_{\sigma\le\beta}\frac{1}{(\beta-\sigma+\iota)^2(\sigma+\iota)^2}\\ &\le cC_o\left(\frac{cC_o}{C}\right)^{|\gamma|}C^{|\beta|+s}(|\beta|+s+1)!. \qquad\blacksquare\end{aligned}$$

9 Proof of the Cauchy–Kowalewski Theorem (Concluded)

To prove (6.3) return to (6.6) and estimate

$$\begin{aligned}|D^\beta D_t^{s+1}\mathbf{u}(0,0)| \le&\, 2(cC_o+1)M_oN\ell^2C^{|\beta|+s}\\ &\times\frac{(|\beta|+s+1)!}{(\beta+\iota)^2(s+2)^2}\sum_{|\gamma|\ge1}\left(\frac{cC_oM}{C}\right)^{|\gamma|}\end{aligned}$$

where M and M_o are the constants appearing in (6.5). It remains to choose C so large that

$$2(C_o+1)M_oN\ell^2\sum_{|\gamma|\ge1}\left(\frac{cC_oM}{C}\right)^{|\gamma|}\le C_o. \tag{9.1}$$

Remark 9.1 The choice of C gives a lower estimate of the radius of convergence of the series (6.1) and (6.4).

9.1 Proof of Lemma 6.1

Given the inequality (6.3), the proof of the Cauchy–Kowalewski theorem is a consequence of the following algebraic lemma

Lemma 9.1 *Let* α *be a* N*-dimensional multi-index. Then* $|\alpha|!\le N^{|\alpha|}\alpha!$.

Proof. Let x_i for $i=1,\dots,N$ be given real numbers, and let k be a positive integer. If α denotes an N-dimensional multi-index of size k, by the Leibniz version of Newton's formula (**9.2** of the Complements)

$$\left(\sum_{i=1}^N x_i\right)^k=\sum_{|\alpha|=k}\frac{k!}{\alpha!}\prod_{i=1}^N x_i^{\alpha_i}. \tag{9.2}$$

From this, taking $x_i=1$ for all $i=1,\dots,N$

$$N^{|\alpha|}=\sum_{|\alpha|=k}\frac{|\alpha|!}{\alpha!}. \qquad\blacksquare$$

10 Holmgren's Uniqueness Theorem

As we have seen in Section 5, if Γ is noncharacteristic and analytic about one of its points x_o, the Cauchy data (4.2) satisfy the compatibility conditions (4.3) and are analytic at x_o, the coefficients A_α and the free term F are analytic about the point $(x_o, u(x_o), D^{m-1}u(x_o))$, then the Cauchy–Kowalewski theorem ensures that the Cauchy problem (4.1)–(4.2) has a solution u, analytic at x_o. Moreover, the solution is unique within the class of analytic solutions at x_o. However, the possibility that a non-analytic solution to (4.1)–(4.2) exists is left open.

For linear equations, Holmgren's theorem denies this possibility. We shall use this result in Chapter 15, and therefore, we are going to present a fairly general version of it.

Before stating the theorem, and giving its full proof, let us briefly discuss the heuristic idea behind it. Let

$$L = \sum_{|\alpha| \le m} A_\alpha(x) D^\alpha \qquad x \in \mathbb{R}^N, \tag{10.1}$$

where the coefficients A_α are $N \times N$ matrices, having C^{m+2} entries (we will be more precise about the regularity of these elements later on). Extending what we have already seen in Section 4, we associate with (10.1) the characteristic matrix

$$\mathscr{L}(x, \xi) = \sum_{|\alpha| = m} A_\alpha(x) \xi^\alpha,$$

where $\xi = (\xi_1, \dots, \xi_N)$ and $\xi^\alpha = \xi_1^{\alpha_1} \cdot \cdots \cdot \xi_N^{\alpha_N}$, and the characteristic form

$$\mathcal{L}(x, \xi) = \det \mathscr{L}(x, \xi),$$

which is a form of degree Nm.

The extension of the notion of characteristic surface to the vector-valued framework is quite straightforward. Indeed, a surface Γ, which is the zero-level set of a function $\Phi = \Phi(x)$, is called characteristic with respect to L at $x_o \in \Gamma$, if

$$\mathcal{L}(x_o, D\Phi(x_o)) = 0.$$

Moreover, we say that Γ is characteristic with respect to L, if it is characteristic at each single point. For a scalar operator, it is apparent that the previous condition reduces to (4.5).

Now consider $G \subset \mathbb{R}^N$ with boundary $\partial G = \Gamma_1 \cup \Gamma_2$, and let $\mathbf{u}$, $\mathbf{v}$ be two vectors such that

$$D^\beta \mathbf{u} = 0 \qquad \text{on } \Gamma_1, \quad |\beta| \le m-1, \tag{10.2}$$

$$D^\beta \mathbf{v} = 0 \qquad \text{on } \Gamma_2, \quad |\beta| \le m-1. \tag{10.3}$$

We are interested in the uniqueness of the solutions to the problem

$$\begin{cases} L\mathbf{u} = 0 & \text{in } \ G, \\ D^\beta \mathbf{u} = 0 & \text{on } \ \mathit{\Gamma}_1, \quad |\beta| \leq m-1. \end{cases} \tag{10.4}$$

In particular, we would like to conclude that $u \equiv 0$ in G. If we integrate the first one of (10.4) by parts, we have

$$\int_G \mathbf{v} \sum_{|\alpha| \leq m} A_\alpha D^\alpha \mathbf{u}\, dx = \int_G \mathbf{u} \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha (\mathbf{v} A_\alpha)\, dx. \tag{10.5}$$

We let $M\mathbf{v}$ be given by $\sum\limits_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha (\mathbf{v} A_\alpha)$. Let us now assume that for any $\mathbf{w}$ the problem

$$\begin{cases} M\mathbf{v} = \mathbf{w} & \text{in } \ G, \\ D^\beta \mathbf{v} = 0 & \text{on } \ \mathit{\Gamma}_2, \quad |\beta| \leq m-1, \end{cases} \tag{10.6}$$

has a solution. In such a case, relying on (10.5) yields

$$\int_G \mathbf{w}\, \mathbf{u}\, dx = \int_G (M\mathbf{v})\mathbf{u}\, dx = \int_G \mathbf{v}(L\mathbf{u})\, dx = 0,$$

whence we conclude that $\mathbf{u} \equiv 0$ in G. Therefore, if (10.6) has a solution for any $\mathbf{w}$, we conclude that the solution of (10.4) is uniquely determined and identically vanishes.

As a matter of fact, we do not need to have existence for any function $\mathbf{w}$; assuming that $\mathbf{w} \in C^o(\mathbb{R}^N)$, it suffices to show that we have existence for each element $\mathbf{w}_k$ of a sequence $\{\mathbf{w}_k\}$, which converges uniformly to $\mathbf{w}$.

Indeed, if the problem

$$\begin{cases} M\mathbf{v}_k = \mathbf{w}_k & \text{in } \ G, \\ D^\beta \mathbf{v}_k = 0 & \text{on } \ \mathit{\Gamma}_2, \quad |\beta| \leq m-1, \end{cases} \tag{10.7}$$

has a solution $\mathbf{v}_k$, then

$$\int_G \mathbf{u}\, \mathbf{w}\, dx = \lim_{k \to \infty} \int_G \mathbf{u}\, \mathbf{w}_k\, dx = 0,$$

and we conclude as we want.

The set of function $\{\mathbf{w}\}_k$ dense in the set of continuous functions has been left unspecified here. In the proof, the approximating functions will be assumed to be analytic, so that it will be possible to apply Theorem 5.1.

Theorem 10.1 (Holmgren). *The equation*

$$L\mathbf{u} = \sum_{|\alpha| \leq m} A_\alpha(x) D^\alpha \mathbf{u} = \mathbf{f}(x), \qquad x \in \mathbb{R}^N,$$

with analytic coefficients A_α *and data*

$$D^\beta \mathbf{u} = \mathbf{g}_\beta \quad on \quad \Gamma, \quad |\beta| \le m-1,$$

on an analytic, noncharacteristic surface Γ *has at most one solution in a neighborhood of* Γ*.*

Remark 10.1 As we have already discussed above, owing to the linearity of the equation, it suffices to show that for $\mathbf{f} = \mathbf{g} = 0$, the only solution is $\mathbf{u} = 0$.

Remark 10.2 The theorem was originally proved by Holmgren (see [121]) in 1901; here, we follow a modern presentation of the original argument as given, for example, in Smoller [239].

Remark 10.3 Notice that we are not assuming $\mathbf{f}$ or $\mathbf{g}$ to be analytic.

Remark 10.4 It is not difficult to see that Γ is characteristic for the operator L if and only if Γ is characteristic for the operator M.

Remark 10.5 As already hinted at before, the analyticity of the coefficients A_α suggest taking $\mathbf{w}$ analytic as well, so that we can apply Theorem 5.1. However, this raises a problem: the Cauchy–Kowalewski theorem ensures that for $\mathbf{w}$ analytic, the problem

$$\begin{cases} M\mathbf{v} &= \mathbf{w} \qquad \text{in} \quad G, \\ D^\beta \mathbf{v} &= 0 \qquad \text{on} \quad \Gamma, \quad |\beta| \le m-1, \end{cases}$$

has a local solution on Γ if such a surface is analytic, but the region where $\mathbf{v}$ is defined, depends on $\mathbf{w}$. Therefore, we need to determine a region of existence, which is independent of $\mathbf{w}$.

11 Proof of the Holmgren Uniqueness Theorem

Let

$$\Gamma = \{x \in \mathbb{R}^N : \ \Phi(x) = 0\}$$

be a noncharacteristic surface with respect to L, and denote by Γ_o a compact subset. We now deform Γ_o by an analytic homotopy, keeping its boundary $\partial\Gamma_o$ fixed.

In order to do that, we consider a family of surfaces

$$\Gamma_\lambda = \{x \in \mathbb{R}^n : \ \Phi(x, \lambda) = 0\}$$

with $\lambda \in [0, \bar{\lambda}]$, all having the same boundary and such that all the functions Φ are analytic with respect to both x and λ.

We choose λ_1, λ_2 such that $0 \le \lambda_1 < \lambda_2 \le \bar{\lambda}$ and we let $E_{\lambda_1\lambda_2}$ be the region determined by Γ_{λ_1} and Γ_{λ_2} as in Figure 11.1.

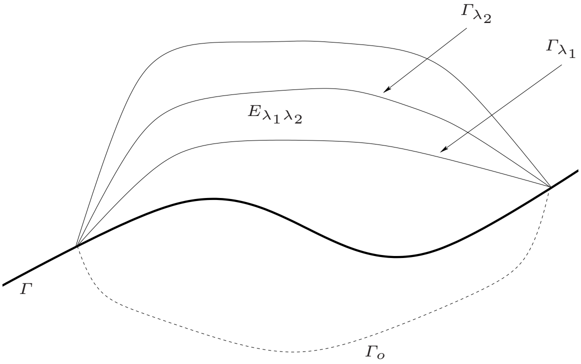

Fig. 11.1

If we let

$$\partial\Gamma_o = \{x \in \mathbb{R}^N : \ \Phi(x) = 0\} \cap \{x \in \mathbb{R}^N : \ \alpha(x) = 0\},$$

where α is a proper analytic function, then the family of surfaces Γ_λ can be built by

$$\Phi(x, \lambda) \overset{\text{def}}{=} (1 - \lambda)\Phi(x) + \lambda\alpha(x).$$

It is apparent that for $\lambda = 0$, we have precisely Γ_o; the issue at hand here is the choice of $\bar{\lambda}$. Provided λ is taken sufficiently small, we have that

$$\mathscr{L}(x, D\Phi(x, \lambda)) \neq 0 \qquad \text{for} \ \ x \in \Gamma_\lambda,$$

if

$$\mathscr{L}(x, D\Phi(x, \lambda)) \neq 0 \qquad \text{for} \ \ x \in \Gamma_o.$$

Hence, $\bar{\lambda}$ is chosen in such a way that for any $\lambda \in [0, \bar{\lambda}]$ all the surfaces Γ_λ are noncharacteristic. We also let $E \overset{\text{def}}{=} E_{0\bar{\lambda}}$. What we want to show is that the problem

$$L\mathbf{u} = \mathbf{f} \qquad \text{in} \ \ E$$

is uniquely determined by its data on Γ_o.

Let us first state the following auxiliary result.

Lemma 11.1 *There exists $\epsilon > 0$ such that for λ, μ satisfying*

$$0 \leq \lambda < \mu < \bar{\lambda}, \qquad \mu < \lambda + \epsilon,$$

the problem

$$\begin{cases} M\mathbf{v} & = \mathbf{w} \qquad \mathit{in} \quad E_{\lambda\mu}, \\ D^\beta\mathbf{v} & = 0 \qquad \mathit{on} \quad \Gamma_\lambda, \quad |\beta| \leq m-1, \end{cases} \tag{11.1}$$

has solutions $\mathbf{v}$ for $\mathbf{w}$ in a dense set of $C^o(E_{\lambda\mu})$.

We postpone the proof of the lemma, and we first show how it allows us to conclude.

Suppose that $\mathbf{u}$ is a solution of the problem

$$\begin{cases} L\mathbf{u} &= 0 \qquad \text{in} \quad E, \\ D^\beta \mathbf{u} &= 0 \qquad \text{on} \quad \Gamma_o, \quad |\beta| \le m-1. \end{cases}$$

Since $\bar{\lambda} > 0$ is given, once we fix $\epsilon > 0$, we can find a finite set

$$S = \{0 < \lambda_1 < \lambda_2 < \cdots < \lambda_j = \bar{\lambda}\}$$

such that $\lambda_{i+1} - \lambda_i < \epsilon$ for $i = 1, \ldots, j-1$. If ϵ is the value given by Lemma 11.1, we know that for $\mathbf{w}$ in a dense set of $C^o(E_{0\lambda_1})$ the problem

$$\begin{cases} M\mathbf{v} &= \mathbf{w} \qquad \text{in} \quad E_{0\lambda_1}, \\ D^\beta \mathbf{v} &= 0 \qquad \text{on} \quad \Gamma_{\lambda_1}, \quad |\beta| \le m-1 \end{cases}$$

has solutions. Since $\partial E_{0\lambda_1} = \Gamma_o \cup \Gamma_{\lambda_1}$, $\mathbf{u}$ vanishes with its derivatives on Γ_o, and $\mathbf{v}$ vanishes with its derivatives on Γ_{λ_1}, working as in the heuristic argument, we conclude that $\mathbf{u} = 0$ in $E_{0\lambda_1}$. We can now repeat the previous argument finitely many times, to conclude that $u \equiv 0$ in E. ■.

The proof has an intrinsic local feature, since $\mathbf{u} \equiv 0$ in a small region around Γ_o. It might be interesting to study conditions, that ensure that $\mathbf{u}$ vanishes in a prescribed region, but this goes beyond the scope of these notes.

On the other hand, we have shown that the solution of the initial value problem vanishes in a region, which is obtained by holomorphic deformation of Γ_o; therefore, the region of uniqueness is determined by the noncharacteristic surface.

11.1 Proof of Lemma 11.1

We consider an entire function $\mathbf{w} = \mathbf{w}(x, \xi)$, where $\xi = (\xi_1, \ldots, \xi_N)$ is a parameter.

As above, Γ_λ is a family of noncharacteristic surfaces, with $\lambda \in [0, \bar{\lambda}]$. For any parameter ξ with $|\xi| \le 1$ and any $\mu \in [0, \bar{\lambda}]$ we want to solve the problem

$$\begin{cases} M\mathbf{v} &= \mathbf{w}(x, \lambda) \qquad \text{in} \quad \mathbb{R}^{2N+1}, \\ D^\beta \mathbf{v} &= 0 \qquad \text{on} \quad \Gamma_\mu, \quad |\beta| \le m-1. \end{cases} \tag{11.2}$$

We can consider $M\mathbf{v} = \mathbf{w}$ as a system for the unknown $\mathbf{v}$, where $\mathbf{v}$ is a function of $2N+1$ variables, namely $x_1, \ldots, x_N, \xi_1, \ldots, \xi_N, \mu$ and there are no derivatives with respect to the $N+1$ variables $\xi_1, \ldots, \xi_N, \mu$.

Since

$$S = \{(x, \xi, \mu) \in \mathbb{R}^{2N+1} : \ x \in \Gamma_\mu, \ 0 \le \mu \le \bar{\lambda}, \ |\xi| \le 1\}$$

is a compact, noncharacteristic, analytic surface in the variables (x, ξ, μ), by the Cauchy–Kowalewski theorem there exists an analytic solution to (11.2) in a neighborhood of S. In other words, there exists $\delta > 0$ such that a solution $\mathbf{v} = \mathbf{v}(x, \xi, \mu)$ with $|\xi| \le 1$ and $\mu \in [0, \bar{\lambda}]$ is defined with x in a neighborhood of width δ of the surface $\varGamma_\mu$.

Provided we choose μ sufficiently close to λ such that the neighborhoods of $\varGamma_\lambda$ and $\varGamma_\mu$ intersect each other, we conclude that $\mathbf{v}(x, \xi, \mu)$ is defined in $E_{\lambda\mu}$ for all $|\xi| \le 1$. The closeness of μ and λ is precisely quantified by ϵ, that is, $0 < \mu - \lambda < \epsilon$.

Altogether, we have found a solution $\mathbf{v}$ of

$$\begin{cases} M\mathbf{v} &= \mathbf{w} \qquad \text{in} \quad E_{\lambda\mu}, \\ D^\beta \mathbf{v} &= 0 \qquad \text{on} \quad \varGamma_\mu, \quad |\beta| \le m - 1, \end{cases} \tag{11.3}$$

with $\mu - \lambda < \epsilon$ and $|\lambda| \le 1$. In particular, ξ is still completely arbitrary. If we can choose $|\xi| \le 1$ in such a way that the corresponding data $\mathbf{w} = \mathbf{w}(x, \xi)$ give linear combinations that are dense in C^o, then the problem (11.3) has a solution for a set of $\mathbf{w}$ that is dense in $C^o(E_{\lambda\mu})$, and this would finish the proof.

In order to achieve this, we pick

$$\mathbf{w}(x, \xi) = e^{x \cdot \xi}.$$

This allows us to uniformly approximate every monomial $x^\alpha = x_1^{\alpha_1} \cdot \dots \cdot x_N^{\alpha_N}$ in every bounded set by linear combinations of such functions $\mathbf{w}(x, \xi)$ with $|\xi| \le 1$. Indeed,

$$x^\alpha = \lim_{\xi \to 0} \prod_{k=1}^{N} \left(\frac{e^{x_k \xi_k} - 1}{\xi_k} \right)^{\alpha_k}.$$

Moreover, since the linear combinations are dense in the space of continuous functions, the same occurs for linear combinations of our $\mathbf{w}(x, \xi)$ with $|\xi| \le 1$. ■

Problems and Complements

1c Quasi-Linear Second-Order Equations in Two Variables

1.1. Assume that the functions A, B, C, D in (1.1) are of class C^∞. Assume also that $\varGamma$ is of class C^∞. Prove that all the derivatives

$$k, h, \ell \in \mathbb{N}, \qquad \frac{\partial^k u}{\partial x^h \partial y^\ell}, \quad h + \ell = k$$

can be computed on Γ provided (1.4) holds.

1.2. Assume that in (1.1), A, B, C are constants and $D = 0$. Introduce an affine transformation of the coordinate variables that transforms (1.1) into either the Laplace equation, the heat equation, or the wave equation.

1.3. Prove the last statement of Section 2.1. Discuss the case of constant coefficients.

1.4 Consider the equation

$$(y-1)^2 u_{xx} - y^2 u_{yy} = 0.$$

a) Classify the equation and determine its characteristics.

b) Show that if data are prescribed on the line $y = 2$, the Cauchy problem (1.2) can be solved.

1.5 Consider the equation

$$x\, u_{xx} + 2\sqrt{x(1+y)}\, u_{xy} + y\, u_{yy} = 0$$

in the open set $E = \{(x,y) \in \mathbb{R}^2 : x(1+y) > 0\}$. Classify the equation and determine its characteristics in the hyperbolic region.

1.6 Consider the equation

$$x^3 y\, u_{xx} - 4xy^3\, u_{yy} = 0.$$

a) Classify the equation and determine its characteristics.

b) Show that if data are prescribed on the half-line $y = \frac{1}{4}$, $x > 0$, the Cauchy problem (1.2) can be solved.

1.7 Consider the equation

$$x^2\, u_{xx} + 4xy\, u_{xy} + (4y^2 - x^2 + 2x - 1)\, u_{yy} = 0.$$

a) Classify the equation and determine its characteristics.

b) Show that if data are prescribed on the line $x = \frac{1}{4}$, the Cauchy problem (1.2) can be solved.

5c Analytic Data and the Cauchy–Kowalewski Theorem

5.1. Denote points in $\mathbb{R}^{N+1}$ by (x,t) where $x \in \mathbb{R}^N$ and $t \in \mathbb{R}$. Let φ and ψ be analytic in $\mathbb{R}^N$. Find an analytic solution, about $t = 0$ of

$$\Delta u = 0, \quad u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x).$$

5.2. Let f_1 and f_2 be analytic and periodic of period 2π in $\mathbb{R}$. Solve the problem

$$\Delta u = 0 \quad \text{in} \quad 1 - \varepsilon < |x| < 1 + \varepsilon$$

$$u \Big|_{|x|=1} = f_1(\theta), \qquad \frac{\partial u}{\partial |x|} \Big|_{|x|=1} = f_2(\theta)$$

for some $\varepsilon \in (0,1)$. Compare with the Poisson integral (3.11) of Chapter 2.

6c Proof of the Cauchy–Kowalewski Theorem

6.1. Prove that (6.2) ensures the convergence of the series (6.1) and give an estimate of the radius of convergence.

6.2. Let β be a $N-$dimensional multi-index of size $|\beta|$. Prove that the number of derivatives D^β of order $|\beta|$ does not exceed $|\beta|^N$.

6.3. Let $\mathbf{u} : \mathbb{R}^N \to \mathbb{R}^\ell$ be analytic at some point $x_o \in \mathbb{R}^N$. Prove that there exist constants C_o and C such that for all $N-$dimensional multi-indices β

$$|D^\beta \mathbf{u}(x_o)| \le C_o C^{|\beta|-1} \frac{|\beta|!}{(\beta+\iota)^2}.$$

6.4. Prove (6.4)–(6.4).

8c The Generalized Leibniz Rule

8.1. Let $u, v \in C^\infty(\mathbb{R})$ be real-valued. The Leibniz rule states that for every $n \in \mathbb{N}$

$$D^n(uv) = \sum_{i=0}^{n} \binom{n}{i} D^{n-i} u D^i v.$$

In particular if $u, v \in C^\infty(\mathbb{R}^N)$, then

$$D^n_{x_r}(uv) = \sum_{i=0}^{n} \binom{n}{i} D^{n-i}_{x_r} u D^i_{x_r} v.$$

Prove, by induction, the generalized Leibniz rule

$$D^\beta(uv) = \sum_{\sigma \le \beta} \binom{\beta}{\sigma} D^{\beta-\sigma} u D^\sigma v.$$

9c Proof of the Cauchy–Kowalewski Theorem Concluded

9.1. Prove that C can be chosen such that (9.1) holds.

9.2. Prove (9.2) by induction, starting from the binomial formula

$$(x_1 + x_2)^k = \sum_{j=0}^{k} \binom{k}{k-j} x_1^j x_2^{k-j}.$$

2

THE LAPLACE EQUATION

1 Preliminaries

Let E be a domain in $\mathbb{R}^N$ for some $N \ge 2$, with boundary ∂E of class C^1. Points in E are denoted by $x = (x_1, \dots, x_N)$. A function $u \in C^2(E)$ is *harmonic* in E if

$$\Delta u = \operatorname{div} \nabla u = \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} u = 0 \quad \text{in} \quad E. \tag{1.1}$$

The formal operator Δ is called the *Laplacean.*[1] The interest in this equation stems from its connection to physical phenomena such as

1. Steady-state heat conduction in a homogeneous body with constant heat capacity and constant conductivity.
2. Steady-state potential flow of an incompressible fluid in a porous medium with constant permeability.
3. Gravitational potential in $\mathbb{R}^N$ generated by a uniform distribution of masses.

The interest is also of pure mathematical nature in view of the rich structure exhibited by (1.1). The formal operator in (1.1) is invariant under rotations or translations of the coordinate axes. Precisely, if A is a (unitary, orthonormal) rotation matrix and $y = A(x - \xi)$ for some fixed $\xi \in \mathbb{R}^N$, then formally

$$\Delta_x = \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} = \sum_{i=1}^N \frac{\partial^2}{\partial y_i^2} = \Delta_y.$$

This property is also called *spherical symmetry* of the Laplacean in $\mathbb{R}^N$.

---

[1] Pierre Simon, Marquis de Laplace, 1749–1827. Author of *Traité de Mécanique Céleste* (1799–1825). Also known for the frequent use of the phrase *il est aisé de voir* which has unfortunately become all too popular in modern mathematical writings. The same equation had been introduced, in the context of potential fluids, by Joseph Louis, Compte de Lagrange, 1736–1813, author of *Traité de Mécanique Analytique* (1788).

© Springer Nature Switzerland AG 2023

E. DiBenedetto, U. Gianazza, *Partial Differential Equations*,
Cornerstones, https://doi.org/10.1007/978-3-031-46618-2_3

1.1 The Dirichlet and Neumann Problems

Given $\varphi \in C(\partial E)$, the Dirichlet problem for the operator Δ in E consists in finding a function $u \in C^2(E) \cap C(\bar{E})$ satisfying

$$\Delta u = 0 \text{ in } E, \quad \text{and} \quad u\big|_{\partial E} = \varphi. \tag{1.2}$$

Given $\psi \in C(\partial E)$, the Neumann problem consists in finding a function $u \in C^2(E) \cap C^1(\bar{E})$ satisfying

$$\Delta u = 0 \text{ in } E, \quad \text{and} \quad \frac{\partial}{\partial \mathbf{n}} u = \nabla u \cdot \mathbf{n} = \psi \text{ on } \partial E \tag{1.3}$$

where $\mathbf{n}$ denotes the outward unit normal to ∂E. The Neumann datum ψ is also called *variational*.

We will prove that if E is bounded, the Dirichlet problem is always uniquely solvable. The Neumann problem, on the other hand, is not always solvable. Indeed, integrating the first of (1.3) in E, we arrive at the necessary condition

$$\int_{\partial E} \psi \, d\sigma = 0 \tag{1.4}$$

where $d\sigma$ denotes the surface measure on ∂E. Thus ψ cannot be assigned arbitrarily.

Lemma 1.1 *Let E be a bounded open set with boundary ∂E of class C^1 and assume that (1.2) and (1.3) can both be solved within the class $C^2(\bar{E})$. Then the solution of (1.2) is uniquely determined by φ, and the solution of (1.3) is uniquely determined by ψ up to a constant.*

Proof. We prove only the statement regarding the Dirichlet problem. If u_i for $i = 1, 2$, are two solutions of (1.2), the difference $w = u_1 - u_2$ is a solution of the Dirichlet problem with homogeneous data

$$\Delta w = 0 \quad \text{in } E, \qquad w\big|_{\partial E} = 0.$$

Multiplying the first of these by w and integrating over E gives

$$\int_E |\nabla w|^2 dx = 0. \qquad \blacksquare$$

Remark 1.1 Arguments of this kind are referred to as *energy methods*. The assumption $w \in C^2(\bar{E})$ is used to justify the various calculations in the integration by parts. The lemma continues to hold for solutions in the class $C^2(E) \cap C^1(\bar{E})$. Indeed, one might first carry the integration over an open, proper subset $E' \subset E$, with boundary $\partial E'$ of class C^1, and then let E' expand to E. We will show later that uniqueness for the Dirichlet problem holds within the class $C^2(E) \cap C(\bar{E})$, required by the formulation (1.2).

Remark 1.2 A consequence of the lemma is that the problem

$$\Delta u = 0 \text{ in } E \quad \text{and } u\big|_{\partial E} = \varphi, \;\; \nabla \cdot \mathbf{n} = \psi$$

in general is not solvable.

1.2 The Cauchy Problem

Let Γ be an $(N-1)$-dimensional surface of class C^1 contained in E and prescribe $N+1$ functions $\psi_i \in C^2(\Gamma)$, for $i = 0, 1, \dots, N$. The Cauchy problem consists in finding $u \in C^2(E)$ satisfying

$$\Delta u = 0 \text{ in } E \quad \text{and} \quad u = \psi_o,\ u_{x_i} = \psi_i,\ i = 1, \dots, N \text{ on } \Gamma. \tag{1.5}$$

The Cauchy problem is not always solvable. First the data ψ_i, must be compatible, i.e., derivatives of u along Γ computed using ψ_o and computed using ψ_i must coincide. Even so, in general, the solution, if any, can only be found near Γ. The Cauchy–Kowalewski theorem gives some sufficient conditions to ensure *local* solvabilty of (1.5).

1.3 Well-Posedness and a Counterexample of Hadamard

A boundary value problem for the Laplacean, say the Dirichlet, Neumann or Cauchy problem, is *well posed* in the sense of Hadamard if one can identify a class of boundary data, say $\mathcal{C}$, such that each datum in $\mathcal{C}$ yields a *unique* solution, and small variations of the data within $\mathcal{C}$ yield small variations on the corresponding solutions. The meaning of *small variation* is made precise in terms of the topology suggested by the problem. This is referred to as the problem of *stability.* A problem that does not meet any one of these criteria is called *ill posed.*

Consider the problem of finding a harmonic function in E taking either Dirichlet data or Neumann data on a portion Σ_1 of ∂E and both Dirichlet and variational data on the remaining part $\Sigma_2 = \partial E - \Sigma_1$. Such a problem is ill posed. Even if a solution exists, in general it is not stable in any reasonable topology, as shown by the following example due to Hadamard ([111]).

The boundary value problem

$$\begin{array}{ll} u_{xx} + u_{yy} = 0 & \text{in } \left(-\frac{\pi}{2} < x < \frac{\pi}{2}\right) \times (y > 0) \\ u(\pm\frac{\pi}{2}, y) = 0 & \text{for } y > 0 \\ u(x, 0) = 0 & \text{for } -\frac{\pi}{2} < x < \frac{\pi}{2} \\ u_y(x, 0) = e^{-\sqrt{n}} \cos nx & \text{for } -\frac{\pi}{2} < x < \frac{\pi}{2} \end{array}$$

admits the family of solutions

$$u_n(x, y) = \frac{1}{n} e^{-\sqrt{n}} \cos nx \sinh ny, \qquad \text{where } n \text{ is an odd integer.}$$

One verifies that

$$\|u_{n,y}(\cdot, 0)\|_{\infty,(-\frac{\pi}{2},\frac{\pi}{2})} \to 0 \text{ as } n \to \infty$$

and that for all $y > 0$

$$\|u_n(\cdot, y)\|_{\infty,(-\frac{\pi}{2},\frac{\pi}{2})}, \quad \|u_n(\cdot, y)\|_{2,(-\frac{\pi}{2},\frac{\pi}{2})} \to \infty \text{ as } n \to \infty.$$

1.4 Radial Solutions

The invariance of Δ under orthonormal linear transformations suggests that we look for solutions of (1.1) in $\mathbb{R}^N$ depending only on $\rho = |x-y|$, for any fixed $y \in \mathbb{R}^N$. Any such solution $\rho \to V(\rho;y)$ must satisfy

$$V'' + \frac{N-1}{\rho} V' = 0, \qquad y \in \mathbb{R}^N \text{ fixed}$$

where the derivatives are meant with respect to ρ. By integration this gives, up to additive and multiplicative constants

$$\mathbb{R}^N - \{y\} \ni x \to \begin{cases} \dfrac{1}{|x-y|^{N-2}} & \text{if } N \geq 3 \\ \\ \ln|x-y| & \text{if } N = 2. \end{cases} \tag{1.6}$$

These are the *potentials* of the Laplacean in $\mathbb{R}^N$ with a *pole* at y. Consider a finite distribution $\{(e_i, y_i)\}$ for $i = 1, \dots, n$, of electrical charges e_i, concentrated at the points y_i. The function

$$\mathbb{R}^N - \{y_1, \dots, y_n\} \ni x \to \sum_{i=1}^{n} \frac{e_i}{|x-y_i|^{N-2}}, \qquad x \neq y_i, \;\; i = 1, 2, \dots, n$$

is harmonic, and it represents the potential generated by the charges (e_i, y_i) outside them.

Let E be a bounded, Lebesgue measurable set in $\mathbb{R}^N$, and let $\mu \in C(\bar{E})$. The function

$$\mathbb{R}^N - \bar{E} \ni x \to \int_E \frac{\mu(y)}{|x-y|^{N-2}} dy, \qquad N \geq 3$$

is harmonic in $\mathbb{R}^N - \bar{E}$, and it represents the *Newtonian potential* generated outside $\bar{E}$, by the distribution of *masses* (or charges) $\mu(y)dy$ in E. Let Σ be an $(N-1)$-dimensional bounded surface of class C^1 in $\mathbb{R}^N$, for some $N \geq 3$, and let $\mathbf{n}(y)$ denote the unit normal at $y \in \Sigma$. The orientation of $\mathbf{n}(y)$ is arbitrary but fixed, so that $y \to \mathbf{n}(y)$ is continuous on Σ. Given $\varphi, \psi \in C(\Sigma)$ the two functions

$$\mathbb{R}^N - \bar{\Sigma} \ni x \to \begin{cases} \displaystyle\int_\Sigma \frac{\varphi(y)}{|x-y|^{N-2}} d\sigma \\ \\ \displaystyle\int_\Sigma \frac{\psi(y)}{|x-y|^{N}} (x-y) \cdot \mathbf{n}(y) d\sigma \end{cases}$$

are harmonic in $\mathbb{R}^N - \bar{\Sigma}$. The first is called *single-layer potential*, and it gives the potential generated, outside $\bar{\Sigma}$, by a distribution of charges (or masses) on Σ, of density $\varphi(\cdot)$. The second is called *double-layer potential* and it represents the electrical potential generated, outside Σ, by a distribution of dipoles on Σ, with density $\psi(\cdot)$.

Analogous harmonic functions can be constructed for $N = 2$, by using the second of (1.6). These would be called logarithmic potentials.

2 The Green and Stokes Identities

Let E be a bounded open set in $\mathbb{R}^N$ with boundary ∂E of class C^1, and let $u, v \in C^2(\bar{E})$. By the divergence theorem we obtain the Green's identities

$$\int_E v\Delta u\,dx = -\int_E \nabla v\cdot\nabla u\,dx + \int_{\partial E} v\frac{\partial u}{\partial \mathbf{n}}\,d\sigma \tag{2.1}$$

$$\int_E (v\Delta u - u\Delta v)\,dx = \int_{\partial E}\left(v\frac{\partial u}{\partial \mathbf{n}} - u\frac{\partial v}{\partial \mathbf{n}}\right)d\sigma. \tag{2.2}$$

Remark 2.1 By approximation, (2.1)–(2.2) continue to hold for functions $u, v \in C^2(E)\cap C^1(\bar{E})$ such that Δu and Δv are essentially bounded in E.

Remark 2.2 If u is harmonic in E, then

$$\int_{\partial E}\frac{\partial u}{\partial \mathbf{n}}d\sigma = 0 \quad\text{and}\quad \int_E |\nabla u|^2 dx = \int_{\partial E} u\frac{\partial u}{\partial \mathbf{n}}d\sigma.$$

2.1 The Stokes Identities

Let $u \in C^2(\bar{E})$ and let ω_N denote the area of the unit sphere in $\mathbb{R}^N$ for $N \ge 3$. Then for all $x \in E$

$$\begin{aligned} u(x) =& \frac{1}{\omega_N(N-2)}\int_{\partial E}\left(|x-y|^{2-N}\frac{\partial u}{\partial \mathbf{n}} - u(y)\frac{\partial |x-y|^{2-N}}{\partial \mathbf{n}}\right)d\sigma \\ &- \frac{1}{\omega_N(N-2)}\int_E |x-y|^{2-N}\Delta u\,dy. \end{aligned} \tag{2.3}$$

If $N = 2$

$$\begin{aligned} u(x) =& \frac{1}{2\pi}\int_{\partial E}\left(u\frac{\partial \ln|x-y|}{\partial \mathbf{n}} - \ln|x-y|\frac{\partial u}{\partial \mathbf{n}}\right)d\sigma \\ &+ \frac{1}{2\pi}\int_E \ln|x-y|\Delta u\,dy. \end{aligned} \tag{2.4}$$

Remark 2.3 These are implicit representation formulas of smooth functions in $\bar{E}$.

Proof. We prove only (2.3). Fix $x \in E$ and let $B_\varepsilon(x)$ be the ball of radius ε centered at x. Assume that ε is so small that $B_\varepsilon(x) \subset E$, and apply (2.2) in $E - B_\varepsilon(x)$ for $y \to v(y)$ equal to the potential with pole at x introduced in (1.6). Since $V(\cdot;x)$ is harmonic in $E - B_\varepsilon(x)$, (2.2) yields

$$\begin{aligned} \frac{N-2}{\varepsilon^{N-1}}\int_{|x-y|=\varepsilon} u(y)\,d\sigma =& \int_{\partial E}\left(|x-y|^{2-N}\frac{\partial u}{\partial \mathbf{n}} - u\frac{\partial |x-y|^{2-N}}{\partial \mathbf{n}}\right)d\sigma \\ &+ \varepsilon^{2-n}\int_{|x-y|=\varepsilon}\nabla u\cdot\frac{x-y}{|x-y|}\,d\sigma - \int_{E-B_\varepsilon(x)}|x-y|^{2-N}\Delta u\,dy. \end{aligned} \tag{2.5}$$

As $\varepsilon \to 0$

$$\int_{E-B_\varepsilon(x)} |x-y|^{2-N} \Delta u \, dy \longrightarrow \int_E |x-y|^{2-N} \Delta u \, dx$$

and

$$\varepsilon^{2-N} \int_{|x-y|=\varepsilon} \nabla u \cdot \frac{x-y}{|x-y|} d\sigma \longrightarrow 0.$$

As for the left-hand side of (2.5)

$$\frac{1}{\varepsilon^{N-1}} \int_{|x-y|=\varepsilon} u(y) d\sigma = \omega_N u(x) + \frac{1}{\varepsilon^{N-1}} \int_{|x-y|=\varepsilon} [u(y)-u(x)] d\sigma.$$

The last integral tends to zero as $\varepsilon \to 0$, since

$$\begin{aligned} \frac{1}{\varepsilon^{N-1}} \int_{|x-y|=\varepsilon} |u(y)-u(x)| d\sigma &\le \frac{\|\nabla u\|_{\infty,E}}{\varepsilon^{N-1}} \int_{|x-y|=\varepsilon} |x-y| d\sigma \\ &\le \varepsilon \omega_N \|\nabla u\|_{\infty,E}. \end{aligned}$$

These remarks in (2.5) prove (2.3) after we let $\varepsilon \to 0$.

Motivated by the Stokes identities, set

$$F(x;y) = \begin{cases} \dfrac{1}{\omega_N(N-2)} \dfrac{1}{|x-y|^{N-2}} & \text{if } N \ge 3 \\ \\ \dfrac{-1}{2\pi} \ln|x-y| & \text{if } N = 2. \end{cases} \tag{2.6}$$

The function $F(\cdot;y)$ is called the fundamental solution of the Laplacean with pole at y.

Corollary 2.1 *Let E be a bounded open set in $\mathbb{R}^N$ with boundary ∂E of class C^1 and let $u \in C^2(\bar{E})$ be harmonic in E. Then for all $x \in E$*

$$u(x) = \int_{\partial E} \left(F(x;\cdot) \frac{\partial u}{\partial \mathbf{n}} - u \frac{\partial F(x;\cdot)}{\partial \mathbf{n}} \right) d\sigma. \tag{2.7}$$

A consequence of this corollary, and the structure of the fundamental solution $F(\cdot, y)$ is the following

Proposition 2.1 *Let E be an open set in $\mathbb{R}^N$ and let $u \in C^2(E)$ be harmonic in E. Then $u \in C^\infty(E)$, and for every multi-index α, the function $D^\alpha u$ is harmonic in E.*

Proof. If E is bounded, ∂E is of class C^1, and $u \in C^2(\bar{E})$, the statement follows from the representation (2.7). Otherwise, apply (2.7) to any bounded open subset $E' \subset E$ with boundary of class C^1. ■

Corollary 2.2 $u \in C_o^2(E) \Longrightarrow u(x) = -\int_E F(x;y) \Delta u \, dy$ *for all* $x \in E$.

3 Green's Function and the Dirichlet Problem for a Ball

Given a bounded open set $E \subset \mathbb{R}^N$ with boundary ∂E of class C^1, consider the problem of finding, for each fixed $x \in E$, a function $y \to \Phi(x;y) \in C^2(\bar{E})$ satisfying

$$\Delta_y \Phi(x;\cdot) = 0 \;\text{ in }\; E, \qquad \text{and} \qquad \Phi(x;\cdot)\big|_{\partial E} = F(x;\cdot) \tag{3.1}$$

where $F(x;y)$ is the fundamental solution of the Laplacean introduced in (2.6). Assume for the moment that (3.1) has a solution. Assume also that the Dirichlet problem (1.2) has a solution $u \in C^2(\bar{E})$. Then the second Green's identity (2.2) written for $y \to u(y)$ and $y \to \Phi(x;y)$ gives

$$0 = \int_{\partial E} \left(\Phi(x;\cdot)\frac{\partial u}{\partial \mathbf{n}} - u\frac{\partial \Phi(x;\cdot)}{\partial \mathbf{n}} \right) d\sigma \qquad \text{for all fixed } \; x \in E.$$

Subtract this from the implicit representation (2.7), to obtain

$$u(x) = -\int_{\partial E} \varphi \frac{\partial G(x;\cdot)}{\partial \mathbf{n}} d\sigma \tag{3.2}$$

where

$$(x,y) \to G(x;y) = F(x;y) - \Phi(x;y). \tag{3.3}$$

The function $G(\cdot;\cdot)$ is the Green's function for the Laplacean in E. Its relevance is in that *every* solution $u \in C^2(\bar{E})$ of the Dirichlet problem (1.2) admits the explicit representation (3.2), through the Dirichlet data φ and $G(\cdot;\cdot)$. Its relevance is also in that it permits a pointwise representation of a smooth function u defined in E and vanishing near ∂E.

Corollary 3.1 $u \in C^2_o(E) \Longrightarrow u(x) = -\int_E G(x;y)\Delta u \, dy$ *for all* $x \in E$.

Lemma 3.1 *The Green's function is symmetric, i.e.,* $G(x;y) = G(y;x)$.

Proof. Fix $x_1, x_2 \in E$ and let $\varepsilon > 0$ be small enough that

$$B_\varepsilon(x_i) \subset E \;\text{ for }\; i = 1,2, \;\text{ and }\; B_\varepsilon(x_1) \cap B_\varepsilon(x_2) = \emptyset.$$

Apply Green's identity (2.2) to the pair of functions $G(x_i;\cdot)$ for $i = 1,2$ in the domain $E - [B_\varepsilon(x_1) \cup B_\varepsilon(x_2)]$. Since $G(x_i;\cdot)$ are harmonic in such a domain, and vanish on ∂E

$$\begin{aligned} &-\int_{\partial B_\varepsilon(x_1)} \left[G(x_1;y)\frac{\partial}{\partial \mathbf{n}(y)}G(x_2;y) - G(x_2;y)\frac{\partial}{\partial \mathbf{n}(y)}G(x_1;y) \right] d\sigma \\ &\quad = \int_{\partial B_\varepsilon(x_2)} \left[G(x_1;y)\frac{\partial}{\partial \mathbf{n}(y)}G(x_2;y) - G(x_2;y)\frac{\partial}{\partial \mathbf{n}(y)}G(x_1;y) \right] d\sigma. \end{aligned}$$

Let $\varepsilon \to 0$ and observe that

$$\lim_{\varepsilon\to 0}\int_{\partial B_\varepsilon(x_1)} G(x_1;y)\frac{\partial}{\partial\mathbf{n}(y)}G(x_2;y)d\sigma = 0$$
$$\lim_{\varepsilon\to 0}\int_{\partial B_\varepsilon(x_2)} G(x_2;y)\frac{\partial}{\partial\mathbf{n}(y)}G(x_1;y)d\sigma = 0.$$

Therefore

$$\lim_{\varepsilon\to 0}\int_{\partial B_\varepsilon(x_1)} G(x_2;y)\frac{\partial G(x_1;y)}{\partial\mathbf{n}(y)}d\sigma = \lim_{\varepsilon\to 0}\int_{\partial B_\varepsilon(x_2)} G(x_1;y)\frac{\partial G(x_2;y)}{\partial\mathbf{n}(y)}d\sigma.$$

From the definition of $G(\cdot;\cdot)$

$$\frac{\partial G(x_i;y)}{\partial\mathbf{n}(y)} = \frac{\partial F(x_i;y)}{\partial\mathbf{n}(y)} - \frac{\partial \Phi(x_i;y)}{\partial\mathbf{n}(y)},$$

and observe that

$$\lim_{\varepsilon\to 0}\int_{\partial B_\varepsilon(x_1)} G(x_1;y)\frac{\partial}{\partial\mathbf{n}(y)}\Phi(x_2;y)d\sigma = 0$$
$$\lim_{\varepsilon\to 0}\int_{\partial B_\varepsilon(x_2)} G(x_2;y)\frac{\partial}{\partial\mathbf{n}(y)}\Phi(x_1;y)d\sigma = 0$$

since $y \to \Phi(x_i;y)$ are regular. This implies that

$$\lim_{\varepsilon\to 0}\int_{\partial B_\varepsilon(x_1)} G(x_2;y)\frac{\partial F(x_1;y)}{\partial\mathbf{n}(y)}d\sigma = \lim_{\varepsilon\to 0}\int_{\partial B_\varepsilon(x_2)} G(x_1;y)\frac{\partial F(x_2;y)}{\partial\mathbf{n}(y)}d\sigma.$$

Computing the limits as in the the proof of the Stokes identity gives

$$\lim_{\varepsilon\to 0}\int_{\partial B_\varepsilon(x_j)} G(x_i;y)\frac{\partial F(x_j;y)}{\partial\mathbf{n}(y)}d\sigma = G(x_i;x_j) \qquad \text{for } \ x_i \neq x_j.$$

Thus $G(x_1;x_2) = G(x_2;x_1)$. ■

Corollary 3.2 *The functions $G(\cdot;y)$ and $\partial G(\cdot;y)/\partial\mathbf{n}(y)$, for fixed $y \in \partial E$, are harmonic in E.*

To solve the Dirichlet problem (1.2) we may find $G(\cdot;\cdot)$, and write down (3.2). This would be a candidate for a solution. By Corollary 3.2 it is harmonic. It would remain to show that

$$\lim_{x\to x_*} u(x) = \varphi(x_*) \qquad \text{for all } \ x_* \in \partial E. \tag{3.4}$$

We will show that this is indeed the case if (3.1) has a solution, i.e., if the Green's function for Δ in E can be determined. Thus solving the Dirichlet problem (1.2) reduces to solving the family of Dirichlet problems (3.1). The advantage in dealing with the latter is that the boundary datum $F(x;\cdot)$ is specific and given by (2.6). Nevertheless, (3.1) can be solved explicitly only for domains E exhibiting a simple geometry.

3.1 Green's Function for a Ball

Let B_R be the ball of radius R centered at the origin of $\mathbb{R}^N$. The map[2]

$$\xi = \frac{R^2}{|x|^2}x, \qquad x \neq 0 \tag{3.5}$$

transforms $B_R - \{0\}$ into $\mathbb{R}^N - B_R$, and ∂B_R into itself. Referring to Figure 3.1, the two triangles $\varDelta(y,0,\xi)$ and $\varDelta(x,0,y)$, are similar whenever $y \in \partial B_R$. Indeed, they have in common the angle θ, and in view of (3.5), the ratios $|x|/|y|$ and $|y|/|\xi|$ are equal, provided $|y| = R$. Therefore

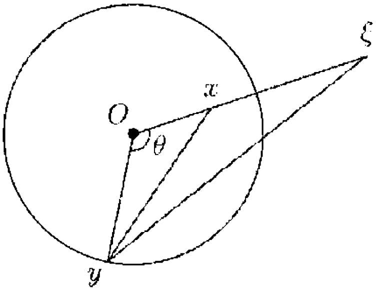

Fig. 3.1

$$|x-y| = |\xi - y|\frac{|x|}{R} = |\xi - y|\frac{R}{|\xi|}, \qquad x \in B_R \text{ and } y \in \partial B_R. \tag{3.6}$$

Then for each fixed $x \in B_R - \{0\}$, the solution of (3.1) is given by

$$\varPhi(x;y) = \begin{cases} \dfrac{1}{\omega_N(N-2)}\left(\dfrac{R}{|x|}\right)^{N-2} \dfrac{1}{|\xi - y|^{N-2}} & \text{if } N \geq 3 \\[2ex] \dfrac{-1}{2\pi}\ln|\xi - y|\dfrac{|x|}{R} & \text{if } N = 2. \end{cases} \tag{3.7}$$

While constructed for $x \neq 0$, the function $\varPhi(x;\cdot)$ is well defined also for $x = 0$, modulo taking the limit as $|x| \to 0$. For all $x \in B_R$, the function $\varPhi(x;\cdot)$ is harmonic in B_R since its pole ξ lies outside B_R. Moreover, by virtue of (3.6), the boundary conditions in (3.1) are satisfied. Thus the Green's function for the ball B_R is

$$G(x;y) = \frac{1}{\omega_N(N-2)}\left[\frac{1}{|x-y|^{N-2}} - \left(\frac{R}{|x|}\right)^{N-2}\frac{1}{|\xi - y|^{N-2}}\right] \tag*{$(3.8)_{N\geq 3}$}$$

---

[2] called also the Kelvin transform ([257]).

for $N \geq 3$, and

$$G(x;y) = \frac{1}{2\pi}\left[\ln|\xi - y|\frac{|x|}{R} - \ln|x-y|\right] \quad \text{for } \; N = 2. \tag{3.8$_{N=2}$}$$

The derivative of $G(x;\cdot)$, normal to the sphere $|y| = R$, is computed from

$$\frac{\partial}{\partial|y|}G(x;y)\big|_{y\in\partial B_R}.$$

From Figure 3.1, by elementary trigonometry

$$\begin{aligned}|x-y|^2 &= |y|^2 + |x|^2 - 2|y||x|\cos\theta\\ |\xi - y|^2 &= |y|^2 + \frac{R^4}{|x|^2} - 2|y|\frac{R^2}{|x|}\cos\theta.\end{aligned}$$

Therefore for θ fixed and $y \in \partial B_R$

$$\frac{\partial|x-y|}{\partial|y|} = \frac{|y| - |x|\cos\theta}{|x-y|}, \qquad \frac{\partial|\xi-y|}{\partial|y|} = \frac{|y| - \dfrac{R^2}{|x|}\cos\theta}{|\xi - y|}.$$

First let $N \geq 3$. Computing from $(3.8)_{N=2}$ and $(3.8)_{N\geq 3}$, and using (3.6), gives

$$-\frac{\partial}{\partial|y|}G(x;y)\big|_{y\in\partial B_R} = \frac{1}{R\omega_N}\frac{R^2 - |x|^2}{|x-y|^N}.$$

Such a formula also holds for $N = 2$ with $\omega_2 = 2\pi$. Put this in (3.3) to derive the following Poisson representation.

Lemma 3.2 *Let $u \in C^2(\bar{E})$ be a solution of the Dirichlet problem (1.2) in the ball B_R. Then*

$$u(x) = \frac{1}{\omega_N R}\int_{\partial B_R}\varphi(y)\frac{R^2 - |x|^2}{|x-y|^N}d\sigma, \qquad N \geq 2. \tag{3.9}$$

Setting $u \equiv 1$ in (3.9) gives

$$\frac{1}{\omega_N R}\int_{\partial B_R}\frac{R^2 - |x|^2}{|x-y|^N}d\sigma = 1 \quad \text{for all } \; x \in B_R. \tag{3.10}$$

Even though the representation (3.9) has been derived for solutions of (1.2) of class $C^2(\bar{B}_R)$, it actually gives the unique solution of the Dirichlet problem for the sphere, as shown by the following existence theorem.

Theorem 3.1. *The Dirichlet problem (1.2) for $E = B_R$ has a unique solution given by (3.9).*

Proof (existence). By Corollary 3.2, the function u given by (3.9) is harmonic in B_R. To prove (3.4), fix $x_* \in \partial B_R$, choose an arbitrarily small positive number ε, and let $\delta \in (0,1)$ be so small that

$$|\varphi(y) - \varphi(x_*)| < \varepsilon \quad \text{for all } \; y \in \Sigma_\delta = \big\{ y \in \partial B_R \; \big| \; |y - x_*| < \delta \big\}. \tag{3.11}$$

By (3.10)

$$\varphi(x_*) = \frac{1}{R\omega_N} \int_{\partial B_R} \varphi(x_*) \frac{R^2 - |x|^2}{|x-y|^N} d\sigma.$$

Therefore

$$\begin{aligned} u(x) - \varphi(x_*) &= \frac{1}{R\omega_N} \int_{\partial B_R} [\varphi(y) - \varphi(x_*)] \frac{R^2 - |x|^2}{|x-y|^N} d\sigma \\ &= \frac{1}{R\omega_N} \int_{\Sigma_\delta} [\varphi(y) - \varphi(x_*)] \frac{R^2 - |x|^2}{|x-y|^N} d\sigma \\ &\quad + \frac{1}{R\omega_N} \int_{\partial B_R - \Sigma_\delta} [\varphi(y) - \varphi(x_*)] \frac{R^2 - |x|^2}{|x-y|^N} d\sigma \\ &= I_\delta^{(1)}(x, x_*) + I_\delta^{(2)}(x, x_*). \end{aligned}$$

For δ fixed, $I_\delta^{(2)}(x, x_*) \to 0$ as $x \to x_*$. Moreover, in view of (3.10) and (3.11), $|I_\delta^{(1)}(x, x_*)| < \varepsilon$. Therefore $\lim_{x \to x_*} |u(x) - \varphi(x_*)| \le \varepsilon$ for all $\varepsilon > 0$. ∎

Remark 3.1 By Lemma 1.1 and the Remark 1.1, such a solution is unique in the class $C^2(B_R) \cap C^1(\bar{B}_R)$. It will be shown in the next section that uniqueness holds for solutions $u \in C^2(B_R) \cap C(\bar{B}_R)$.

4 Sub-Harmonic Functions and the Mean Value Property

Let E be a bounded open set in $\mathbb{R}^N$ with boundary ∂E of class C^1. Let $u \in C^2(\bar{E})$, and assume that the solution $\Phi(x; \cdot)$ of (3.1) exists for all $x \in E$. Subtracting the second Green's identity (2.2) for u and $\Phi(x; \cdot)$ from the Stokes identity (2.3) gives

$$u(x) = -\int_{\partial E} u \frac{\partial G(x;y)}{\partial \mathbf{n}} d\sigma - \int_E G(x;y) \Delta u(y) dy \qquad \text{for all } \; x \in E.$$

In particular, if E is a ball $B_R(x_o)$ of radius R centered at x_o, by setting $x = x_o$, we obtain

$$u(x_o) = \fint_{\partial B_R(x_o)} u d\sigma - \int_{B_R(x_o)} G(0; y - x_o) \Delta u(y) dy \tag{4.1}$$

where for a measurable set $D \subset \mathbb{R}^N$ of finite measure and $f \in L^1(D)$

$$\fint_D f\,dy = \frac{1}{|D|}\int_D f\,dy, \qquad |D| = \text{meas}\,(D).$$

Let $E \subset \mathbb{R}^N$ be open. A function $u \in C(E)$ is *sub-harmonic* in E if

$$u(x_o) \le \fint_{\partial B_R(x_o)} u\,d\sigma \qquad \text{for all } B_R(x_o) \subset E. \tag{4.2}$$

This implies that if $u \in C(E)$ is sub-harmonic in E, then (**4.1** of the Complements)

$$u(x_o) \le \fint_{B_R(x_o)} u\,dy \qquad \text{for all } B_R(x_o) \subset E. \tag{4.3}$$

A function $u \in C(E)$ is *super-harmonic* if $-u$ is sub-harmonic in E.

The Green's function $G(\cdot;\cdot)$ for a ball, as defined in $(3.8)_{N=2}$ and $(3.8)_{N\ge3}$, is non-negative. A consequence is that if $u \in C^2(E)$ is such that $\Delta u \ge (\le)0$ in E, then, by (4.1), u is sub(super)-harmonic in E. Conversely if $u \in C^2(E)$ is sub(super)-harmonic, then $\Delta u \ge (\le)0$ in E. Indeed, (4.1) implies

$$\int_{B_R(x_o)} G(0; y - x_o)\Delta u(y)dy \ge 0 \qquad \text{for all } B_R(x_o) \subset E.$$

From this

$$\Delta u(x_o)\int_{B_R(x_o)} G(0; y - x_o)dy \ge \int_{B_R(x_o)} G(0; y - x_o)[\Delta u(x_o) - \Delta u(y)]dy$$

and the assertion follows upon dividing by the coefficient of $\Delta u(x_o)$, and letting $R \to 0$.

Lemma 4.1 *Let E be a bounded, connected, open set in $\mathbb{R}^N$. If $u \in C(\bar{E})$ is sub-harmonic in E, then either u is constant, or*

$$u(x) < \sup_{\partial E} u \qquad \textit{for all } x \in E.$$

Proof. Let $x_o \in \bar{E}$ be a point where $u(x_o) = \sup_{\bar{E}} u$, and assume that u is not identically equal to $u(x_o)$. If $x_o \in E$, for every ball $B_R(x_o) \subset E$

$$u(y) \le u(x_o) \qquad \text{for all } y \in B_R(x_o)$$

which implies

$$\fint_{\partial B_R(x_o)} [u(y) - u(x_o)]d\sigma \le 0.$$

On the other hand, since u is sub-harmonic, this same integral must be non-negative. Therefore

$$\fint_{\partial B_R(x_o)} [u(y) - u(x_o)]d\sigma = 0 \quad \text{and} \quad u(y) \le u(x_o).$$

Thus there exists a ball $B_r(x_o)$ for some $r > 0$ such that $u(y) = u(x_o)$ for all $y \in B_r(x_o)$. Consider the set $\mathcal{E}_o = \{y \in E | u(y) = u(x_o)\}$. The previous remarks prove that $\mathcal{E}_o$ is open. By the continuity of u, it is closed in the relative topology of E. Therefore, since E is connected, $\mathcal{E}_o = E$ and $u \equiv u(x_o)$. The contradiction implies that $x_o \in \partial E$. ■

A function $u \in C(E)$ satisfies the *mean value property* in E if

$$u(x_o) = \fint_{\partial B_R(x_o)} u \, d\sigma \qquad \text{for all } B_R(x_o) \subset E, \tag{4.4}$$

equivalently if (**4.2** of the Complements)

$$u(x_o) = \fint_{B_R(x_o)} u \, dy \qquad \text{for all } B_R(x_o) \subset E. \tag{4.5}$$

Functions satisfying such a property are both sub- and super-harmonic. By (4.1), harmonic functions in E satisfy the mean value property.

Lemma 4.2 *Let E be a bounded, connected, open set in $\mathbb{R}^N$. If $u \in C(\bar{E})$ satisfies the mean value property in E, then either it is constant or*

$$\sup_E |u| = \sup_{\partial E} |u|.$$

Proof (Theorem 3.1, uniqueness). If u, v are two solutions of the Dirichlet problem (1.2) for $E = B_R$, the difference $w = u - v$ is harmonic in B_R and vanishes on ∂B_R. Thus $w \equiv 0$ by Lemma 4.2. ■

Lemma 4.3 *The following are equivalent:*

$$u \in C(E) \ \textit{satisfies the mean value property} \tag{i}$$

$$u \in C^2(E) \ \textit{and} \quad \Delta u = 0. \tag{ii}$$

Proof. We have only to prove (i)$\Longrightarrow$(ii). Having fixed $B_R(x_o) \subset E$, let $v \in C^2(B_R(x_o)) \cap C(\bar{B}_R(x_o)$ be the unique solution of the Dirichlet problem

$$\Delta v = 0 \ \text{ in } \ B_R(x_o) \qquad \text{and} \qquad v\big|_{\partial B_R(x_o)} = u.$$

Such a solution is given by the Poisson formula (3.9) up to a change of variables that maps x_o into the origin. The difference $w = u - v$ satisfies the mean value property in $B_R(x_o)$, and by Lemma 4.2, $w \equiv 0$. ■

4.1 The Maximum Principle

We restate some of these properties in a commonly used form. Let E be a bounded, connected, open set in $\mathbb{R}^N$ with boundary ∂E of class C^1, and let $u \in C^2(E) \cap C(\bar{E})$ be nonconstant in E. Then

$$\begin{array}{lll} \Delta u \ge 0 \ \text{ in } \ E \implies u(x) < \sup_{\partial E} u & \forall x \in E \\ \Delta u \le 0 \ \text{ in } \ E \implies u(x) > \inf_{\partial E} u & \forall x \in E \\ \Delta u = 0 \ \text{ in } \ E \implies |u(x)| < \sup_{\partial E} |u| & \forall x \in E. \end{array}$$

Remark 4.1 The assumption that ∂E is of class C^1 can be removed by applying the maximum principle to a family of expanding connected open sets with smooth boundary, exhausting E.

Remark 4.2 The assumption of E being bounded cannot be removed, as shown by the following counterexample. Let E be the sector $x_2 > |x_1|$ in $\mathbb{R}^2$. The function $u(x) = x_2^2 - x_1^2$ is harmonic in E, vanishes on ∂E, and takes arbitrarily large values in E.

4.2 Structure of Sub-Harmonic Functions

Set

$$\begin{aligned} \sigma(E) &= \{v \in C(E) \bigm| v \text{ is sub-harmonic in } E\} \\ \varSigma(E) &= \{v \in C(E) \bigm| v \text{ is super-harmonic in } E\}. \end{aligned} \tag{4.6}$$

Proposition 4.1 *Let $v, v_i \in \sigma(E)$ and $c_i \in \mathbb{R}^+$ for $i = 1, \dots, n$. Then*

$$v \in \sigma(E') \quad \textit{for every open subset } E' \subset E \tag{i}$$

$$\sum_{i=1}^{n} c_i v_i \in \sigma(E) \tag{ii}$$

$$\max\{v_1, v_2, \dots, v_n\} \in \sigma(E) \tag{iii}$$

$$\begin{aligned} &\textit{For every nondecreasing convex function } f(\cdot) \textit{ in } \mathbb{R} \\ &v \in \sigma(E) \Longrightarrow f(v) \in \sigma(E). \end{aligned} \tag{iv}$$

Proof. The statements (i)–(ii) are obvious. To prove (iii), observe that having fixed $B_R(x_o) \subset E$, for some $1 \le i \le n$

$$\begin{aligned} \max\{v_1(x_o), \dots, v_n(x_o)\} = v_i(x_o) &\le {\int\!\!\!\!\!\!-}_{\partial B_R(x_o)} v_i \, d\sigma \\ &\le {\int\!\!\!\!\!\!-}_{\partial B_R(x_o)} \max\{v_1, \dots, v_n\} d\sigma. \end{aligned}$$

To prove (iv), write (4.2) for v and apply $f(\cdot)$ to both sides. By Jensen's inequality

$$f\big(v(x_o)\big) \le f\bigg({\int\!\!\!\!\!\!-}_{\partial B_R(x_o)} v d\sigma\bigg) \le {\int\!\!\!\!\!\!-}_{\partial B_R(x_o)} f(v) d\sigma. \qquad \blacksquare$$

Remark 4.3 For simplicity, (iii) and (iv) have been stated separately. In fact (iv) implies (iii).

An important subclass of $\sigma(E)$ is that of the sub-harmonic functions in E that actually are harmonic in some sphere contained in E. Given $v \in \sigma(E)$, fix $B_\rho(\xi) \subset E$ and solve the Dirichlet problem

$$\varDelta H_v = 0 \text{ in } B_\rho(\xi) \quad \text{and} \quad H_v \big|_{\partial B_\rho(\xi)} = v.$$

The unique solution H_v is the harmonic extension of $v\big|_{\partial B_\rho(\xi)}$ into $B_\rho(\xi)$. The function that coincides with v in $E - B_\rho(\xi)$ and that equals H_v in $B_\rho(\xi)$ is denoted by $v_{\xi,\rho}$, i.e.,

$$v_{\xi,\rho}(x) = \begin{cases} v(x) & \text{if } x \in E - B_\rho(\xi) \\ H_v(x) & \text{if } x \in B_\rho(\xi). \end{cases} \tag{4.7}$$

Since $v \in \sigma(B_\rho(\xi))$ and H_v is harmonic in $B_\rho(\xi)$, we have $v - H_v \in \sigma(B_\rho(\xi))$. Therefore $v \le H_v$ in $B_\rho(\xi)$. The definition of $v_{\xi,\rho}$ then implies

$$v \le v_{\xi,\rho} \quad \text{in } E. \tag{4.8}$$

Proposition 4.2 *Let $v \in \sigma(E)$. Then $v_{\xi,\rho} \in \sigma(E)$.*

Proof. One needs to verify that $v_{\xi,\rho}$ satisfies (4.2) for all $B_R(x_o) \subset E$. This is obvious for $x_o \in E - B_\rho(\xi)$ in view of (4.8). Fix $x_o \in B_\rho(\xi)$ and assume, by contradiction, that there is a ball $B_R(x_o) \subset E$ such that

$$v_{\xi,\rho}(x_o) > \fint_{\partial B_R(x_o)} v_{\xi,\rho}\, d\sigma.$$

Construct the function

$$w = (v_{\xi,\rho})_{x_o,R} = \begin{cases} v_{\xi,\rho} & \text{in } E - B_R(x_o) \\ H_{v_{\xi,\rho}} & \text{in } B_R(x_o). \end{cases}$$

Since $v_{\xi,\rho} \ge v$, by the maximum principle $w \ge v_{x_o,R}$. Since w satisfies the mean value property in $B_R(x_o)$, the contradiction assumption implies that

$$v_{\xi,\rho}(x_o) - w(x_o) > 0. \tag{4.9}$$

The difference $v_{\xi,\rho} - w$ is harmonic in $B_R(x_o) \cap B_\rho(\xi)$. The boundary of such a set is the union of ∂_1 and ∂_2, where

$$\partial_1 = \partial B_R(x_o) \cap \bar{B}_\rho(\xi) \quad \text{and} \quad \partial_2 = \partial B_\rho(\xi) \cap \bar{B}_R(x_o).$$

Because of (4.9), the function $x \to (v_{\xi,\rho} - w)(x)$, restricted to $B_R(x_o) \cap B_\rho(\xi)$, must take its positive maximum at some point $x_* \in \partial_1 \cup \partial_2$. Since it vanishes on ∂_1, there exists some $x_* \in \partial_2$ such that $v_{\xi,\rho}(x_*) > w(x_*)$. By construction, $v_{\xi,\rho} = v$ on ∂_2. Therefore $v(x_*) > w(x_*)$. Since $w \ge v_{x_o,R}$, we conclude that $v(x_*) > v_{x_o,R}(x_*)$. This contradicts (4.8) and proves the proposition. ■

Remark 4.4 Analogous facts hold for super-harmonic functions.

5 Estimating Harmonic Functions and Their Derivatives

We will prove that if u is harmonic and is non-negative in E, then in any compact subset $K \subset E$, its maximum and minimum value are comparable. We also establish sharp estimates for the derivatives of u in the interior of E.

5.1 The Harnack Inequality and the Liouville Theorem

Theorem 5.1 (Harnack ([118])). *Let u be a non-negative harmonic function in E. Then for all $x \in B_\rho(x_o) \subset B_R(x_o) \subset E$*

$$\left(\frac{R}{R+\rho}\right)^{N-2} \frac{R-\rho}{R+\rho} u(x_o) \le u(x) \le \left(\frac{R}{R-\rho}\right)^{N-2} \frac{R+\rho}{R-\rho} u(x_o). \tag{5.1}$$

Proof. Modulo a translation, we may assume that $x_o = 0$. By the Poisson formula (3.9) and the mean value property (4.7), for all $x \in B_R$

$$\begin{aligned} u(x) &= \frac{R^2 - |x|^2}{R\omega_N} \int_{\partial B_R} \frac{u(y)}{|x-y|^N} d\sigma \le \frac{R^2 - |x|^2}{R\omega_N} \int_{\partial B_R} \frac{u(y)}{(|y|-|x|)^N} d\sigma \\ &= \frac{R^2 - |x|^2}{(R-|x|)^N} R^{N-2} \fint_{\partial B_R} u \, d\sigma = \left(\frac{R}{R-|x|}\right)^{N-2} \frac{R+|x|}{R-|x|} u(0). \end{aligned}$$

This proves the estimate above in (5.1). For the estimate below, observe that

$$u(x) = \frac{R^2 - |x|^2}{R\omega_N} \int_{\partial B_R} \frac{u(y)}{|x-y|^N} d\sigma \ge \frac{R^2 - |x|^2}{R\omega_N} \int_{\partial B_R} \frac{u(y)}{(|y|+|x|)^N} d\sigma$$

and conclude as above. ■

Corollary 5.1 (Harnack Inequality ([118])) *For every compact and connected subset $K \subset E$, there exists a constant C depending only on N and* $\operatorname{dist}(K; \partial E)$, *such that*

$$C \min_K u \ge \max_K u. \tag{5.2}$$

Proof. Let $x_1, x_2 \in K$ be such that $\min_K u = u(x_1)$ and $\max_K u = u(x_2)$. Fix a path Γ in K connecting x_1 and x_2, and cover Γ with finitely many spheres for each of which (5.1) holds. ■

Corollary 5.2 (Liouville Theorem) *A non-negative harmonic function in $\mathbb{R}^N$ is constant.*

Proof. In (5.1) fix $x_o \in \mathbb{R}^N$ and $\rho > 0$. Letting $R \to \infty$ gives $u(x) = u(x_o)$ for all $x \in B_\rho(x_o)$. Since x_o and $\rho > 0$ are arbitrary, $u = \text{const}$ in $\mathbb{R}^N$. ■

Corollary 5.3 *Let u be harmonic in $\mathbb{R}^N$ and such that $u \ge k$ for some constant k. Then u is constant.*

Proof. The function $u - k$ is harmonic and non-negative in $\mathbb{R}^N$. ■

Remark 5.1 The proof of Theorem 5.1 shows that in (5.1), for $x \in B_R(x_o)$ fixed, the number ρ can be taken to be $|x|$. This permits us to estimate from below the normal derivative of any harmonic function u in $B_R(x_o)$ at points $x_* \in \partial B_R(x_o)$ where u attains its minimum.

Proposition 5.1 *Let $u \in C^2(B_R(x_o)) \cap C(\bar{B}_R(x_o))$ be harmonic in $B_R(x_o)$, let $x_* \in \partial B_R(x_o)$ be a minimum point of u in $\bar{B}_R(x_o)$, and set*

$$u(x_*) = \min_{\bar{B}_R(x_o)} u \qquad \textit{and} \qquad \mathbf{n} = \frac{x_* - x_o}{|x_* - x_o|}.$$

Then

$$-\frac{\partial u}{\partial \mathbf{n}}(x_*) \ge 2^{1-N} \frac{u(x_o) - u(x_*)}{R}. \tag{5.3}$$

Proof. The function $u - u(x_*)$ is harmonic and non-negative in $B_R(x_o)$. Apply (5.1) to such a function, with $\rho = |x|$, to get

$$\frac{u(x) - u(x_*)}{R - |x|} \ge 2^{1-N} \frac{u(x_o) - u(x_*)}{R}.$$

Letting now $x \to x_*$ along $\mathbf{n}$ proves (5.3). ■

5.2 Analyticity of Harmonic Functions

If u is harmonic in E, by Proposition 2.1, $D^\alpha u$ is also harmonic in E, for every multi-index α. Therefore $D^\alpha u$ satisfies the mean value property (4.5) for all multi-indices α. In particular, for all $i = 1, \dots, N$ and all $B_R(x_o) \subset E$

$$\frac{\partial u}{\partial x_i}(x_o) = \fint_{B_R(x_o)} u_{x_i}(y) dy = \frac{N}{\omega_N R^N} \int_{\partial B_R(x_o)} u \frac{(y - x_o)_i}{|y - x_o|} d\sigma.$$

From this

$$\Big| \frac{\partial u}{\partial x_i}(x_o) \Big| \le \frac{N}{R} \sup_{B_R(x_o)} |u|. \tag{5.4}$$

This estimate is a particular case of the following

Theorem 5.2. *Let u be harmonic in E. Then for all $B_R(x_o) \subset E$, and for all multi-indices α*

$$|D^\alpha u(x_o)| \le \left(\frac{Ne}{R} \right)^{|\alpha|} \frac{|\alpha|!}{e} \sup_{B_R(x_o)} |u|. \tag{5.5}$$

Proof. By (5.4) the estimate holds for multi-indices of size 1. It will be shown by induction that if (5.5) holds for multi-indices of size $|\alpha|$, it continues to hold for multi-indices β of size $|\beta| = |\alpha| + 1$. For any such β

$$D^\beta u = \frac{\partial}{\partial x_i} D^\alpha u \quad \text{for some } 1 \le i \le N.$$

Fix $\tau \in (0,1)$ and apply (4.5) to $D^\beta u$ in the ball $B_{\tau R}(x_o)$. This gives

$$\begin{aligned} D^\beta u(x_o) &= \fint_{B_{\tau R}(x_o)} \frac{\partial}{\partial x_i} D^\alpha u \, dy \\ &= \frac{N}{\omega_N \tau^N R^N} \int_{\partial B_{\tau R}(x_o)} D^\alpha u(y) \frac{(y-x_o)_i}{|y-x_o|} \, d\sigma. \end{aligned}$$

By (5.5) applied over balls centered at $y \in \partial B_{\tau R}(x_o)$ and radius $(1-\tau)R$

$$|D^\alpha u(y)| \le \left(\frac{Ne}{(1-\tau)R} \right)^{|\alpha|} \frac{|\alpha|!}{e} \sup_{B_R(x_o)} |u| \quad \text{for all } \ y \in \partial B_{\tau R}(x_o).$$

Therefore

$$|D^\beta u(x_o)| \le \left(\frac{Ne}{R} \right)^{|\alpha|+1} \frac{1}{(1-\tau)^{|\alpha|}\tau} \frac{|\alpha|!}{e^2} \sup_{B_R(x_o)} |u|.$$

To prove the theorem, choose

$$\tau = \frac{1}{|\alpha|+1} = \frac{1}{|\beta|} \qquad \text{so that} \qquad (1-\tau)^{-|\alpha|} \le \left(1 + \frac{1}{|\alpha|}\right)^{|\alpha|} \le e. \qquad \blacksquare$$

Corollary 5.4 *Let u be harmonic in E. Then u is locally analytic in E.*

Proof. Let k be a positive number to be chosen, and having fixed $x_o \in E$, let R be so small that $B_{(k+1)R}(x_o) \subset E$. The Taylor expansion of u in $B_R(x_o)$ about x_o is

$$u(x) = \sum_{|\alpha| \le n} \frac{D^\alpha u(x_o)}{\alpha!} (x-x_o)^\alpha + \sum_{|\beta|=n+1} \frac{D^\beta u(\xi)}{\beta!} (x-x_o)^\beta$$

for some $\xi \in B_R(x_o)$. Estimate the terms of the remainder by applying (5.5) to the ball centered at ξ and radius kR. This gives

$$\begin{aligned} \frac{|D^\beta u(\xi)|}{\beta!} |(x-x_o)^\beta| &\le \left(\frac{Ne}{kR} \right)^{|\beta|} \frac{|\beta|!}{\beta!} \frac{R^{|\beta|}}{e} \sup_{B_{(k+1)R}(x_o)} |u| \\ &\le \left(\frac{Ne^{N+1}}{k} \right)^{|\beta|} \sup_{B_{(k+1)R}(x_o)} |u| \end{aligned}$$

where we have also used the inequality $|\beta|! \le e^{N|\beta|}\beta!$. Set

$$\frac{Ne^{N+1}}{k} = \theta \qquad \text{and} \qquad \sup_{B_{(k+1)R}(x_o)} |u| = M.$$

Choose k so that $\theta < 1$, and majorize the remainder of the Taylor series by

$$\left| \sum_{|\beta|=n+1} \frac{D^\beta u(\xi)}{\beta!} (x-x_o)^\beta \right| \le M \sum_{|\beta|=n+1} \theta^{|\beta|} \le M \, |\beta|^N \theta^{|\beta|}.$$

Since this tends to zero as $|\beta| \to \infty$, the Taylor series of u about x_o converges to u uniformly in $B_R(x_o)$. $\blacksquare$

6 The Dirichlet Problem

We will establish that the boundary value problem (1.2) has a unique solution for any given $\varphi \in C(\partial E)$. In the statement of the Dirichlet problem (1.2), the boundary ∂E was assumed to be of class C^1. In particular ∂E satisfies the *exterior sphere condition*, i.e.,

$$\begin{gathered}\text{for all } x_* \in \partial E \text{ there exists an exterior ball} \\ B_R(x_o) \subset \mathbb{R}^N - \bar{E} \text{ such that } \partial B_R(x_o) \cap \partial E = x_*.\end{gathered} \tag{6.1}$$

The ball $B_R(x_o)$ is exterior to E, and its boundary $\partial B_R(x_o)$ touches ∂E only at x_*. Such a property is shared by domains whose boundary could be irregular. For example, it is satisfied if ∂E exhibits corners or even spikes pointing outside E.

Theorem 6.1. *Let E be a bounded domain in $\mathbb{R}^N$ whose boundary ∂E satisfies the exterior sphere condition (6.1). Then for every $\varphi \in C(\partial E)$ there exists a unique solution to the Dirichlet problem*

$$u \in C^2(E) \cap C(\bar{E}), \quad \Delta u = 0 \ \text{ in } \ E, \quad \text{ and } \quad u\big|_{\partial E} = \varphi. \tag{6.2}$$

Proof (Perron ([201])). Recall the definition (4.6) of the classes $\sigma(E)$ and $\Sigma(E)$ and for a fixed $\varphi \in C(\partial E)$, consider the two classes

$$\begin{aligned}\sigma(\varphi; E) &= \big\{ v \in \sigma(E) \cap C(\bar{E}) \ \text{ and } \ v \,\big|_{\partial E} \le \varphi \big\} \\ \Sigma(\varphi; E) &= \big\{ v \in \Sigma(E) \cap C(\bar{E}) \ \text{ and } \ v \,\big|_{\partial E} \ge \varphi \big\}.\end{aligned}$$

Any constant $k \le \min_{\partial E} \varphi$ is in $\sigma(\varphi; E)$, and any constant $h \ge \max_{\partial E} \varphi$ is in $\Sigma(\varphi; E)$. Therefore $\sigma(\varphi; E)$ and $\Sigma(\varphi; E)$ are not empty. If a solution u to (6.2) exists, it must satisfy

$$v \le u \le w \quad \text{ for all } \ v \in \sigma(\varphi; E) \ \text{ and } \ \text{ for all } \ w \in \Sigma(\varphi; E).$$

This suggests to look for u as the unique element of separation of the two classes $\sigma(\varphi; E)$ and $\Sigma(\varphi; E)$, i.e.,

$$\sup_{v \in \sigma(\varphi; E)} v(x) \overset{\text{def}}{=} u(x) \overset{\text{def}}{=} \inf_{w \in \Sigma(\varphi; E)} w(x), \qquad \forall x \in \bar{E}. \tag{6.3}$$

To prove the theorem we have to prove the following two facts.

Lemma 6.1 *The function u defined by (6.3) is harmonic in E.*

Lemma 6.2 *$u \in C(\bar{E})$ and $u|_{\partial E} = \varphi$.*

Proof (of Lemma 6.1). Fix $x_o \in E$ and select a sequence $\{v_n\} \subset \sigma(\varphi; E)$ such that $v_n(x_o) \to u(x_o)$. The functions

$$V_n = \max\{v_1, v_2, \dots, v_n\} \tag{6.4}$$

belong to $\sigma(\varphi; E)$, and the sequence $\{V_n\}$ satisfies

$$V_n \le V_{n+1} \quad \text{and} \quad \lim_{n\to\infty} V_n(x_o) = u(x_o).$$

Let $B_\rho(\xi) \subset E$ be a ball containing x_o, and construct the functions $V_{n;\xi,\rho}$ as described in (4.7). By Proposition 4.2, $V_{n;\xi,\rho} \in \sigma(\varphi; E)$, and by the previous remarks

$$V_{n;\xi,\rho} \le V_{n+1;\xi,\rho} \quad \text{and} \quad V_{n;\xi,\rho}(x_o) \to u(x_o).$$

Thus $\{V_{n;\xi,\rho}\}$ converges monotonically to some function $z(\cdot)$, which we claim is harmonic in $B_\rho(\xi)$. Indeed, $V_{n;\xi,\rho} - V_{1;\xi,\rho}$ are all harmonic and non-negative in $B_\rho(\xi)$, and the sequence $\{V_{n;\xi,\rho}(x_o) - V_{1;\xi,\rho}(x_o)\}$ is equi-bounded. Therefore by the Harnack inequality (5.1), $\{V_{n;\xi,\rho} - V_{1;\xi,\rho}\}$ is equi-bounded on compact subsets of $B_\rho(\xi)$. By Theorem 5.2, also all the derivatives $D^\alpha(V_{n;\xi,\rho} - V_{1;\xi,\rho})$ are equi-bounded on compact subsets of $B_\rho(\xi)$. Therefore, by possibly passing to a subsequence, $\{D^\alpha(V_{n;\xi,\rho} - V_{1;\xi,\rho})\}$ converge uniformly on compact subsets of $B_\rho(\xi)$, for all multi-indices α. Thus $z(\cdot)$ is infinitely differentiable in $B_\rho(\xi)$ and $\{D^\alpha V_{n;\xi,\rho}\} \to D^\alpha z$ uniformly on compact subsets of $B_\rho(\xi)$, for all multi-indices α. Since all $V_{n;\xi,\rho}$ are harmonic in $B_\rho(\xi)$, also z is harmonic in $B_\rho(\xi)$.

By construction, $z(x_o) = u(x_o)$. To prove that $z(x) = u(x)$ for all $x \in B_\rho(\xi)$, fix $\tilde{x} \in B_\rho(\xi)$ and construct sequences $\{\tilde{v}_n\}$ and $\{\tilde{V}_n\}$ as follows:

$$\tilde{v}_n \in \sigma(\varphi; E) \quad \text{and} \quad \tilde{v}_n(\tilde{x}) \to u(\tilde{x})$$

$$\tilde{V}_n(x) = \max\{V_n(x); \tilde{v}_1(x), \tilde{v}_2(x), \dots, \tilde{v}_n(x)\} \quad \forall x \in E$$

where V_n are defined in (6.4). Starting from $\tilde{V}_n$, construct the corresponding functions $\tilde{V}_{n;\xi,\rho}$ as indicated in (4.7). Arguing as before, these satisfy

$$\tilde{V}_n \le \tilde{V}_{n;\xi,\rho}, \qquad \tilde{V}_{n;\xi,\rho} \le \tilde{V}_{n+1;\xi,\rho}, \qquad \tilde{V}_{n;\xi,\rho}(\tilde{x}) \to u(\tilde{x}).$$

Moreover, $\{\tilde{V}_{n;\xi,\rho}\}$ converges monotonically in $B_\rho(\xi)$ to a harmonic function $\tilde{z}(\cdot)$ satisfying

$$\tilde{z}(x) \ge z(x) \quad \text{for all } x \in B_\rho(\xi) \quad \text{and} \quad \tilde{z}(\tilde{x}) = u(\tilde{x}).$$

By the construction (6.3) of u

$$u(x_o) = z(x_o) \le \tilde{z}(x_o) = u(x_o).$$

Thus the function $\tilde{z} - z$ is non-negative and harmonic in $B_\rho(\xi)$, and it vanishes in an interior point x_o of $B_\rho(\xi)$. This is impossible unless $\tilde{z}(x) = z(x)$ for all $x \in B_\rho(\xi)$. In particular

$$\tilde{z}(\tilde{x}) = z(\tilde{x}) = u(\tilde{x}).$$

Since $\tilde{x} \in B_\rho(\xi)$ is arbitrary, we conclude that u is harmonic in a neighborhood of x_o and hence in the whole of E, since x_o is an arbitrary point of E. ■

Proof (of Lemma 6.2). Fix $x_* \in \partial E$ and let $B_R(x_o)$ be the ball exterior to E and touching ∂E only at x_* claimed by (6.1). The function

$$H(x) = \begin{cases} \dfrac{1}{R^{N-2}} - \dfrac{1}{|x-x_o|^{N-2}} & \text{if } N \ge 3 \\[2ex] \ln \dfrac{|x-x_o|}{R} & \text{if } N = 2 \end{cases} \tag{6.5}$$

is harmonic in a neighborhood of E and positive on ∂E except at x_*, where it vanishes. Fix an arbitrarily small positive number ε and determine $\delta = \delta(\varepsilon) \in (0,1)$ so that

$$|\varphi(x) - \varphi(x_*)| \le \varepsilon \qquad \forall [|x - x_*| \le \delta] \cap \partial E.$$

We claim that for all $\varepsilon > 0$ there exists a constant C_ε, depending only on $\|\varphi\|_{\infty,\partial E}$, R, N, and $\delta(\varepsilon)$, such that

$$|\varphi(x) - \varphi(x_*)| < \varepsilon + C_\varepsilon H(x) \qquad \forall x \in \partial E. \tag{6.6}$$

This is obvious if $|x - x_*| \le \delta$. If $x \in \partial E$ and $|x - x_*| > \delta$

$$|\varphi(x) - \varphi(x_*)| \le 2\|\varphi\|_{\infty,\partial E} \frac{H(x)}{H_\delta}, \quad \text{where} \quad H_\delta = \min_{[|x-x_*| \ge \delta] \cap \partial E} H(x).$$

To prove (6.6), we have only to observe that $H_\delta > 0$. It follows from (6.6) that for all $x \in \partial E$

$$\varphi(x_*) - \varepsilon - C_\varepsilon H(x) \le \varphi(x) \le \varphi(x_*) + \varepsilon + C_\varepsilon H(x).$$

This implies that

$$\varphi(x_*) - \varepsilon - C_\varepsilon H \in \sigma(\varphi; E)$$

and

$$\varphi(x_*) + \varepsilon + C_\varepsilon H \in \Sigma(\varphi; E).$$

Therefore for all $x \in \bar{E}$

$$\varphi(x_*) - \varepsilon - C_\varepsilon H(x) \le u(x) \le \varphi(x_*) + \varepsilon + C_\varepsilon H(x).$$

This in turn implies

$$|u(x) - \varphi(x_*)| \le \varepsilon + C_\varepsilon H(x) \qquad \forall x \in \bar{E}.$$

We now let $x \to x_*$ for $\varepsilon \in (0,1)$ fixed. Since $H \in C(\bar{E})$ and $H(x_*) = 0$

$$\limsup_{x \to x_*} |u(x) - \varphi(x_*)| \le \varepsilon \qquad \forall \varepsilon \in (0,1). \qquad \blacksquare$$

7 About the Exterior Sphere Condition

The existence theorem is based on an *interior* statement (Lemma 6.1) and a *boundary* statement concerning the behavior of u near ∂E (Lemma 6.2). The first can be established regardless of the structure of ∂E. The second relies on the construction of the function $H(\cdot)$ in (6.5). Such a construction is made possible by the exterior sphere condition (6.1). Indeed, this is the only role played by (6.1). Keeping this in mind, we might impose on ∂E the

Barrier Postulate: $\forall x_* \in \partial E,\ \exists H(x_*;\cdot) \in C(\bar{E})$ satisfying

$$\begin{array}{l} H(x_*;\cdot) \text{ is super-harmonic in a neighborhood of } E \\ H(x_*;x) > 0 \quad \forall x \in \bar{E} - \{x_*\}, \text{ and } H(x_*;x_*) = 0. \end{array} \tag{7.1}$$

Any such function $H(x_*;\cdot)$ is a *barrier* for the Dirichlet problem (6.2) at x_*.

Assume that ∂E satisfies the barrier postulate. Arguing as in the proof of Lemma 6.2, having fixed $x_* \in \partial E$, for all $\varepsilon > 0$ there exists a constant

$$C_\varepsilon = C_\varepsilon(\|\varphi\|_{\infty,\partial E}, N, H(x_*;\cdot), \varepsilon)$$

such that

$$|\varphi(x) - \varphi(x_*)| \le \varepsilon + C_\varepsilon H(x_*;x) \qquad \forall x \in \partial E.$$

Therefore, for all $x \in \partial E$

$$\varphi(x_*) - \varepsilon - C_\varepsilon H(x_*;x) \le \varphi(x) \le \varphi(x_*) + \varepsilon + C_\varepsilon H(x_*;x).$$

Since $H(x_*;\cdot)$ is super-harmonic

$$\varphi(x_*) - \varepsilon - C_\varepsilon H(x_*;x) \in \sigma(\varphi;E)$$

and

$$\varphi(x_*) + \varepsilon + C_\varepsilon H(x_*;x) \in \Sigma(\varphi;E).$$

Therefore for all $x \in \bar{E}$

$$\varphi(x_*) - \varepsilon - C_\varepsilon H(x_*;x) \le u(x) \le \varphi(x_*) + \varepsilon + C_\varepsilon H(x_*;x)$$

and

$$|u(x) - \varphi(x_*)| < \varepsilon + C_\varepsilon H(x_*;x) \qquad \forall x \in \bar{E}.$$

This proves Lemma 6.2 if the exterior sphere condition (6.1) is replaced by the barrier postulate (7.1). We conclude that the Dirichlet problem (6.2) is uniquely solvable for every domain E satisfying the barrier postulate.

7.1 The Case $N = 2$ and ∂E Piecewise Smooth

Let E be a bounded domain in $\mathbb{R}^2$ whose boundary ∂E is the finite union of portions of curves of class C^1. Domains of this kind permit corners and even spikes pointing outside or inside E. Fix $x_* \in \partial E$ and assume, modulo a translation, that x_* coincides with the origin. We may also assume, up to a homothetic transformation, that E is contained in the unit disc about the origin. Identifying $\mathbb{R}^2$ with the complex plane $\mathbb{C}$, points $z = \rho e^{i\theta}$ of E, are determined by a unique value of the argument $\theta \in (-\pi, \pi)$. Therefore $\ln z$ is uniquely defined in E. A barrier at the origin is

$$H(x) = -\mathrm{Re}\left(\frac{1}{\ln z}\right) = -\frac{\ln \rho}{\ln^2 \rho + \theta^2}.$$

7.2 A Counterexample of Lebesgue for $N = 3$ ([163])

If $N \ge 3$, spikes pointing outside E are permitted, since any such point would satisfy the exterior sphere condition (6.1). Spikes pointing inside E are, in general, not permitted as shown by the following example of Lebesgue.

Denote points in $\mathbb{R}^3$ by (x, z), where $x = (x_1, x_2)$ and $z \in \mathbb{R}$. The function

$$v(x, z) = \int_0^1 \frac{s\,ds}{\sqrt{|x|^2 + (s - z)^2}} \tag{7.2}$$

is harmonic outside $[|x| = 0] \cap [0 \le z \le 1]$. By integration by parts one computes

$$\begin{aligned} v(x, z) =& \sqrt{|x|^2 + (1 - z)^2} - \sqrt{|x|^2 + z^2} \\ &+ z \ln \left| \left[(1 - z) + \sqrt{|x|^2 + (1 - z)^2}\right] \left[z + \sqrt{|x|^2 + z^2}\right] \right| \\ &- 2z \ln |x|. \end{aligned}$$

As $(x, z) \to 0$, the sum of the first three terms on the right-hand tends to 1, whereas the last term is discontinuous at zero. It tends to zero if $(x, z) \to 0$ along the curve $|z|^\beta = |x|$ for all $\beta > 0$. However, if $(x, z) \to 0$ along $|x| = e^{-\gamma/2z}$, for $z > 0$ and $\gamma > 0$, it converges to γ. We conclude that

$$\lim_{\substack{(x,z)\to 0 \\ \text{along } |x| = e^{-\gamma/2z}}} v(x, z) = 1 + \gamma.$$

Therefore, all the level surfaces $[v = 1 + \gamma]$ for all $\gamma > 0$ go through the origin, and as a consequence, v is not continuous at the origin.

Fix $c > 0$ and consider the domain

$$E = [v < 1 + c] \cap [|x, z| < 1].$$

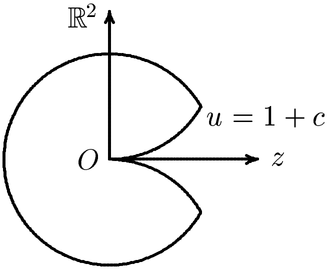

Fig. 7.1

There exists no solution to the Dirichlet problem

$$u \in C^2(E) \cap C(\bar{E}), \quad \Delta u = 0 \;\text{ in }\; E, \quad \text{and} \quad u\big|_{\partial E} = v\big|_{\partial E}. \tag{7.3}$$

Notice that even though v is not continuous in $\bar{E}$, the restriction $v|_{\partial E}$ is continuous on ∂E. The idea of the counterexample is based on showing that any solution of (7.3) must coincide with v, which itself is not a solution.

Fix any $\varepsilon > 0$, and consider the domain $E_\varepsilon = E \cap [|x, z| > \varepsilon]$. Assume that u is a solution of (7.3) and let C be a constant such that $|u - v| < C$ in E. The functions

$$w_\varepsilon = C \frac{\varepsilon}{|x, z|} \pm (u - v)$$

are harmonic in E_ε and non-negative on ∂E_ε. Thus by the maximum principle

$$|u(x, z) - v(x, z)| \le C \frac{\varepsilon}{|x, z|} \qquad \text{in} \quad E_\varepsilon.$$

8 The Poisson Integral for the Half Space

Denote points in $\mathbb{R}^{N+1}$ by (x, t), where $x \in \mathbb{R}^N$ and $t \in \mathbb{R}$. Consider the Dirichlet problem

$$\begin{cases} u \in C^2(\mathbb{R}^N \times \mathbb{R}^+) \cap C(\overline{\mathbb{R}^N \times \mathbb{R}^+}) \\ \Delta u = 0 \quad \text{in } \; \mathbb{R}^N \times \mathbb{R}^+ \\ u(x, 0) = \varphi(x) \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N). \end{cases} \tag{8.1}$$

A solution to (8.1) is called the harmonic extension of φ in the upper half-space $\mathbb{R}^N \times \mathbb{R}^+$. Consider the fundamental solution (2.6) of the Laplacean in $\mathbb{R}^{N+1}$ with pole at $(y, 0)$

$$F(x, t; y) = \begin{cases} \dfrac{1}{(N-1)\omega_{N+1}} \dfrac{1}{\left[|x - y|^2 + t^2\right]^{\frac{N-1}{2}}} & \text{if } \; N \ge 2 \\[2ex] -\dfrac{1}{2\pi} \ln \left[|x - y|^2 + t^2\right]^{1/2} & \text{if } \; N = 1. \end{cases}$$

The Poisson kernel for the half-space is defined for all $N \geq 1$ by

$$K(x;y) = -2\frac{\partial F(x,t;y)}{\partial t} = \frac{2t}{\omega_{N+1}\left[|x-y|^2+t^2\right]^{\frac{N+1}{2}}}. \tag{8.2}$$

Theorem 8.1. *Every $\varphi \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ has a unique bounded harmonic extension H_φ in $\mathbb{R}^N \times \mathbb{R}^+$, given by*

$$H_\varphi(x,t) = \frac{2t}{\omega_{N+1}}\int_{\mathbb{R}^N} \frac{\varphi(y)}{\left[|x-y|^2+t^2\right]^{\frac{N+1}{2}}}dy. \tag{8.3}$$

Proof (Uniqueness). If u and v are both bounded solutions of (8.1), the difference $w = u - v$ is harmonic in $\mathbb{R}^N \times \mathbb{R}^+$ and vanishes for $t = 0$. By reflection about the hyperplane $t = 0$, the function

$$\tilde{w}(x,t) = \begin{cases} w(x,t) & \text{if } t > 0 \\ -w(x,-t) & \text{if } t \leq 0 \end{cases}$$

is bounded and harmonic in $\mathbb{R}^{N+1}$. Therefore, by Liouville's theorem (Corollary 5.3), it is constant. Since $w(\cdot,0) = 0$, it vanishes identically. ■

Remark 8.1 The statement of uniqueness in Theorem 8.1 holds only within the class of bounded solutions. Indeed, the two functions $u = 0$ and $v = t$ are both harmonic extensions of $\varphi = 0$.

Proof (Existence). The function H_φ defined in (8.3) is harmonic in $\mathbb{R}^N \times \mathbb{R}^+$. The boundedness of H_φ follows from the boundedness of φ and the following lemma.

Lemma 8.1 *For all $\varepsilon > 0$ and all $x \in \mathbb{R}^N$*

$$\frac{2\varepsilon}{\omega_{N+1}}\int_{\mathbb{R}^N} \frac{dy}{\left[|x-y|^2+\varepsilon^2\right]^{\frac{N+1}{2}}} = 1. \tag{8.4}$$

Proof. Assume $N \geq 2$. The change of variables $y - x = \varepsilon\xi$ transforms the integral in (8.4) into

$$\frac{2}{\omega_{N+1}}\int_{\mathbb{R}^N} \frac{d\xi}{(1+|\xi|^2)^{\frac{N+1}{2}}} = 2\frac{\omega_N}{\omega_{N+1}}\int_0^\infty \frac{\rho^{N-1}}{(1+\rho^2)^{\frac{N+1}{2}}}d\rho = 1. \tag{8.5}$$

The case $N = 1$ is treated analogously (**8.1** of the Complements). ■

To conclude the proof of Theorem 8.1, it remains to show that for all $x_* \in \mathbb{R}^N$

$$\lim_{(x,t)\to x_*} H_\varphi(x,t) = \varphi(x_*).$$

This is established as in Theorem 3.1 by making use of (8.4). ■

9 Schauder Estimates of Newtonian Potentials

Let E be a bounded open set in $\mathbb{R}^N$ and continue to denote by $F(\cdot;\cdot)$ the fundamental solution of the Laplacean, introduced in (2.6). The Newtonian potential generated in $\mathbb{R}^N$ by a density distribution $f \in L^p(E)$ for some $p > 1$ is defined by

$$A_F f = \int_E F(\cdot; y) f(y) dy \tag{9.1}$$

provided the right-hand side is finite. If $f \in L^\infty(E)$

$$\|A_F f\|_{\infty,\mathbb{R}^N} + \|\nabla A_F f\|_{\infty,\mathbb{R}^N} \le \gamma \|f\|_{\infty,E} \tag{9.2}$$

where γ is a constant depending only on N and $\operatorname{diam}(E)$. Further regularity of $A_F f$ can be established if f is Hölder continuous and compactly supported in E. For $m \in \mathbb{N} \cup \{0\}$, $\eta \in (0,1)$, and $\varphi \in C^\infty(E)$, set

$$|\!|\!|\varphi|\!|\!|_{m,\eta;E} \overset{\text{def}}{=} \sum_{|\alpha| \le m} \|D^\alpha \varphi\|_{\infty,E} + \sum_{|\alpha|=m} \sup_{x,y \in E} \frac{|D^\alpha \varphi(x) - D^\alpha \varphi(y)|}{|x-y|^\eta}. \tag{9.3}$$

Denote by $C^{m,\eta}(E)$ the space of functions $\varphi \in C^m(E)$ with finite norm $|\!|\!|\varphi|\!|\!|_{m,\eta;E}$ and by $C_o^{m,\eta}(E)$ the space of functions $\varphi \in C^{m,\eta}(E)$ compactly supported in E. If $m = 0$, we let $C^{0,\eta}(E) = C^\eta(E)$ and $|\!|\!|\varphi|\!|\!|_{0,\eta;E} = |\!|\!|\varphi|\!|\!|_{\eta;E}$.

Proposition 9.1 *Let $f \in C_o^{m,\eta}(E)$. Then $A_F f \in C^{m+2,\eta}(E)$, and there exists a constant γ depending upon N, m, η, and* diam(E)*, such that*

$$|\!|\!|A_F f|\!|\!|_{m+2,\eta;E} \le \gamma |\!|\!|f|\!|\!|_{m,\eta;E}. \tag{9.4}$$

Proof. It suffices to prove (9.4) for $m = 0$ and for $f \in C_o^\infty(E)$. Assume $N \ge 3$, the proof for $N = 2$ being analogous, and rewrite (9.1) as

$$\omega_N (N-2) A_F f = v(\cdot) = \int_{\mathbb{R}^N} |\xi|^{2-N} f(\cdot + \xi) d\xi$$

and compute

$$\begin{aligned} v_{x_i x_j}(x) &= \int_{\mathbb{R}^N} |\xi|^{2-N} f_{\xi_i \xi_j}(x+\xi) d\xi = -\int_{\mathbb{R}^N} (|\xi|^{2-N})_{\xi_j} f_{\xi_i}(x+\xi) d\xi \\ &= -\int_{|\xi|>r} (|\xi|^{2-N})_{\xi_j} f_{\xi_i}(x+\xi) d\xi \\ &\quad - \int_{|\xi|<r} (|\xi|^{2-N})_{\xi_j} [f(x+\xi) - f(x)]_{\xi_i} d\xi \\ &= \int_{|\xi|>r} (|\xi|^{2-N})_{\xi_j \xi_j} f(x+\xi) d\xi \\ &\quad + \int_{|\xi|<r} (|\xi|^{2-N})_{\xi_j \xi_j} (f(x+\xi) - f(x)) d\xi \\ &\quad + f(x) \int_{|\xi|=r} (|\xi|^{2-N})_{\xi_j} \frac{\xi_i}{|\xi|} d\sigma. \end{aligned} \tag{9.5}$$

In this representation, r is any positive number, $d\sigma$ is the surface measure over the sphere ∂B_r, and the integral extended over the ball $|\xi| < r$ is meant in the sense of the limit

$$\begin{aligned}\int_{|\xi|<r} (|\xi|^{2-N})_{\xi_j\xi_j}(f(x+\xi)-f(x))d\xi \\ \overset{\text{def}}{=} \lim_{\varepsilon\to 0}\int_{\varepsilon<|\xi|<r} (|\xi|^{2-N})_{\xi_j\xi_j}(f(x+\xi)-f(x))d\xi.\end{aligned}$$

Such a limit exists, since f is Hölder continuous. From (9.5), by taking $r = \operatorname{diam}(E)$, we estimate

$$\begin{aligned}\sum_{|\alpha|=2}\|D^\alpha v\|_{\infty,\mathbb{R}^N} &\le \gamma\|f\|_{\infty,E}\left(1+\int_{r<|\xi|<2r}|\xi|^{-N}d\xi\right) \\ &\quad + \gamma\int_{|\xi|<r}|\xi|^{-N+\eta}\frac{|f(x+\xi)-f(x)|}{|\xi|^\eta}d\xi \\ &\le \gamma\left(1+\operatorname{diam}(E)\right)|\!|\!|f|\!|\!|_{\eta;E}.\end{aligned} \tag{9.6}$$

Next we fix $y \in \mathbb{R}^N$ and represent $v_{x_ix_j}(y)$. By calculations analogous to those leading to (9.5)

$$\begin{aligned}v_{x_ix_j}(y) &= \int_{\mathbb{R}^N}|\xi|^{2-N}f_{\xi_i\xi_j}(y+\xi)d\xi \\ &= \int_{\mathbb{R}^N}|(x-y)+\xi|^{2-N}f_{\xi_i\xi_j}(x+\xi)d\xi \\ &= \int_{|\xi|>r}\left(|(x-y)+\xi|^{2-N}\right)_{\xi_i\xi_j}f(x+\xi)d\xi \\ &\quad + \int_{|\xi|<r}\left(|(x-y)+\xi|^{2-N}\right)_{\xi_i\xi_j}[f(x+\xi)-f(y)]d\xi \\ &\quad + f(y)\int_{|\xi|=r}\left(|(x-y)+\xi|^{2-N}\right)_{\xi_j}\frac{\xi_i}{|\xi|}d\sigma.\end{aligned} \tag{9.7}$$

From the representations (9.5) and (9.7), we obtain by difference

$$
\begin{aligned}
&v_{x_ix_j}(x) - v_{x_ix_j}(y)\\
&\qquad = \int_{|\xi|>r} \left(|\xi|^{2-N} - |(x-y)+\xi|^{2-N}\right)_{\xi_i\xi_j} [f(x+\xi) - f(y)]d\xi\\
&\qquad\quad + \int_{|\xi|<r} (|\xi|^{2-N})_{\xi_j\xi_j} [f(x+\xi) - f(x)]d\xi\\
&\qquad\quad - \int_{|\xi|<r} \left(|(x-y)+\xi|^{2-N}\right)_{\xi_i\xi_j} [f(x+\xi) - f(y)]d\xi\\
&\qquad\quad + [f(x) - f(y)] \int_{|\xi|=r} (|\xi|^{2-N})_{\xi_j} \frac{\xi_i}{|\xi|} d\sigma\\
&\qquad\quad + f(y) \int_{|\xi|=r} \left(|\xi|^{2-N} - |(x-y)+\xi|^{2-N}\right)_{\xi_j} \frac{\xi_i}{|\xi|} d\sigma\\
&\qquad\quad + f(y) \int_{|\xi|>r} \left(|\xi|^{2-N} - |(x-y)+\xi|^{2-N}\right)_{\xi_i\xi_j} d\xi.
\end{aligned}
$$

The sum of the last two integrals is zero, and the integral extended over the shell $|\xi| = r$ is majorized by

$$\gamma \|f\|_{\eta;E} |x-y|^\eta$$

where γ is a constant depending only upon the dimension N. In the estimates below we denote by γ a constant that can be different in different contexts, and it can be computed quantitatively a priori in terms of N alone. Estimating the first integral extended over the ball $|\xi| < r$ we have

$$
\begin{aligned}
\left| \int_{|\xi|<r} (|\xi|^{2-N})_{\xi_j\xi_j} [f(x+\xi) - f(x)]d\xi \right| &\le \gamma(N) \|f\|_{\eta;E} \int_{|\xi|<r} |\xi|^{-N+\eta} d\xi\\
&\le \gamma \|f\|_{\eta;E} r^\eta.
\end{aligned}
$$

Analogously

$$
\begin{aligned}
&\left| \int_{|\xi|<r} \left(|(x-y)+\xi|^{2-N}\right)_{\xi_i\xi_j} [f(x+\xi) - f(y)]d\xi \right|\\
&\qquad \le \gamma \|f\|_{\eta;E} \int_{|\xi|<r} |(x-y)+\xi|^{-N+\eta} d\xi\\
&\qquad \le \gamma \|f\|_{\eta;E} \int_{|z|<r+|x-y|} |z|^{-N+\eta} dz\\
&\qquad \le \gamma \|f\|_{\eta;E} (r^\eta + |x-y|^\eta).
\end{aligned}
$$

Combining these estimates, we conclude that there is a constant γ depending only upon N, such that for every $r > 0$

$$
\begin{aligned}
&|v_{x_ix_j}(x) - v_{x_ix_j}(y)| \le \gamma \|f\|_{\eta;E} (r^\eta + |x-y|^\eta)\\
&\quad + \gamma \left| \int_{|\xi|>r} \left(|\xi|^{2-N} - |(x-y)+\xi|^{2-N}\right)_{\xi_i\xi_j} [f(x+\xi) - f(y)]d\xi \right|.
\end{aligned}
\tag{9.8}
$$

Choose $r = 2|x-y|$ so that over the set $|\xi| > r$

$$2|\xi| \ge |(x-y)+\xi| \ge \frac{1}{2}|\xi|.$$

Then by direct calculation and the mean value theorem

$$\begin{aligned}\Big|\big(|\xi|^{2-N} - |(x-y)+\xi|^{2-N}\big)_{\xi_i\xi_j}\Big| &\le \left|\frac{\delta_{ij}}{|\xi|^N} - \frac{\delta_{ij}}{|(x-y)+\xi|^N}\right| \\ &\quad + \left|\frac{\xi_i\xi_j}{|\xi|^{N+2}} - \frac{[(x-y)+\xi]_i[(x-y)+\xi]_j}{|(x-y)+\xi|^{N+2}}\right| \\ &\le \gamma\frac{|x-y|}{|\xi|^{N+1}}.\end{aligned}$$

Therefore the last integral in (9.8) can be majorized by

$$\begin{aligned}&\gamma|x-y|\int_{|\xi|>r}\frac{|\xi|^\eta}{|\xi|^{N+1}}\frac{|f(x+\xi)-f(y)|}{|(x-y)+\xi|^\eta}d\xi \\ &\qquad \le \gamma|\!|\!| f |\!|\!|_{\eta;E}\, r\int_{|\xi|>r}|\xi|^{-(N+1)+\eta}d\xi = \gamma|\!|\!| f |\!|\!|_{\eta;E}\, r^\eta.\end{aligned}$$

Combining this with (9.2) and (9.6) proves the proposition. ■

Remark 9.1 The proof shows that

$$\sup_{x,y\in\mathbb{R}^N}\frac{|v_{x_ix_j}(x)-v_{x_ix_j}(y)|}{|x-y|^\eta} \le \gamma\sup_{x,y\in E}\frac{|f(x)-f(y)|}{|x-y|^\eta}$$

where the constant γ depends only on N and η and is independent of $|E|$.

Remark 9.2 The dependence on $\operatorname{diam}(E)$ on the right-hand side of (9.3) enters only through (9.2) and (9.6). Therefore the constant γ in (9.4) depends on $\operatorname{diam}(E)$ as

$$\gamma = \gamma_o(1+\operatorname{diam}(E)) \qquad \text{for some } \gamma_o = \gamma_o(N, m, \eta).$$

10 Potential Estimates in $L^p(E)$

The estimate (9.2) implies that A, as defined by (9.1), is a map from $L^\infty(E)$ into $L^\infty(E)$. More precisely, it maps $L^\infty(E)$ into the subspace of the Lipschitz continuous functions defined in E. It is natural to ask whether $f \in L^p(E)$ for some $p \ge 1$ would imply that $A_F f \in L^q(E)$ for some $q \ge 1$, and what is the relation between p and q.

If $f \in C_o^\eta(E)$, then $A_F f$, as defined by (9.1), is differentiable and

$$\nabla A_F f(x) = \frac{1}{\omega_N}\int_E \frac{(x-y)}{|x-y|^N}f(y)dy.$$

If however $f \in L^p(E)$ for some $p \ge 1$, the symbol $\nabla A_F f$ does not have the classical meaning of derivative, and it is simply defined by its right-hand side. If this is finite a.e. in E, we say that $\nabla A_F f$ is the weak gradient of the potential $A_F f$. We will give sufficient integrability conditions on f to ensure that the weak gradient $\nabla A_F f$ is in $L^q(E)$ for some $q \ge 1$.

Both issues are addressed by investigating the integrability of the *Riesz potential*

$$E \ni x \to w_\alpha(x) = \int_E \frac{f(y)}{|x-y|^{N-\alpha}} dy \qquad \text{for some } \alpha > 0. \tag{10.1}$$

Proposition 10.1 *Let $f \in L^p(E)$ for some $p \ge 1$. Then*

$$|w_\alpha| \in L^q(E) \quad \textit{where} \quad q \in \begin{cases} \left[1, \dfrac{Np}{N-\alpha p}\right) & \textit{if } p < \dfrac{N}{\alpha} \\ [1, \infty) & \textit{if } p = \dfrac{N}{\alpha} \\ [1, \infty] & \textit{if } p > \dfrac{N}{\alpha}. \end{cases} \tag{10.2}$$

Moreover, there exists a constant γ that can be determined a priori only in terms of N, p, q, α, and diam(E)*, such that*

$$\|w_\alpha\|_{q,E} \le \gamma \|f\|_{p,E}. \tag{10.3}$$

The constant $\gamma \to \infty$ as either $p \to N/\alpha$ or as diam$(E) \to \infty$.

Proof. Assume first $p < N/\alpha$, and choose $s > 1$ from

$$\frac{1}{s} = 1 + \frac{1}{q} - \frac{1}{p}, \qquad 1 < s < \frac{N}{N-\alpha}.$$

By Hölder's inequality

$$\begin{aligned} |w_\alpha(x)| &= \int_E \left(|x-y|^{(\alpha-N)s}|f|^p\right)^{\frac{1}{q}} |f|^{1-\frac{p}{q}} |x-y|^{(\alpha-N)(1-\frac{s}{q})} dy \\ &\le \left(\int_E |x-y|^{(\alpha-N)s}|f|^p dy\right)^{\frac{1}{q}} \left(\int_E |x-y|^{(\alpha-N)\frac{q-s}{q-1}} |f|^{\frac{q-p}{q-1}} dy\right)^{\frac{q-1}{q}} \\ &\le \left(\int_E |x-y|^{(\alpha-N)s}|f|^p dy\right)^{\frac{1}{q}} \|f\|_{p,E}^{p(1-\frac{1}{s})} \left(\int_E |x-y|^{(\alpha-N)s} dy\right)^{\frac{1}{s}-\frac{1}{q}}. \end{aligned}$$

The last integral involving $|x-y|^{(\alpha-N)s}$ is estimated above by extending the domain of integration to the ball of center x and radius diam(E). It gives

$$\begin{aligned}\int_E |x-y|^{(\alpha-N)s}dy &\le \omega_N \int_0^{\operatorname{diam}(E)} \rho^{(N-1)-(N-\alpha)s}d\rho \\ &= \frac{\omega_N \operatorname{diam}(E)^{N(1-s\frac{N-\alpha}{N})}}{N\left(1-s\frac{N-\alpha}{N}\right)} = \gamma(N,s)\end{aligned}$$

provided

$$1 < s < \frac{N}{N-\alpha} \qquad \text{i.e.,} \qquad \frac{1}{p}-\frac{1}{q} < \frac{\alpha}{N}.$$

This determines the range of q in (10.1). In the estimates below, denote by γ a generic positive constant that can be determined a priori only in terms of N, p, q, and $\operatorname{diam}(E)$. To proceed, carry this estimate into the right-hand side of the estimation of $|w_\alpha(x)|$, take the qth power of both sides and integrate over E. Interchanging the order of integration with the aid of Fubini's theorem

$$\begin{aligned}\int_E |w_\alpha|^q dx &\le \gamma\bigg(\int_E\int_E |x-y|^{(\alpha-N)s}|f(y)|^p dydx\bigg)\|f\|_{p,E}^{pq(1-\frac{1}{s})} \\ &\le \gamma\bigg[\int_E |f(y)|^p\bigg(\int_E |x-y|^{(\alpha-N)s}dx\bigg)dy\bigg]\|f\|_{p,E}^{pq(1-\frac{1}{s})} \\ &\le \gamma\|f\|_{p,E}^q.\end{aligned}$$

Let now $p > N/\alpha$. Then by Hölder's inequality

$$|w_\alpha(x)| \le \|f\|_{p,E}\bigg(\int_E |x-y|^{(\alpha-N)\frac{p}{p-1}}dy\bigg)^{\frac{p-1}{p}}. \qquad \blacksquare$$

Corollary 10.1 *Let $f \in L^p(E)$ for some $p > N$. There exists a constant γ depending only upon N, p and diam(E), such that*

$$\|v\|_{\infty,E} + \|\nabla v\|_{\infty,E} \le \gamma\|f\|_{p,E}.$$

The constant $\gamma \to \infty$ as $p \to N$.

Let E a bounded open set in $\mathbb{R}^N$ with boundary ∂E of class C^1. For $f \in L^p(\partial E)$ set

$$v_\alpha(x) = \int_{\partial E} \frac{f(y)d\sigma(y)}{|x-y|^{N-1-\alpha}} \qquad \text{for } \alpha > 0 \tag{10.4}$$

where $d\sigma$ is the Lebesgue surface measure on ∂E.

Corollary 10.2 *Let $f \in L^p(\partial E)$ for some $p \ge 1$. Then*

$$|v_\alpha| \in L^q(E), \qquad \textit{where} \qquad q \in \begin{cases} \left[1, \dfrac{(N-1)p}{(N-1)-\alpha p}\right) & \textit{if } p < \dfrac{N-1}{\alpha} \\ [1,\infty) & \textit{if } p = \dfrac{N-1}{\alpha} \\ [1,\infty] & \textit{if } p > \dfrac{N-1}{\alpha}. \end{cases} \tag{10.5}$$

Moreover, there exists a constant γ *that can be determined a priori in terms of* N, p, q, α, *and* $|\partial E|$ *only, such that*

$$\|v_\alpha\|_{q,\partial E} \le \gamma \|f\|_{p,\partial E}. \tag{10.6}$$

The constant $\gamma \to \infty$ *as either* $p \to (N-1)/\alpha$ *or as* $|\partial E| \to \infty$.

11 Local Solutions

Consider formally local solutions of the Poisson equation

$$\Delta u = f, \quad \text{ in } E \tag{11.1}$$

with no reference to possible boundary data on ∂E. Thus we assume that $u \in C^2(E)$ and $f \in C^{m+\eta}_{\text{loc}}(E)$ for some non-negative integer m and some $\eta \in [0,1)$. This means that $|\!|\!| f |\!|\!|_{m,\eta;K} < \infty$ for every compact set $K \subset E$. Let $K \subset K' \subset E$ be such that $\operatorname{dist}(K;\partial K') > 0$. We will derive estimates of the norm $C^{m+\eta}(K)$ for u in terms of the norms $\|u\|_{\infty,K'}$ and $|\!|\!| f |\!|\!|_{m+\eta;K'}$.

Proposition 11.1 *There exists a constant* γ *depending only upon* N, m, *and* $\operatorname{dist}(K;\partial K')$ *such that*

$$|\!|\!| u |\!|\!|_{m+2,\eta;K} \le \gamma \left(|\!|\!| f |\!|\!|_{m,\eta;K'} + \|u\|_{\infty,K'} \right).$$

Proof. It suffices to prove the proposition for $m = 0$, and for $K \subset K'$ two concentric balls $B_{\sigma\rho}(x_o) \subset B_\rho(x_o) \subset E$, for some $\sigma \in (0,1)$. In such a case the statement takes the following form.

Lemma 11.1 *There exists a constant* γ *depending only upon* N *such that for all* $B_{\sigma\rho}(x_o) \subset B_\rho(x_o) \subset E$

$$\begin{aligned} |\!|\!| u |\!|\!|_{2+\eta;B_{\sigma\rho}(x_o)} \le{}& \gamma\left(1 + \frac{1}{(1-\sigma)\rho^\eta}\right) |\!|\!| f |\!|\!|_{\eta;B_\rho(x_o)} \\ &+ \gamma\left(1 + \frac{1}{(1-\sigma)^{N+2}\rho^{2+\eta}}\right) \|u\|_{\infty,B_\rho(x_o)}. \end{aligned}$$

Proof (of Lemma 11.1). The point $x_o \in E$ being fixed, we may assume after a translation that it coincides with the origin and write $B_\rho(0) = B_\rho$. Construct a smooth non-negative cutoff function $\zeta \in C_o^\infty(E)$ such that

$$\zeta = 1 \text{ in } B_{\frac{(1+\sigma)}{2}\rho} \quad \text{ and } \quad |D^\alpha \zeta| \le \frac{C^{|\alpha|}}{[(1-\sigma)\rho]^{|\alpha|}} \tag{11.2}$$

for all multi-indices α of size $|\alpha| \le 2$ and for some constant C. Multiplying (11.1) by ζ and setting $v = u\zeta$, we find that v satisfies

$$\Delta v = f\zeta + u\Delta\zeta + 2\nabla u \cdot \nabla \zeta, \qquad v \in C_o^2(B_\rho).$$

Let $N \geq 3$, the case $N = 2$ being similar. By the Stokes identity (2.3), we may represent v as the superposition of the two Newtonian potentials

$$\begin{aligned} V_1(x) &= -\frac{1}{\omega_N(N-2)} \int_{\mathbb{R}^N} |x-y|^{2-N}(f\zeta)(y)dy \\ V_2(x) &= -\frac{1}{\omega_N(N-2)} \int_{\mathbb{R}^N} |x-y|^{2-N}(u\Delta\zeta + 2\nabla u \cdot \nabla\zeta)(y)dy. \end{aligned}$$

By virtue of Proposition 9.1 and Remark 9.2

$$|\!|\!| V_1 |\!|\!|_{2,\eta;\mathbb{R}^N} \leq \gamma(1+\rho)|\!|\!| f\zeta |\!|\!|_{\eta;B_\rho} \leq \gamma(1+\rho+\rho^{-\eta})|\!|\!| f |\!|\!|_{\eta;B_\rho}.$$

We estimate $V_2(\cdot)$ within the ball $B_{\sigma\rho}$ by rewriting it as

$$\begin{aligned} V_2(x) =& \frac{1}{\omega_N(N-2)} \int_{\mathbb{R}^N} u(y)|x-y|^{2-N}\Delta\zeta\, dy \\ &+ \frac{2}{\omega_N(N-2)} \int_{\mathbb{R}^N} u(y)\nabla|x-y|^{2-N} \cdot \nabla\zeta\, dy. \end{aligned}$$

Since $\nabla\zeta = 0$ within the ball of radius $\frac{(1+\sigma)}{2}\rho$, these integrals are not singular for $x \in B_{\sigma\rho}$ and we estimate

$$|\!|\!| V_2 |\!|\!|_{2,\eta;B_{\sigma\rho}} \leq \gamma\left(1 + \frac{1}{(1-\sigma)^{N+2}\rho^{2+\eta}}\right)\|u\|_{\infty,B_\rho}. \qquad \blacksquare$$

11.1 Local Weak Solutions

The previous remarks imply that if $f \in C^\eta_{\text{loc}}(E)$, the classical local solution of the Poisson equation (11.1) can be implicitly represented about any point $x_o \in E$ as

$$\begin{aligned} u(x) =& -\int_{B_\rho(x_o)} F(x;y)f\zeta dy \\ &+ \int_{B_\rho(x_o)} u(y)\left(F(x;y)\Delta\zeta + 2\nabla F(x;y)\cdot\nabla\zeta\right)dy \end{aligned} \tag{11.3}$$

where $F(\cdot;\cdot)$ is the fundamental solution of the Laplace equation, introduced in (2.6), and ζ satisfies (11.2). Consider now the various integrals in (11.3), regardless of their derivation. The second is well defined for all $x \in B_{\sigma\rho}(x_o)$ if $u \in L^1_{\text{loc}}(E)$. The first defines a function

$$x \to \int_{B_\rho(x_o)} F(x;y)f\zeta\, dy \in L^1_{\text{loc}}(E)$$

provided $f \in L^p_{\text{loc}}(E)$ for some $p \geq 1$. Since $B_\rho(x_o) \subset E$ is arbitrary, this suggests the following

Definition: Let $f \in L^p_{\text{loc}}(E)$ for some $p \geq 1$. A function $u \in L^1_{\text{loc}}(E)$ is a *weak solution* to the Poisson equation (11.1) in E if it satisfies (11.3).

The estimates of Section 10 imply:

Proposition 11.2 *Let $f \in L^p_{\rm loc}(E)$ for $p > 1$ and let $u \in L^1_{\rm loc}(E)$ be a local weak solution of (11.1) in E. There exists a constant γ depending only on N and p such that for all $B_{\sigma\rho}(x_o) \subset B_\rho(x_o) \subset E$*

$$\|\nabla u\|_{q,B_{\sigma\rho}(x_o)} \le \gamma \|f\|_{p,B_\rho(x_o)} + \gamma[(1-\sigma)\rho]^{-(N+1)}\rho^{\frac{N}{q}}\|u\|_{1,B_\rho(x_o)}$$

where

$$q \in \begin{cases} \left[1, \dfrac{N}{N-p}\right) & \textit{if} \quad 1 \le p < N \\[2ex] [1,\infty) & \textit{if} \qquad p = N \\[2ex] [1,\infty] & \textit{if} \qquad p > N. \end{cases}$$

If $p \in [1,N]$, the constant $\gamma \to \infty$ as $q \to Np/(N-p)$. Moreover, if $p > N/2$

$$\|u\|_{\infty,B_{\sigma\rho}(x_o)} \le \gamma\rho^{2-\frac{N}{p}}\|f\|_{p,B_\rho(x_o)} + \gamma[(1-\sigma)\rho]^{-N}\|u\|_{1,B_\rho(x_o)}.$$

The constant $\gamma \to \infty$ as $p \to N/2$.

12 Inhomogeneous Problems

12.1 On the Notion of Green's Function

Let E be a bounded domain in $\mathbb{R}^N$ with boundary ∂E of class C^1. The construction of the Green's function for E, introduced in (3.3), hinges on solving the family of Dirichlet problems (3.1). These solutions $y \to \Phi(x;y)$ were required to be of class $C^2(\bar{E})$. Such a regularity has been used to justify intermediate calculations, and it appears naturally in the explicit construction of Green's function for a ball.

However for each fixed $x \in E$, the Dirichlet problem (3.1) has a unique solution if one merely requires that ∂E satisfies the barrier postulate and that

$$\Phi(x;\cdot) \in C^2(E) \cap C(\bar{E}) \qquad \text{for all fixed } x \in E.$$

This is the content of Theorem 6.1 and the remarks of Section 7.

Proposition 12.1 *Every bounded open set $E \subset \mathbb{R}^N$ with boundary ∂E satisfying the barrier postulate admits a Green's function $G(\cdot;\cdot)$. Moreover, for all $(x;y) \in E \times E$,*

$$\begin{aligned} &0 \le G(x;y) \le F(x;y) && \textit{for } N \ge 3 \\ &0 \le G(x;y) \le \frac{1}{2\pi}\ln\frac{\operatorname{diam}(E)}{|x-y|} && \textit{for } N = 2. \end{aligned} \tag{12.1}$$

Proof. For fixed $x \in E$, let $\varepsilon > 0$ be so small that $B_\varepsilon(x) \subset E$. The function $G(x;\cdot)$ is harmonic in $E - B_\varepsilon(x)$, and it vanishes on ∂E. The number ε can be chosen to be so small that $G(x;\cdot) > 0$ on $\partial B_\varepsilon(x)$. Therefore, by the maximum principle, $G(x;\cdot) \ge 0$ in $E - B_\varepsilon(x)$ and hence in E since ε is arbitrary. The function $\Phi(x;\cdot)$ is harmonic in E, and by the maximum principle, it takes its maximum and minimum values on ∂E. If $N \ge 3$ it is positive on ∂E and thus $\Phi(\cdot;\cdot) > 0$ in E. This proves the first of (12.1). If $N = 2$ rewrite the Green's function as

$$G(x;y) = \frac{1}{2\pi} \ln \frac{\operatorname{diam}(E)}{|x-y|} - \left[\Phi(x;y) + \frac{1}{2\pi} \ln \operatorname{diam}(E) \right].$$

For fixed $x \in E$, the function of y in $[\cdots]$ is harmonic in E and non-negative on ∂E. ■

Remark 12.1 The estimate roughly asserts that the singularity of $G(\cdot;\cdot)$ is of the same nature as the singularity of the fundamental solution $F(\cdot;\cdot)$.

Corollary 12.1 *$G(x;\cdot) \in L^p(E)$ uniformly in x, for all $p \in [1, \frac{N}{N-2})$.*

12.2 Inhomogeneous Problems

Given $f \in C^\eta(\bar{E})$ for some $\eta \in (0,1)$, consider the boundary value problem

$$u \in C^2(E) \cap C(\bar{E}), \quad \Delta u = f \text{ in } E, \quad \text{and} \quad u\big|_{\partial E} = 0. \tag{12.2}$$

Theorem 12.1. *The boundary value problem (12.2) has a unique solution.*

Proof (Uniqueness). If u_i for $i = 1, 2$ solve (12.2), their difference is harmonic in E and vanishes on ∂E. Thus it vanishes identically in E, by the maximum principle. ■

Proof (Existence). Assume momentarily that (12.2) has a solution $u \in C^2(\bar{E})$ and that the Green's function $G(x;\cdot)$ for E is of class $C^2(\bar{E})$. Then subtracting Green's identity (2.2) written for the pair of functions u and $\Phi(x;\cdot)$ from the Stokes identity (2.3)–(2.3) gives

$$\begin{aligned} u(x) &= -\int_E G(x;y) f(y) dy \\ &= -\int_E F(x;y) f(y) dy + \int_E \Phi(x;y) f(y) dy. \end{aligned} \tag{12.3}$$

This is a candidate for a solution of (12.2). To show that it is indeed a solution, we have to show that it takes zero boundary data in the sense of continuous functions in $\bar{E}$, it is of class $C^2(E)$, and it satisfies the PDE. Fix $x_* \in \partial E$ and write

$$\begin{aligned}\lim_{x\to x_*} u(x) &= -\lim_{x\to x_*}\int_E G(x;y)f(y)dy\\ &= -\lim_{x\to x_*}\int_{E\cap B_\varepsilon(x_*)} G(x;y)f(y)dy\\ &\quad -\lim_{x\to x_*}\int_{E-B_\varepsilon(x_*)} G(x;y)f(y)dy.\end{aligned}$$

The second integral tends to zero by the property of the Green's function, since y is away from the singularity $x = x_*$. The first integral is estimated by means of (12.1), and it yields

$$\int_{E\cap B_\varepsilon(x_*)} G(x;y)|f(y)|dy \le 2\sup_E|f|\int_{E\cap B_\varepsilon(x_*)} |F(x;y)|dy \le O(\varepsilon)$$

uniformly in x. ■

To establish in what sense the PDE is satisfied, we assume $N \ge 3$, the arguments for $N = 2$ being similar.

12.3 The Case $f \in C_o^\infty(E)$

The set $K = \mathrm{supp}(f)$ is a compact proper subset of E, and $\Phi(x;\cdot)$ is in $C^\infty(E)$. Therefore by symmetry

$$\Delta\int_E \Phi(x;y)f(y)dy = \int_E \Delta_y\Phi(x;y)f(y)dy = 0.$$

Calculating the Laplacean of the first term on the right-hand side of (12.3) gives

$$\begin{aligned}\Delta\int_E |x-y|^{2-N}f(y)dy &= \int_{\mathbb{R}^N} |\xi|^{2-N}\Delta_\xi f(x-\xi)d\xi\\ &= \lim_{\varepsilon\to 0}\int_{|\xi|>\varepsilon} |\xi|^{2-N}\Delta_\xi f(x\xi)d\xi.\end{aligned}$$

Perform a double integration by parts on the last integral using that f is compactly supported in $\mathbb{R}^N$. Taking into account that $|\xi|^{2-N}$ is harmonic in $|\xi| > \varepsilon$, and proceeding as in the proof of the Stokes identity (2.3) gives

$$\frac{1}{\omega_N(N-2)}\Delta\int_E |x-y|^{2-N}f(y)dy = f(x).$$

Combining these calculations shows that u defined by (12.3) satisfies the Poisson equation (12.2) in the classical sense.

12.4 The Case $f \in C^{\eta}(\bar{E})$

Let $\{K_j\}$ be a family of nested compact subsets of E exhausting E. Construct a sequence of functions $\{f_j\} \subset C_o^{\infty}(E)$ satisfying

$$|\!|\!| f_j |\!|\!|_{\eta;K_j} \le |\!|\!| f |\!|\!|_{\eta;E} \quad \text{and} \quad \lim_{j\to\infty} |\!|\!| f_j - f |\!|\!|_{\eta;K} = 0$$

for every compact subset $K \subset E$. Let u_j be the unique classical solution of

$$u_j \in C^2(E) \cap C(\bar{E}), \quad \Delta u_j = f_j \text{ in } E, \quad \text{and} \quad u_j\big|_{\partial E} = 0. \tag{12.4}$$

From the representation formula (12.3)

$$\|u_j\|_{\infty,E} \le \gamma \|f_j\|_{\infty;E} \le \gamma |\!|\!| f |\!|\!|_{\eta;E}, \qquad \forall j \in \mathbb{N}.$$

Combining this with Proposition 11.1 gives

$$|\!|\!| u_j |\!|\!|_{2,\eta;K_1} \le \gamma |\!|\!| f_j |\!|\!|_{\eta;K_1} \le \gamma |\!|\!| f |\!|\!|_{\eta;E} \qquad \forall j \in \mathbb{N}$$

where γ depends on N and $\text{dist}\{K_1; \partial E\}$ and is independent of j. By the Ascoli–Arzelà theorem, we may select a subsequence $\{u_{j_1}\}$ out of $\{u_j\}$ converging in $C^{2,\eta}(K_1)$ to a function $u_1 \in C^{2,\eta}(K_1)$. By the same process, we may select a subsequence $\{u_{j_2}\}$ out of $\{u_{j_1}\}$ converging in $C^{2,\eta}(K_2)$ to a function $u_2 \in C^{2,\eta}(K_2)$, which coincides with u_1 within K_1. Continuing this diagonalization process, we obtain a function $u \in C^{2,\eta}_{\text{loc}}(E)$ and a subsequence $\{u_{j'}\}$ out of the original sequence $\{u_j\}$ such that $\{u_{j'}\} \to u$ in $C^{2,\eta}_{\text{loc}}(E)$. Letting $j \to \infty$ in (12.4) along such a subsequence proves that $u \in C^{2,\eta}(E)$ is a classical solution of the PDE. To verify that $u \in C(\bar{E})$ and that it vanishes on ∂E in the sense of continuous functions, we have only to observe that u satisfies (12.2) by the same limiting process. ■

Problems and Complements

1c Preliminaries

1.1c Newtonian Potentials on Ellipsoids

Compute the Newtonian potential generated by a uniform distribution of masses, or charges, on the surface of an ellipsoid. Verify that such a potential is constant inside the ellipsoid ([136], pages 22 and 193).

Theorem 1.1c. *Let E be a bounded domain in $\mathbb{R}^N$ of boundary ∂E of class C^2. The Newtonian potential $V(\cdot)$, generated by a uniform distribution of masses on ∂E, is constant in E if and only if E is an ellipsoid.*

The sufficient part of the theorem is due to Newton. The necessary part in dimension $N = 2$ was established in 1931 by Dive [60]. The necessary part for all $N \geq 2$ has been recently established in [52]. The assumption of uniform distribution cannot be removed as shown in [237].

1.2c Invariance Properties

1.4. Prove that the Laplacean is invariant under a unitary affine transformation of coordinates in $\mathbb{R}^N$.

1.5. Find all second-order rotation invariant operators of the type

$$L(u) = \sum_{i,j,h,k=1}^{N} a_{ijhk} u_{x_i x_j} u_{x_h x_k}.$$

1.6. Prove that Δ is the only second-order, linear operator invariant under orthogonal linear transformation of the coordinates axes.

1.7. Find all homogeneous harmonic polynomials of degree n in two and three variables ([120]).
Hint: For $N = 2$ attempt z^n and $\bar{z}^n$, where $z = x_1 + ix_2$ and $\bar{z} = x_1 - ix_2$. For $N = 3$ attempt polynomials of the type $z^j P_{n-j}(|z|^2, x_3)$, where P_{n-j} is a polynomial of degree $n - j$ in the variables $|z|^2 = x_1^2 + x_2^2$ and x_3.

1.8. Let $N = 2$, and identify E with a portion of the complex plane $\mathbb{C}$. Then the real and imaginary part of a holomorphic function in E are harmonic in E ([31], pages 124-125).

2c The Green and Stokes Identities

2.1. Prove that if $u \in C^2(E) \cap C^1(\bar{E})$ is harmonic in E, then it is locally analytic in E. It will be a consequence of the estimates in Section 5 that the hypothesis $u \in C^1(\bar{E})$, can be removed.

2.2. It follows from the Stokes identity (2.3)–(2.3) that if $u \in C^2(E) \cap C^1(\bar{E})$ is harmonic in E, it can be represented as the sum of a single-layer, and a double-layer potential.

2.3. Let ω_N denote the surface area of the unit sphere in $\mathbb{R}^N$. Prove that for $N = 2, 3, \ldots,$

$$\omega_{N+1} = 2\omega_N \int_0^{\pi/2} (\sin t)^{N-1} dt.$$

3c Green's Function and the Dirichlet Problem for the Ball

3.1. Prove that the Green's function in (3.3) is non-negative.

3.2. Verify by direct calculation that the kernel in (3.9) is harmonic.
Hint: One needs to prove that for all $y \in \partial B_R$

$$\Delta_x \frac{R^2 - |x|^2}{|x-y|^N} = 0 \ \text{ in } \ B_R.$$

From (3.8)

$$\begin{aligned}\frac{2|x|}{2-N}\frac{\partial}{\partial|x|}|x-y|^{2-N} &= \frac{2|x|}{|x-y|^N}(|x| - |y|\cos\theta)\\ &= \frac{|x|^2 - |y|^2 + |x-y|^2}{|x-y|^N}\end{aligned}$$

and for $y \in \partial B_R$

$$\frac{R^2 - |x|^2}{|x-y|^N} = |x-y|^{2-N} + \frac{2|x|}{N-2}\frac{\partial}{\partial|x|}|x-y|^{2-N}.$$

Therefore it suffices to show that the second term on the right-hand side is harmonic.

3.3. Study the Dirichlet problem

$$\Delta u = 0 \ \text{ in } \ B_R, \qquad \text{and } u|_{\partial B_R} = \begin{cases} 0 & \text{if } x_N < 0 \\ 1 & \text{if } x_N \ge 0. \end{cases}$$

Examine the behavior of the solution for $x_N = 0$ near ∂B_R.

3.4. Find Green's function for the half-space $x_N > 0$.
Hint: Set $\bar{x} = (x_1, \dots, x_{N-1})$ and consider the reflection map analogous to (3.5), i.e., $\xi(x) = (\bar{x}, -x_N)$.

3.5. Using the results of **3.4**, discuss the solvability of the Dirichlet problem

$$\begin{aligned}&\Delta u = 0 \ \text{ in } \ \mathbb{R}^{N-1} \times [x_N > 0]\\ &u(\bar{x}, 0) = \varphi(\bar{x}) \in C(\mathbb{R}^{N-1}) \cap L^\infty(\mathbb{R}^{N-1}).\end{aligned}$$

3.6. Construct Green's function for the quadrant $[x > 0] \cap [y > 0]$ in $\mathbb{R}^2$.

3.7. Find Green's function for the half-ball in $\mathbb{R}^N$.

3.1c Separation of Variables

3.8. Solve the Dirichlet problem for the rectangle $[0 < x < a] \times [0 < y < b]$ in $\mathbb{R}^2$ by looking for "separated" solutions of the form $u(x, y) = X(x)Y(y)$. Enforcing the PDE, derive the ODEs

$$X'' = (\text{const})X, \qquad Y'' = -(\text{const})Y.$$

Superpose the families of solutions X_n, Y_n of these ODEs, by writing

$$u = \sum A_n X_n(x) Y_n(y), \qquad A_n \in \mathbb{R},\ n = 0, 1, 2, \dots,$$

Finally, determine the coefficients A_n from the prescribed boundary data. In the actual calculations, it is convenient to split the problem into the sum of Dirichlet problems for each of which the data are zero on three sides of the rectangle. For example, the problem

$$\begin{gathered} \Delta u = 0 \ \text{ in } \ R = [0 < x < 1] \times [0 < y < 1] \\ u(0, y) = \cos \frac{\pi}{2} y, \quad u(x, 0) = 1 - x, \quad u(1, y) = u(x, 1) = 0 \end{gathered}$$

can be solved by superposing the solutions of the two problems

$$\begin{gathered} \Delta u_1 = 0 \ \text{ in } \ R, \qquad \text{and} \qquad u_1(0, y) = \cos \frac{\pi}{2} y \\ u_1(x, 0) = u_1(1, y) = u_1(x, 1) = 0, \end{gathered}$$

and

$$\begin{gathered} \Delta u_2 = 0 \ \text{ in } \ R, \qquad \text{and} \qquad u_2(x, 0) = 1 - x \\ u_2(1, y) = u_2(x, 1) = u_2(0, y) = 0. \end{gathered}$$

Even though the boundary data are not continuous on ∂R, one might solve formally for u_i, $i = 1, 2$ and verify that $u = u_1 + u_2$ is indeed the unique solution of the given problem.

3.9. Solve the Dirichlet problem for the disc $x^2 + y^2 < 1$ by separation of variables, and show that this produces the same solution as that obtained by the Poisson formula (3.9).
Hint: Write Δu in terms of polar coordinates (ρ, θ) to arrive at

$$u_{\rho\rho} + \frac{1}{\rho} u_\rho + \frac{1}{\rho^2} u_{\theta\theta} = 0 \ \text{ in } \ [0 < \rho < 1] \times [0 \le \theta < 2\pi].$$

To this equation apply the method of separation of variables.

3.10. Use a modification of this technique to solve the Dirichlet problem for the annulus $r < |x| < R$ in $\mathbb{R}^2$.

3.11. Solve the Dirichlet problem for the rectangle with vertices $A = (1, 0)$, $B = (2, 1)$, $C = (1, 2)$, $D = (0, 1)$.

4c Sub-Harmonic Functions and the Mean Value Property

4.1. Prove that (4.2) implies (4.3). *Hint*: (4.2) implies

$$\omega_N r^{N-1} u(x_o) \le \int_{\partial B_r(x_o)} u(y)\, d\sigma \qquad \text{for all } \ B_r(x_o) \subset E.$$

Integrate both sides in dr for $r \in (0, R)$.

4.2. Prove that (4.4) and (4.5) are equivalent. *Hint*: Write (4.5) in the form

$$\frac{\omega_N}{N} r^N u(x_o) = \int_{B_r(x_o)} u\,dy$$

and take the derivative of both sides with respect to r.

4.3. Let $u \in C^2(E)$ satisfy $\Delta u = u$ in E. Prove that u has neither a positive maximum nor a negative minimum in E.

4.4. Let u be harmonic in E. Prove that $|\nabla u|^2$ is sub-harmonic in E.

4.1c Reflection and Harmonic Extension

4.5. Denote points in $\mathbb{R}^N$ by $x = (\bar{x}, x_N)$ where $\bar{x} = (x_1, \dots, x_{N-1})$, and let u be the unique solution of

$$\Delta u = 0 \ \text{ in } \ B_1, \qquad \text{and} \qquad u\big|_{\partial B_1} = \varphi \in C(\partial B_1).$$

Prove that $\varphi(\bar{x}, x_N) = -\varphi(\bar{x}, -x_N)$ implies $u(\bar{x}, x_N) = -u(\bar{x}, -x_N)$.

4.6. Let u be harmonic in $B_1^+ = B_1 \cap [x_N > 0]$, and vanishing for $x_N = 0$. Extend it with a harmonic function in the whole of B_1.

4.7. Solve explicitly the Dirichlet problem

$$\Delta u = 0 \ \text{ in } \ B_1^+, \qquad \text{and} \qquad u\big|_{\partial B_1^+} = \begin{cases} x_N^3 & \text{if } \ x_N > 0 \\ 0 & \text{if } \ x_N = 0. \end{cases}$$

4.2c The Weak Maximum Principle

Consider the formal differential operator

$$\mathcal{L}_o = a_{ij}(x) \frac{\partial^2}{\partial x_i x_j} + b_i(x) \frac{\partial}{\partial x_i} \tag{4.1c}$$

where a_{ij}, $b_i \in C(\bar{E})$ and the matrix (a_{ij}) is symmetric and positive definite in E, that is there exists $\lambda > 0$ such that $a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2$ for all $\xi \in \mathbb{R}^N$ and all $x \in E$.

Theorem 4.1c. *Let $u \in C^2(E) \cap C(\bar{E})$ satisfy $\mathcal{L}_o(u) \geq 0$ in E. Then*

$$u(x) \leq \max_{\partial E} u \qquad \textit{for all } \ x \in E.$$

Proof. Fix $y \in \mathbb{R}^N - \bar{E}$, let γ be a constant to be chosen later, and consider the function

$$v = u + \varepsilon e^{\gamma|x-y|^2} \qquad \text{for } \ \varepsilon > 0.$$

It satisfies

$$\begin{aligned}\mathcal{L}_o(v) &\ge 2\gamma\varepsilon\left[a_{ij}\delta_{ij} + 2\gamma a_{ij}(x-y)_i(x-y)_j\right]e^{\gamma|x-y|^2}\\ &\quad + 2\gamma\varepsilon\left[b_i(x-y)_i\right]e^{\gamma|x-y|^2}\\ &\ge 2\gamma\varepsilon\left[N\lambda + 2\gamma\lambda|x-y|^2 - B|x-y|\right]e^{\gamma|x-y|^2}\end{aligned}$$

where $B = \max_{1\le i\le n}\max_{\bar{E}}|b_i|$. By the Cauchy inequality

$$N\lambda + 2\gamma\lambda|x-y|^2 - B|x-y| \ge N\lambda + \left(2\gamma\lambda - \frac{B^2}{4N\lambda}\right)|x-y|^2 - N\lambda.$$

Therefore γ can be chosen a priori dependent only upon B, N, and λ, such that $\mathcal{L}_o(v) > 0$ in E. If x_o is an interior maximum for v, $b_i(x_o)v_{x_i}(x_o) = 0$ and

$$a_{ij}v_{x_ix_j}\Big|_{x=x_o} > 0 \quad \text{and} \quad v_{x_ix_j}\Big|_{x=x_o} \le 0. \tag{$*$}$$

Next observe that $a_{ij}v_{x_ix_j} = \operatorname{trace}\big((a_{ij})(v_{x_ix_j})\big)$. Using that (a_{ij}) and $(v_{x_ix_j})$ are symmetric, and that the trace is invariant under orthogonal linear transformations, prove that

$$a_{ij}(x_o)v_{x_ix_j}(x_o) \le 0.$$

This and $(*)$ give a contradiction. Therefore

$$\max_{\bar{E}} v \le \max_{\partial E} v$$

and the theorem follows on letting $\varepsilon \to 0$. ■

4.8. Let $u \in C^2(E)\cap C(\bar{E})$ satisfy

$$\mathcal{L}(u) \equiv \mathcal{L}_o(u) + c(x)u \ge 0 \quad \text{in } E.$$

Prove that if $c < 0$, then u cannot have a positive maximum in the interior of E. Give a counterexample to show that the assumption $c < 0$ cannot be removed.

4.3c Sub-Harmonic Functions

4.9. Prove that $v \in \sigma(E)$ if and only if for every open set $E' \subset E$ and every harmonic function u such that $u\big|_{\partial E'} = v$, then $v \le u$ in E'.

4.10. Prove that $x \to \ln|x|$ is sub-harmonic in $\mathbb{R}^N - \{0\}$.

4.11. Give examples of nondifferentiable sub-harmonic functions.

4.12. Let $u \in C^2(E)\cap C(\bar{E})$ be a solution of

$$\Delta u = -1 \ \text{ in } E, \qquad \text{and} \qquad u\big|_{\partial E} = 0.$$

Prove that for all $x_o \in E$

$$u(x_o) \ge \frac{1}{2N}\inf_{x\in\partial E}|x-x_o|^2.$$

4.3.1c A More General Notion of Sub-Harmonic Functions

Certain questions in potential theory require a notion of sub-harmonic functions that does not assume continuity. First, by a real valued function in E is meant $u : E \to [-\infty, +\infty)$. Thus u is defined everywhere in E and is permitted to take the "value" $-\infty$. A real-valued function $u : E \to [-\infty, +\infty)$ is *upper semi-continuous* if $[u < s]$ is open for all $s \in \mathbb{R}$.

Definition of F. Riesz ([214], see also [209]): Let E be a connected, open subset of $\mathbb{R}^N$. A real valued function $u : E \to [-\infty, +\infty)$, is sub-harmonic in E, if it is upper semi-continuous, and if for every compact subset $K \subset E$ and for every function $H \in C(K)$ and harmonic in the interior of K

$$u \big|_{\partial K} \le H \big|_{\partial K} \quad \Longrightarrow \quad u \le H \;\text{ in }\; K.$$

4.13. Prove that this notion of upper semi-continuity is equivalent to

$$\limsup_{x \to x_o} u(x) \le u(x_o) \qquad \text{for all }\; x_o \in E.$$

4.14. Prove that the function

$$u(x) = \begin{cases} \ln |x| & \text{if }\; x \neq 0 \\ -\infty & \text{if }\; x = 0 \end{cases}$$

is upper semi-continuous. Such a function is sub-harmonic in the sense of F. Riesz, and it is not sub-harmonic in the sense of Section 4.

4.15. Prove that apart from the continuity requirement, Riesz notion of a sub-harmonic function is equivalent to the notion of Section 4.

4.16. Let $\{u_n\}$ be a decreasing sequence of sub-harmonic functions, in the sense of F. Riesz and let

$$u(x) = \lim_{n \to \infty} u_n(x) \qquad \forall\, x \in E.$$

Prove that either $u \equiv -\infty$ in E, or u is sub-harmonic in E, in the sense of F. Riesz ([128], page 16).

5c Estimating Harmonic Functions

5.1. Let u be harmonic in E. Estimate the radius of convergence of its Taylor's series about $x_o \in E$.

5.2. (Unique Continuation) Let u be harmonic in E and vanishing in an open subset of E. Prove that if E is connected, u vanishes identically in E. As a consequence, if u and v are harmonic in a connected domain E and coincide in a open subset of E, then $u \equiv v$ in E.

5.3. Find two harmonic functions in the unit ball that coincide on the set $[|x| < \frac{1}{2}] \cap [x_N = 0]$.

5.4. (Phragmen–Lindelöf-Type Theorems) Let $\mathbb{R}^N_+ = \mathbb{R}^N \cap [x_N > 0]$, and let u be a non-negative harmonic function in $\mathbb{R}^N_+$. Prove that if u is bounded and vanishes on the hyperplane $x_N = 0$, then it is identically zero.

Remark 5.1c The function $u = x_N$ shows that the assumption of u being bounded cannot be removed. However, this is in some sense the only counterexample as shown by the following theorem of Serrin ([232]).

Theorem 5.1c. *Let u be a non-negative harmonic function in $\mathbb{R}^N_+$ vanishing for $x_N = 0$. There exists a constant C depending only on N, such that*

$$\limsup_{|x|\to\infty} \frac{u(x)}{|x|} \le C.$$

5.1c Harnack-Type Estimates

5.5. Let u be harmonic in $\mathbb{R}^N$, and let $\mathcal{G}$ be its graph. If $P \in \mathcal{G}$, denote by π_P be the tangent plane to $\mathcal{G}$ at P. Prove that $[\pi_P \cap \mathcal{G}] - P \neq \emptyset$.

5.6. Prove that a non-negative harmonic function in a connected open set E is either identically zero or strictly positive in E.

5.7. Let Q be the rectangle with vertices $(0,0)$, $(nr,0)$, $(nr,2r)$, $(0,2r)$, for some $r > 0$ and $n \in \mathbb{N}$. Let $P_o = (r,r)$ and $P_* = \big((n-1)r, r\big)$. Prove that a non-negative harmonic function u in Q satisfies

$$2^{-2n} u(P_o) \le u(P_*) \le 2^{2n} u(P_o).$$

5.8. Assume that the mixed boundary problem

$$\begin{gathered} u \in C^2(B_1) \cap C^1(\bar{B}_1), \quad \Delta u = -1 \text{ in } B_1 \\ u = 0 \text{ in } \partial B_1 \cap [x_N > 0], \quad \nabla u \cdot \mathbf{n} = -u \text{ on } \partial B_1 \cap [x_N < 0] \end{gathered}$$

has a unique solution. Prove that $u \ge 0$ in $\bar{B}_1$, and that $u \,\big|_{\partial B_1 \cap [x_N > 0]} > 0$.

5.2c Ill Posed Problems. An Example of Hadamard

The following problem is in general ill posed.

$$\begin{gathered} \Delta u = 0 \text{ in } E = [0 < x < 1] \times [0 < y < 1] \\ u(\cdot, 0) = \varphi, \quad u_y(\cdot, 0) = \psi \end{gathered} \tag{$*$}$$

Proposition 5.1c *A solution of $(*)$ exists if and only if the function*

$$(0,1) \ni x \to \varphi(x) - \frac{1}{\pi} \int_0^1 \psi(s) \ln|x - s| ds$$

is analytic.

Proof. If u solves $(*)$, write $u = v + w$, where

$$v(x,y) = \frac{1}{2\pi}\int_0^1 \psi(s)\ln[(x-s)^2 + y^2]ds.$$

Then $\varDelta w = 0$ in E and $w_y(\cdot,0) = 0$. Therefore, by the reflection principle $(x,y) \to w(x,|y|)$ is harmonic in $[0 < x < 1] \times [-1 < y < 1]$, and $x \to w(x,0)$ is analytic. ■

5.9. Prove that the following problem is ill posed.

$$\begin{aligned}&\varDelta u = 0 \ \text{ in } \ [|x| < 1] \times [0 < y < 1]\\ &u(-1,\cdot) = u(\cdot,1) = u(1,\cdot) = 0\\ &u_y(\cdot,0) = 0, \quad u(\cdot,0) = 1 - |x|.\end{aligned}$$

5.3c Removable Singularities

Let $x_o \in E$ and let u be harmonic in $E - \{x_o\}$. The function u is analytic in $E - \{x_o\}$, and it might be singular at x_o. An example is the fundamental solution $F(\cdot;x_o)$ of the Laplace equation with pole at x_o. A point $x_o \in E$ is a *removable singularity* for u if u can be extended continuously in x_o so that the resulting function is harmonic in the whole of E.

The pole x_o is not a removable singularity for $F(\cdot;x_o)$. This suggests that for a singularity at x_o to be removable, the behavior of u near x_o should be better than that of $F(\cdot;x_o)$.

Theorem 5.2c. *Assume that*

$$\lim_{x\to x_o}\frac{u(x)}{F(x;x_o)} = 0. \tag{$**$}$$

Then x_o is a removable singularity.

Proof. Let v be the harmonic extension in the ball $B_\rho(x_o)$ of $u\,\big|_{\partial B_\rho(x_o)}$. Such an extension can be constructed by the Poisson formula (3.9) and Theorem 3.1. The proof consists in showing that $u = v$ in $B_\rho(x_o)$. Assume $N \ge 3$, the proof for $N = 2$ being similar. Consider the ball $B_\varepsilon(x_o) \subset B_\rho(x_o)$ and set

$$M_\varepsilon = \|u - v\|_{\infty,\partial B_\varepsilon(x_o)}.$$

By $(**)$, for every fixed $\eta > 0$, there exists $\varepsilon_o \in (0,\rho)$ such that $M_\varepsilon \le \varepsilon^{2-N}\eta$ for all $\varepsilon \le \varepsilon_o$. The two functions

$$w^\pm = M_\varepsilon\left(\frac{\varepsilon}{|x - x_o|}\right)^{N-2} \pm (u - v)$$

are harmonic in the annulus $\varepsilon < |x - x_o| < \rho$ and non-negative for $|x - x_o| = \rho$. Moreover, on the sphere $|x - x_o| = \varepsilon$

$$w^{\pm}\Big|_{|x-x_o|=\varepsilon} = M_\varepsilon \pm (u-v)\Big|_{|x-x_o|=\varepsilon} \ge 0.$$

Therefore, by the maximum principle, for all $\varepsilon < |x-x_o| < \rho$

$$|u-v|(x) \le \frac{M_\varepsilon \varepsilon^{N-2}}{|x-x_o|^{N-2}} \le \frac{\eta}{|x-x_o|^{N-2}}.$$ ■

Theorem 5.3c. *Assume that*

$$\lim_{x\to x_o} |x-x_o|^{N-1}\nabla u \cdot \frac{x-x_o}{|x-x_o|} = 0.$$

Then x_o is a removable singularity.

Proof (Hint:). Let v be the harmonic extension of $u|_{\partial B_\rho(x_o)}$ into $B_\rho(x_o)$, and for $\varepsilon \in (0,\rho)$ set

$$D_\varepsilon = \sup_{\partial B_\rho(x_o)} \left| \nabla(u-v) \cdot \frac{x-x_o}{|x-x_o|} \right|.$$

Introduce the two functions

$$w^{\pm} = \frac{\varepsilon^{N-1} D_\varepsilon}{(N-2)|x-x_o|^{N-2}} \pm (u-v)$$

and prove that a minimum for $w^{\pm}$ cannot occur on $\partial B_\varepsilon(x_o)$. ■

5.10. Prove that if

$$\lim_{x\to x_o} \frac{u(x)}{F(x;x_o)} = c \qquad \text{for some } c \in \mathbb{R}$$

then $u = cF(\cdot;x_o) + v$, where v is harmonic in E.

5.11. The previous statements assume that u has a limit as $x \to x_o$. Prove that if u is a non-negative harmonic function in the punctured ball $B_1 - \{0\}$, then the limit of $u(x)$ as $|x| \to 0$ exists, finite or infinite ([100]).

7c About the Exterior Sphere Condition

The counterexample of Lebesgue leads to a question that can be roughly formulated as follows. How wide should the cusp in Figure 7.1 be, to ensure the existence of solutions? The correct way of measuring "how wide" a cusp should be is by means of the concept of *capacity* introduced by Wiener ([277]). The capacity of a compact set $K \subset \mathbb{R}^N$ is defined by

$$\operatorname{cap}(K) = \inf_{\substack{v\in C_o^\infty(\mathbb{R}^N)\\ v\ge 1 \text{ on } K}} \int_{\mathbb{R}^N} |\nabla v|^2 dx.$$

Such a definition can be extended to Borel sets. Now consider a domain E whose boundary has a cusp pointing inside E as in Figure 7.1. Let x_* be the “vertex” of the cusp, and consider the compact sets

$$K_n = (\mathbb{R}^N - E) \cap \bar{B}_{2^{-n}}(x_*) \qquad n \in \mathbb{N}$$

obtained by intersecting the region enclosed by the cusp, outside E, with balls centered at x_* and radius 2^{-n}.

Theorem 7.1c (Wiener [278]). *The following are equivalent:*

(i). There exists a barrier $H(x_*;\cdot)$ for the Dirichlet problem (6.2) at x_*
(ii). The series $\sum cap(K_n)$ is divergent.

For a theory of capacity and *capacitable* sets, see [136, 181, 154].

8c Problems in Unbounded Domains

8.1. Compute (8.5) for $N = 2, 3$ first. Then proceed by induction for all N. Use **2.3** and

$$\int_0^\infty \frac{\rho^{N-2}}{(1+\rho^2)^{N/2}} d\rho = \int_0^{\pi/2} (\sin t)^{N-2} dt.$$

8.2. Having in mind **3.7** and the representation formula (3.2), justify the definition (8.2) of the Poisson kernel for the half-space.

8.3. Give a solution formula for the Neumann problem in the half-space. Discuss uniqueness.

8.4. Let $E \subset \mathbb{R}^N$ be bounded, connected, and with boundary ∂E of class C^1. Prove that there exists at most one solution to the boundary value problem in the exterior of E

$$u \in C^2(\mathbb{R}^N - E) \cap C(\overline{\mathbb{R}^N - E}), \quad \Delta u = 0 \ \text{ in } \ \mathbb{R}^N - E$$
$$u\big|_{\partial E} = \varphi \in C(\partial E), \quad \lim_{|x|\to\infty} u(x) = \gamma$$

for a given constant γ.

8.1c The Dirichlet Problem Exterior to a Ball

Let $N \ge 3$, set $E = |x| > R$ for some $R > 0$, and consider the exterior problem

$$\begin{aligned} &u \in C^2(E) \cap C(\bar{E}), \quad \Delta u = 0 \ \text{ in } \ E \\ &u\big|_{|x|=R} = \varphi \in C(\partial E), \quad \lim_{|x|\to\infty} u(x) = \gamma \end{aligned} \tag{8.1c}$$

for a given constant γ.

Step 1. First apply the Kelvin transform to map E into $B_R - \{0\}$. Then introduce the new unknown function

$$v(y) = |y|^{2-N} u\left(\frac{R^2}{|y|^2} y\right) \qquad y \neq 0.$$

With the aid of Theorem 5.2c, verify that the singularity $y = 0$ is removable. Then, in terms of the new coordinates, (8.1c) becomes

$$v \in C^2(B_R) \cap C(\bar{B}_R), \quad \Delta v = 0 \text{ in } B_R \tag{8.2c}$$

$$v\big|_{\partial B_R} = R^{2-N} \varphi \in C(\partial B_R). \tag{8.3c}$$

Step 2. Solve (8.2c) by means of Poisson formula (3.9). Return to the original coordinates x by inverting the Kelvin transform. In this process use formula (3.6). To the function so obtained add a radial harmonic function vanishing for $|x| = R$ and satisfying the last of (8.1c). The solution is

$$u(x) = \gamma\left[1 - \left(\frac{R}{|x|}\right)^{N-2}\right] - \frac{1}{R\omega_N} \int_{\partial B_R} \varphi(y) \frac{R^2 - |x|^2}{|x-y|^N} d\sigma.$$

9c Schauder Estimates up to the Boundary ([222, 223])

Let $u \in C^2(E) \cap C(\bar{E})$ be the unique solution of the Dirichlet problem (1.2). If ∂E and the boundary datum φ are of class $C^{2,\eta}$, it is natural to expect that $u \in C^{2,\eta}(\bar{E})$. Prove the following[3]

Proposition 9.1c *Let $u \in C^2(\mathbb{R}^N \times \mathbb{R}^+) \cap C(\overline{\mathbb{R}^N \times \mathbb{R}^+})$ be the unique bounded solution of the Dirichlet problem (8.1). If $\varphi(x) \in C_o^{2,\eta}(\mathbb{R}^N)$ then $u \in C^{2,\eta}(\overline{\mathbb{R}^N \times \mathbb{R}^+})$ and there exists a constant γ depending only upon N, η and the diameter of the support of φ such that*

$$\|u\|_{2,\eta;\overline{\mathbb{R}^N \times \mathbb{R}^+}} \leq \gamma \|\varphi\|_{2,\eta;\mathbb{R}^N}.$$

Moreover there exist a constant γ depending only upon N and η and independent of the support of φ such that

$$\sup_{\substack{x,y \in \mathbb{R}^N \\ x_{N+1}, y_{N+1} \in \mathbb{R}^+}} \frac{|u_{x_i x_j}(x, x_{N+1}) - u_{x_i x_j}(y, y_{N+1})|}{[|x-y|^2 + (x_{N+1} - y_{N+1})^2]^{\eta/2}} \leq \gamma \sup_{x,y \in \mathbb{R}^N} \frac{|\varphi(x) - \varphi(y)|}{|x-y|^\eta}.$$

Proof (Hint:). Apply the same technique of proof of Theorem 9.1 to the Poisson integral (8.1). ■

The result of Theorem 9.1c is the key step in deriving $C^{2,\eta}$ estimates up to the boundary for solutions of the Dirichlet problem (1.2). The technique consist of performing a local flattening of ∂E.

---

[3] A version of these estimates is in [101, 150]. Their parabolic counterpart is in [83, 151].

10c Potential Estimates in $L^p(E)$

10.1. Compute the last integral in the proof of Proposition 10.1 by introducing polar coordinates. Prove that for $\alpha = 1$, the constant γ in (10.3) is

$$\gamma = \frac{1}{\omega_N^{1/p}(N-2)} \left(\frac{p-1}{2p-N}\right)^{(p-1)/p} \operatorname{diam}(E)^{2-N/p}.$$

10.2. Verify that for $N = 2$ and $\alpha = 1$, the constant γ in (10.3) is

$$\gamma = \frac{1}{(2\pi)^{1/p}} \begin{cases} \dfrac{1}{e}\dfrac{p}{p-1} & \text{if } \operatorname{diam}(E) \le 1 \\ \operatorname{diam}(E)^{2\frac{p-1}{p}} \ln \operatorname{diam}(E) & \text{if } \operatorname{diam}(E) > 1. \end{cases}$$

10.3. Prove the following

Corollary 10.1c *Let $E = B_R$ and $N \ge 3$. Then*

$$\|A_F f\|_{\infty,E} \le \gamma(N,p,R) \left(\fint_{B_R} |f(y)|^p dy\right)^{1/p}$$

where

$$\gamma(N,p,R) = 2R^2 \frac{N^{1/p}2^{2-N/p}}{N-2} \left(\frac{p-1}{2p-N}\right)^{\frac{p-1}{p}}.$$

State and prove a similar corollary for the case $N = 2$.

10.4. In Proposition 10.1 the boundedness of E is essential. For $\alpha = 2$, give an example of $f \in L^p(E)$ for some $p > N/2$ and E unbounded for which $A_F f$ is unbounded.

10.5. If E is unbounded, the boundedness of $A_F f$ can be recovered by imposing on f a fast decay as $|x| \to \infty$. Prove the following

Lemma 10.1c *Assume $f \in L^p(E)$ for some $p > N/2$, and*

$$|f(x)| \le C|x|^{-(2+\varepsilon)} \qquad \text{for } |x| > R_o$$

for given positive constants C, R_o, ε. Then $A_F f \in L^\infty(E)$, and there exists a constant γ depending only on N, C, R_o, ε such that

$$\|A_F f\|_{\infty,E} \le \gamma(1 + \|f\|_{p,E}).$$

10.6. Give an example of $A_F f$ such that $|\nabla A_F f| \in L^2(E)$ and $w_2 \notin L^\infty(E)$.

10.1c Integrability of Riesz Potentials

The proof of Proposition 10.1 shows that the constant γ in (10.3) deteriorates as $q \to Np/(N-\alpha p)_+$. However, (10.3) continues to hold also for the limiting case

$$q = \frac{Np}{(N-\alpha p)_+}, \qquad \text{provided} \qquad \alpha p < N.$$

The proof for such a limiting case however, is rather delicate and is based on Hardy's inequality ([115], also in [50], Chapter VIII, §18).

10.2c Second Derivatives of Potentials

If $f \in C_o^\eta(E)$, then $A_F f$ is twice continuously differentiable, and formally

$$(A_F f)_{x_i x_j}(x) = \int_E F_{x_i x_j}(x;y) f(y) dy. \tag{10.1c}$$

The integral is meant in the sense of the improper integral

$$\int_E F_{x_i x_j}(x;y) f(y) dy = \lim_{\varepsilon \to} \int_{E \cap [|x-y|>\varepsilon]} F_{x_i x_j}(x;y) f(y) dy.$$

The limit exists, since $f \in C_o^\eta(E)$. However, if $f \in L^p(E)$ for some $p > 1$, such a representation loses its classical meaning. It is natural to ask, in analogy with Proposition 10.1, whether one may use (10.1c) to define $(A_F f)_{x_i x_j}$ in a weak sense and whether such weak second derivatives are in $L^q(E)$ for some $q \geq 1$. It turns out that

$$f \in L^p(E) \implies (A_F f)_{x_i x_j} \in L^p(E) \quad \text{for } 1 < p < \infty.$$

The proof of this fact cannot be constructed from (10.1) for $\alpha = 0$, since the latter would be a divergent integral even if $f \in C_o^\infty(E)$. One has to rely instead on cancellation properties of the kernel in (10.1c). These estimates are due to Calderón and Zygmund ([27], see also [246]).

3

BOUNDARY VALUE PROBLEMS BY DOUBLE LAYER POTENTIALS

1 The Double-Layer Potential

Let Σ be an $(N-1)$-dimensional bounded surface of class C^1 in $\mathbb{R}^N$ for $N \ge 2$, whose boundary $\partial\Sigma$ is an $(N-2)$-dimensional surface of class C^1. Fix $x_o \in \mathbb{R}^N - \bar{\Sigma}$ and consider the cone $C(\Sigma, x_o)$ generated by half-lines originating at x_o and passing through points of $\partial\Sigma$. Let $\alpha(x_o)$ denote the solid angle spanned by $C(\Sigma, x_o)$, that is, the area of the portion of the unit sphere centered at x_o, cut by the cone. The double-layer potential generated in x_o by a distribution of dipoles identically equal to 1 on Σ is defined by

$$W(\Sigma, x_o) = -\int_{\Sigma} \frac{\partial F(x_o; y)}{\partial \mathbf{n}(y)} d\sigma = \frac{-1}{\omega_N} \int_{\Sigma} \frac{(x_o - y)\cdot \mathbf{n}(y)}{|x_o - y|^N} d\sigma. \tag{1.1}$$

Here $\mathbf{n}(\cdot)$ is the unit normal to Σ exterior to the cone $C(\Sigma, x_o)$, and $F(\cdot;\cdot)$ is the fundamental solution of the Laplacean, introduced in (2.6) of Chapter 2.

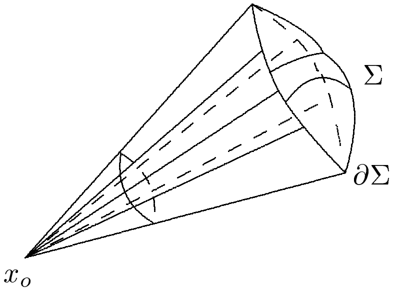

Fig. 1.1

© Springer Nature Switzerland AG 2023
E. DiBenedetto, U. Gianazza, *Partial Differential Equations*, Cornerstones, https://doi.org/10.1007/978-3-031-46618-2_4

The same cone is generated by infinitely many surfaces; however, the double-layer potential depends only on x_o and the solid angle $\alpha(x_o)$. This is the content of the next proposition.

Proposition 1.1 *Let* Σ_1 *and* Σ_2 *be any two surfaces generating the same cone* $C(\Sigma, x_o)$. *Then*

$$W(\Sigma_1, x_o) = W(\Sigma_2, x_o) = W(\Sigma, x_o) = \frac{\alpha(x_o)}{\omega_N}.$$

Proof. Let E be the portion of the cone $C(\Sigma_1, x_o) = C(\Sigma_2, x_o)$ included by the surfaces Σ_1 and Σ_2. Since x_o is outside $\bar{E}$

$$\begin{aligned} 0 = \int_E \Delta_y F(x_o; y) dy &= \frac{1}{\omega_N} \int_{\partial E - (\Sigma_1 \cup \Sigma_2)} \frac{(x_o - y) \cdot \mathbf{n}(y)}{|x_o - y|^N} d\sigma \\ &+ \int_{\Sigma_1} \frac{\partial F(x_o; y)}{\partial \mathbf{n}(y)} d\sigma - \int_{\Sigma_2} \frac{\partial F(x_o; y)}{\partial \mathbf{n}(y)} d\sigma. \end{aligned}$$

The first integral on the right-hand side vanishes since $(x_o - y)$ is tangent to the cone and thus normal to $\mathbf{n}(y)$. Therefore $W(\Sigma_1, x_o) = W(\Sigma_2, x_o)$.

Next, since $W(\Sigma, x_o)$ is independent of Σ, we replace Σ with the portion of the sphere $\partial B_R(x_o)$ cut by the cone, that is

$$\Sigma_o = \partial B_R(x_o) \cap C(\Sigma, x_o) \quad \text{for some } R > 0.$$

The normal to Σ_o exterior to the cone is

$$\mathbf{n}(y) = \frac{y - x_o}{|y - x_o|}.$$

This in (1.1) gives

$$W(\Sigma, x_o) = \frac{1}{\omega_N R^{N-1}} \int_{\Sigma_o} d\sigma = \frac{\alpha(x_o)}{\omega_N}. \qquad \blacksquare$$

In what follows E, is a bounded open set in $\mathbb{R}^N$ with boundary ∂E of class $C^{1,\alpha}$ for some $\alpha \in (0, 1]$. The double-layer potential generated at a point $x_o \in \mathbb{R}^N - \partial E$ by a continuous distribution of dipoles $y \to v(y)$ on ∂E is

$$\begin{aligned} W(\partial E, x_o; v) &= -\int_{\partial E} v(y) \frac{\partial F(x_o; y)}{\partial \mathbf{n}(y)} d\sigma \\ &= \frac{-1}{\omega_N} \int_{\partial E} v(y) \frac{(x_o - y) \cdot \mathbf{n}(y)}{|x_o - y|^N} d\sigma. \end{aligned} \tag{1.2}$$

Proposition 1.2 *In (1.2) let* $v = 1$. *Then*

$$W(\partial E, x_o; 1) = \begin{cases} 1 & \textit{for } x_o \in E \\ 0 & \textit{for } x_o \in \mathbb{R}^N - \bar{E}. \end{cases} \tag{1.3}$$

Proof. The first follows from the Stokes identity (2.3)–(2.4) of Chapter 2, written for $u = 1$. If x_o is outside $\bar{E}$, the function $y \to F(x_o; y)$ is harmonic in E. Therefore

$$\int_E \Delta_y F(x_o; y) dy = \int_{\partial E} \frac{\partial F(x_o; y)}{\partial \mathbf{n}(y)} d\sigma = 0. \qquad \blacksquare$$

2 On the Integral Defining the Double-Layer Potential

As x_o tends to a point $x \in \partial E$ the integrand in (1.2), becomes singular. Such a singularity however is integrable. This is the content of the next lemma.

Lemma 2.1 *There exists a constant C depending only on N, α, and the structure of ∂E such that*

$$\frac{|(x-y)\cdot \mathbf{n}(x)|}{|x-y|^N} \le C \frac{1}{|x-y|^{N-1-\alpha}} \quad \textit{for all } x, y \in \partial E.$$

Proof. It will suffice to prove the lemma for all $|x - y| < \eta$ for some $\eta > 0$. Fix $x \in \partial E$ and assume, after a translation, that it coincides with the origin. Since ∂E is of class $C^{1,\alpha}$, there exists $\eta > 0$ such that the portion of ∂E within the ball B_η, centered at the origin, can be represented, in a local system of coordinates, as the graph of a function φ satisfying

$$\begin{gathered} \xi_N = \varphi(\xi), \quad \xi = (\xi_1, \dots, \xi_{N-1}), \quad |\xi| < \eta \\ \varphi \in C^{1,\alpha}(|\xi| < \eta), \quad \varphi(0) = 0, \quad |\nabla\varphi(0)| = 0 \\ |\nabla\varphi(\xi)| \le C_\varphi |\xi|^\alpha \text{ for a given positive constant } C_\varphi. \end{gathered} \tag{2.1}$$

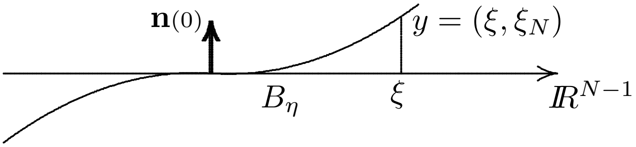

Fig. 2.2

Then $\mathbf{n}(0)$ is the unit vector of the ξ_N-axis, exterior to E, and in the new coordinates

$$|\xi| \le |y| \le (1 + \|\nabla\varphi\|_{\infty, B_\eta})|\xi|$$

and

$$|-y \cdot \mathbf{n}(0)| = |\varphi(\xi)| \le C_\varphi |\xi|^{1+\alpha}. \qquad \blacksquare$$

Let $x \in \partial E$ and for $\varepsilon > 0$ let $S_\varepsilon(x) = \partial E \cap B_\varepsilon(x)$ denote the portion of ∂E within the ball $B_\varepsilon(x)$ centered at x and radius ε.

Lemma 2.2 *There exist constants C and ε_o, depending only on N, α, and the structure of ∂E, such that for every $\varepsilon \le \varepsilon_o$*

$$\int_{S_\varepsilon(x)} \frac{|(z-y)\cdot \mathbf{n}(y)|}{|z-y|^N} d\sigma \le C$$

uniformly for all $z \in B_\varepsilon(x)$.

Proof. Fix such a $z \in B_\varepsilon(x)$. Since ∂E is of class $C^{1+\alpha}$, we may choose ε_o so small that z has a unique projection, say z_p, on ∂E. Set $\delta = |z - z_p|$ and compute

$$|z-y|^2 = |z_p - y|^2 + \delta^2 - 2(z_p - y)\cdot(z_p - z).$$

Since $z_p \in \partial E$, by Lemma 2.1

$$\begin{aligned}|(z_p - y)\cdot(z_p - z)| &= \left|(z_p - y)\cdot \frac{z_p - z}{|z_p - z|}\right|\delta \\ &= |(z_p - y)\cdot \mathbf{n}(z_p)|\delta \le C|z_p - y|^{1+\alpha}\delta.\end{aligned}$$

Therefore

$$|z-y|^2 \ge |z_p - y|^2 + \delta^2 - 2C|z_p - y|^{1+\alpha}\delta \ge \tfrac{1}{4}(|z_p - y| + \delta)^2$$

provided ε_o is chosen sufficiently small. Also

$$|(z-y)\cdot \mathbf{n}(y)| \le |(z_p - y)\cdot \mathbf{n}(y)| + \delta \le C|z_p - y|^{1+\alpha} + \delta.$$

Therefore

$$\frac{|(z-y)\cdot \mathbf{n}(y)|}{|z-y|^N} \le C\frac{|z_p - y|^{1+\alpha} + \delta}{(|z_p - y| + \delta)^N}.$$

From this

$$\begin{aligned}\int_{S_\varepsilon(x)} \frac{|(z-y)\cdot \mathbf{n}(y)|}{|z-y|^N} d\sigma \le &\int_{S_\varepsilon(x)} \frac{1}{|z_p - y|^{N-1-\alpha}} d\sigma \\ &+ C\delta \int_{S_\varepsilon(x)} \frac{d\sigma}{(|z_p - y| + \delta)^N}.\end{aligned}$$

The first integral is convergent, since $z_p \in \partial E$, and it is bounded above by

$$\sup_{x\in\partial E} \int_{\partial E} \frac{1}{|x-y|^{N-1-\alpha}} d\sigma.$$

To estimate the second integral, first extend the integration to the larger set $S_{2\varepsilon}(z_p)$. Then introduce a local system of coordinates with the origin at z_p, so that $S_{2\varepsilon}(z_p)$ is represented as in (2.1). This gives

$$\begin{aligned}\delta \int_{S_\varepsilon(x)} \frac{d\sigma}{(|x-y| + \delta)^N} &\le C_\varphi \delta \int_{|\xi|<2\varepsilon} \frac{d\xi}{(|\xi| + \delta)^N} \\ &\le C'_\varphi \delta \int_0^{2\varepsilon} \frac{dr}{(r+\delta)^2} \le C''_\varphi. \qquad \blacksquare\end{aligned}$$

3 The Jump Condition of $W(\partial E, x_o; v)$ Across ∂E

We will compute the limit of $W(\partial E, x_o; v)$ as $x_o \to \partial E$ either from within E or from outside $\bar{E}$. Having fixed $x \in \partial E$, denote by $\{x^i\} \subset E$ a sequence of points approaching x from the interior of E. Likewise, denote by $\{x^e\} \subset \mathbb{R}^N - \bar{E}$ a sequence of points approaching x from the exterior of $\bar{E}$.

Proposition 3.1 *Let $v \in C(\partial E)$. Then for all $x \in \partial E$*

$$\begin{aligned} \lim_{x^i \to x} W(\partial E, x^i; v) &= \frac{1}{2} v(x) - \frac{1}{\omega_N} \int_{\partial E} v(y) \frac{(x-y)\cdot \mathbf{n}(y)}{|x-y|^N} d\sigma \\ \lim_{x^e \to x} W(\partial E, x^e; v) &= -\frac{1}{2} v(x) - \frac{1}{\omega_N} \int_{\partial E} v(y) \frac{(x-y)\cdot \mathbf{n}(y)}{|x-y|^N} d\sigma. \end{aligned}$$

Combining these limits gives the jump condition of the potential $W(\partial E, x_o; v)$ across ∂E.

Corollary 3.1 *Let $v \in C(\partial E)$. Then for all $x \in \partial E$*

$$\lim_{x^i \to x} W(\partial E, x^i; v) - \lim_{x^e \to x} W(\partial E, x^e; v) = v(x).$$

Proof (of Proposition 3.1). For $\varepsilon > 0$, let $S_\varepsilon(x) = \partial E \cap B_\varepsilon(x)$ and write

$$\begin{aligned} \int_{\partial E} v(y) \frac{(x^i-y)\cdot \mathbf{n}(y)}{|x^i-y|^N} d\sigma = v(x) &\int_{S_\varepsilon(x)} \frac{(x^i-y)\cdot \mathbf{n}(y)}{|x^i-y|^N} d\sigma \\ &+ \int_{S_\varepsilon(x)} [v(y)-v(x)] \frac{(x^i-y)\cdot \mathbf{n}(y)}{|x^i-y|^N} d\sigma \\ &+ \int_{\partial E - S_\varepsilon(x)} v(y) \frac{(x^i-y)\cdot \mathbf{n}(y)}{|x^i-y|^N} d\sigma. \end{aligned}$$

Choose $\varepsilon \le \varepsilon_o$, where ε_o is the number claimed by Lemma 2.2, and set

$$-\int_{S_\varepsilon(x)} \frac{(x^i-y)\cdot \mathbf{n}(y)}{|x^i-y|^N} d\sigma = \alpha(\varepsilon, x^i)$$

where by Proposition 1.1, $\alpha(\varepsilon, x^i)$ is the solid angle of the cone generated by the lines through x^i and the points of $\partial S_\varepsilon(x)$. By Lemma 2.2

$$\begin{aligned} \Big| \int_{S_\varepsilon(x)} [v(y)-v(x)] &\frac{(x^i-y)\cdot \mathbf{n}(y)}{|x^i-y|^N} d\sigma \Big| \\ &\le \sup_{y \in S_\varepsilon(x)} |v(y)-v(x)| \Big| \int_{S_\varepsilon(x)} \frac{(x^i-y)\cdot \mathbf{n}(y)}{|x^i-y|^N} d\sigma \Big| \\ &\le C \sup_{y \in S_\varepsilon(x)} |v(y)-v(x)|. \end{aligned}$$

Therefore

$$\int_{\partial E} v(y) \frac{(x^i - y) \cdot \mathbf{n}(y)}{|x^i - y|^N} d\sigma = - v(x)\alpha(\varepsilon, x^i) + O(\varepsilon) + \int_{\partial E - S_\varepsilon(x)} v(y) \frac{(x^i - y) \cdot \mathbf{n}(y)}{|x^i - y|^N} d\sigma.$$

Now let $x^i \to x$ to obtain

$$\lim_{x^i \to x} W(\partial E, x^i; v) = v(x) \frac{\alpha(\varepsilon; x)}{\omega_N} - O(\varepsilon) - \int_{\partial E - S_\varepsilon(x)} v(y) \frac{(x - y) \cdot \mathbf{n}(y)}{|x - y|^N} d\sigma \tag{3.1}$$

where $\alpha(\varepsilon; x)$ is the solid angle of the cone generated by lines through x and points of $\partial S_\varepsilon(x)$. As $\varepsilon \to 0$, $\alpha(\varepsilon; x) \to \frac{1}{2}\omega_N$. To prove the proposition, we let $\varepsilon \to 0$ in (3.1) with the aid of Lemma 2.1. ■

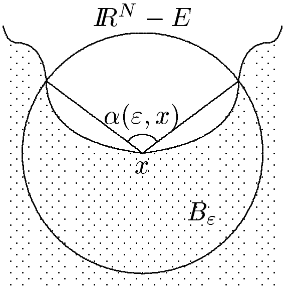

Fig. 3.3

Corollary 3.2 *For all $x \in \partial E$*

$$-\frac{1}{\omega_N} \int_{\partial E} \frac{(x - y) \cdot \mathbf{n}(y)}{|x - y|^N} d\sigma = \frac{1}{2}.$$

Proof. Apply (3.1) with $v = 1$ and use (1.3). ■

Remark 3.1 Combining this with Proposition 1.2, we conclude that the double-layer potential $x \to W(\partial E; x)$ generated by a constant distribution of dipoles on ∂E, at points $x \in \mathbb{R}^N$, is a function that is discontinuous across $\partial\Omega$, and its values are, up to a multiplicative constant

$$W(\partial E; x) = \begin{cases} 1 & \text{for } x \in E \\ \frac{1}{2} & \text{for } x \in \partial E \\ 0 & \text{for } x \in \mathbb{R}^N - \bar{E}. \end{cases} \tag{3.2}$$

4 More on the Jump Condition Across ∂E

Fix $x \in \partial E$ and denote by $\mathbf{n}(x)$ the outward unit normal at x. For $x_o \in \mathbb{R}^N - \partial E$, set

$$\begin{aligned}\tilde{W}(\partial E, x_o; v) &= -\frac{1}{\omega_N}\int_{\partial E} v(y)\frac{(x_o - y)\cdot \mathbf{n}(x)}{|x_o - y|^N} d\sigma \\ &= \nabla_z \left(\int_{\partial E} v(y)F(z;y)d\sigma\right)\cdot \mathbf{n}(x)\bigg|_{z=x_o}.\end{aligned} \tag{4.1}$$

As $x_o \to x$ the behavior of $\tilde{W}(\partial E, x_o; v)$ is similar to that of the double-layer potential $W(\partial E, x_o; v)$, provided x_o approaches x along the normal $\mathbf{n}(x)$. Let $\{x^i\}$ and $\{x^e\}$ denote sequences approaching x from the inside and respectively outside of E, say for example

$$x^i = x - \delta_i \mathbf{n}(x), \qquad x^e = x + \delta_e \mathbf{n}(x)$$

where $\{\delta_i\}$ and $\{\delta_e\}$ are sequences of positive numbers decreasing to zero.

Proposition 4.1 *Let $v \in C(\partial E)$. Then for all $x \in \partial E$*

$$\begin{aligned}\lim_{\delta_i \to 0} \tilde{W}(\partial E, x^i; v) &= \frac{1}{2}v(x) - \frac{1}{\omega_N}\int_{\partial E} v(y)\frac{(x-y)\cdot \mathbf{n}(x)}{|x-y|^N}d\sigma \\ \lim_{\delta_e \to 0} \tilde{W}(\partial E, x^e; v) &= -\frac{1}{2}v(x) - \frac{1}{\omega_N}\int_{\partial E} v(y)\frac{(x-y)\cdot \mathbf{n}(x)}{|x-y|^N}d\sigma.\end{aligned}$$

Corollary 4.1 *Let $v \in C(\partial E)$. Then for all $x \in \partial E$*

$$\lim_{x^i \to x} \tilde{W}(\partial E, x^i; v) - \lim_{x^e \to x} \tilde{W}(\partial E, x^e; v) = v(x).$$

Proof (of Proposition 4.1). We prove only the first statement. Write

$$\begin{aligned}\lim_{\delta_i \to 0} \tilde{W}(\partial E, x^i; v) = &\lim_{\delta_i \to 0} \frac{1}{\omega_N}\int_{\partial E} v(y)\frac{(x^i - y)\cdot(\mathbf{n}(y) - \mathbf{n}(x))}{|x^i - y|^N}d\sigma \\ &+ \lim_{\delta_i \to 0}\frac{-1}{\omega_N}\int_{\partial E} v(y)\frac{(x^i - y)\cdot \mathbf{n}(y)}{|x^i - y|^N}d\sigma.\end{aligned}$$

The second limit is computed by means of Proposition 3.1, and equals

$$\frac{1}{2}v(x) - \frac{1}{\omega_N}\int_{\partial E} v(y)\frac{(x-y)\cdot \mathbf{n}(y)}{|x-y|^N}d\sigma.$$

To compute the first limit, set $S_{2\varepsilon} = \partial E \cap B_{2\varepsilon}(x)$ for $0 < \varepsilon \ll 1$, and write

$$\begin{aligned}&\int_{\partial E} v(y)\frac{(x^i - y)\cdot(\mathbf{n}(y) - \mathbf{n}(x))}{|x^i - y|^N}d\sigma \\ &\qquad = \int_{\partial E - B_{2\varepsilon}(x)} v(y)\frac{(x^i - y)\cdot(\mathbf{n}(y) - \mathbf{n}(x))}{|x^i - y|^N}d\sigma \\ &\qquad\quad + \int_{S_{2\varepsilon}(x)} v(y)\frac{(x^i - y)\cdot(\mathbf{n}(y) - \mathbf{n}(x))}{|x^i - y|^N}d\sigma.\end{aligned}$$

Without loss of generality, we may assume that the sequence $\{x^i\}$ is contained in $B_\varepsilon(x)$. Then for ε fixed

$$\begin{aligned}\lim_{\delta_i\to 0}\int_{\partial E - B_{2\varepsilon}(x)} & v(y)\frac{(x^i-y)\cdot(\mathbf{n}(y)-\mathbf{n}(x))}{|x^i-y|^N}d\sigma \\ &= \int_{\partial E - B_{2\varepsilon}(x)} v(y)\frac{(x-y)\cdot(\mathbf{n}(y)-\mathbf{n}(x))}{|x-y|^N}d\sigma.\end{aligned}$$

It remains to prove that

$$\begin{aligned}\lim_{\delta_i\to 0}\int_{S_{2\varepsilon}(x)} & v(y)\frac{(x^i-y)\cdot(\mathbf{n}(y)-\mathbf{n}(x))}{|x^i-y|^N}d\sigma \\ &= \int_{S_{2\varepsilon}(x)} v(y)\frac{(x-y)\cdot(\mathbf{n}(y)-\mathbf{n}(x))}{|x-y|^N}d\sigma.\end{aligned}$$

Compute

$$|x^i-y|^2 = |x-y|^2 + \delta_i^2 + 2\delta_i(x-y)\cdot\mathbf{n}(x).$$

By Lemma 2.1, since $|x-y|<\varepsilon$

$$|(x-y)\cdot\mathbf{n}(x)| \le \text{const}|x-y|^{1+\alpha} \le \varepsilon^\alpha\text{const}|x-y|.$$

Therefore, if ε is chosen so small that $\varepsilon^\alpha\text{const}\le 1$

$$|x^i-y|^2 \ge |x-y|^2 + \delta_i^2 - 2\varepsilon^\alpha\text{const}|x-y|\delta_i \ge \frac{1}{4}(|x-y|+\delta_i)^2.$$

Next estimate

$$\begin{aligned}\frac{|(x^i-y)\cdot(\mathbf{n}(y)-\mathbf{n}(x))|}{|x^i-y|^N} &\le \gamma\frac{|x-y|^\alpha}{|x^i-y|^{N-1}} \le \gamma\frac{|x-y|^\alpha}{(|x-y|+\delta_i)^{N-1}} \\ &\le \frac{\gamma}{|x-y|^{N-1-\alpha}}\end{aligned}$$

for a constant γ depending only upon N and the structure of ∂E. Therefore to compute the last limit, it suffices to pass to the limit under the integral with the aid of the Lebesgue dominated convergence theorem. ■

5 The Dirichlet Problem by Integral Equations ([192])

Let E be a bounded domain in $\mathbb{R}^N$ with boundary ∂E of class $C^{1,\alpha}$ for some $\alpha\in(0,1]$, and consider the Dirichlet problem (1.2) of Chapter 2. Seek a solution of such a problem in the form of a double-layer potential

$$u(x) = \frac{-1}{\omega_N}\int_{\partial E} v(y)\frac{(x-y)\cdot\mathbf{n}(y)}{|x-y|^N}d\sigma \tag{5.1}$$

for some unknown density $v \in C(\partial E)$. To impose the boundary data $u = \varphi$ on $\partial\Omega$, let x tend to points of ∂E. Using Proposition 3.1, we conclude that $v(\cdot)$ must satisfy the integral equation

$$\frac{1}{2}v = \int_{\partial E} K_D(\cdot; y)v(y)d\sigma + \varphi \tag{5.2}$$

where $K_D(\cdot;\cdot)$ is the Dirichlet kernel

$$K_D(x; y) = \frac{1}{\omega_N}\frac{(x - y)\cdot \mathbf{n}(y)}{|x - y|^N}. \tag{5.3}$$

Proposition 5.1 *Suppose that (5.2) has a solution $v \in C(\partial E)$. Then (5.1) for such a v defines a solution of the Dirichlet problem (1.2) of Chapter 2.*

Proof. The function u defined by (5.1) is harmonic in E, and by Proposition 3.1, it takes the boundary values φ on ∂E. ■

For $v \in L^\infty(\partial E)$, set

$$A_D v = \int_{\partial E} K_D(\cdot; y)v(y)d\sigma. \tag{5.4}$$

Proposition 5.2 *The function $x \to A_D v(x)$ is continuous in ∂E.*

Proof. Let $\{x_n\}$ be a sequence of points in ∂E converging to some $x_o \in \partial E$. First observe that $\{K_D(x_n; \cdot)\} \to K_D(x_o; \cdot)$ a.e. in ∂E. Then write

$$\lim |A_D v(x_n) - A_D v(x_o)| \le \lim \int_{\partial E} |K_D(x_n; y) - K_D(x_o; y)||v(y)|dy.$$

Pass to the limit under integral, with the aid of Lemma 2.1 (**5.2.** of the Complements). ■

Corollary 5.1 *Let $\{v_n\}$ be equi-bounded in $L^\infty(\partial E)$. Then $\{A_D v_n\}$ is equi-bounded and equi-continuous in ∂E (**5.3.** of the Problems and Complements).*

6 The Neumann Problem by Integral Equations ([192])

Consider the Neumann problem (1.3) of Chapter 2 for a datum $\psi \in C(\partial E)$. For $x \in \partial E$, such a datum is taken in the sense

$$\lim_{\delta\to 0} \nabla u(x - \delta\mathbf{n}(x)) \cdot \mathbf{n}(x) = \psi(x). \tag{6.1}$$

Seek solutions of the Neumann problem in the form of a single-layer potential

$$u = \int_{\partial E} F(\cdot; y)v(y)d\sigma \tag{6.2}$$

where $F(\cdot;\cdot)$ is the fundamental solution of the Laplacean, introduced in (2.6) of Chapter 2, and $v(\cdot)$ is an unknown surface density. To compute v, impose the boundary condition in the sense of (6.1). First, for $x^i \in E$ and $x \in \partial E$, compute

$$\nabla u(x^i) \cdot \mathbf{n}(x) = -\frac{1}{\omega_N} \int_{\partial E} v(y) \frac{(x^i - y) \cdot \mathbf{n}(x)}{|x^i - y|^N} d\sigma.$$

Then take $x^i = x - \delta \mathbf{n}(x)$ and take the limit as $\delta \to 0$ of the integral on the right-hand side by making use of Proposition 4.1. We conclude that $v(\cdot)$ must satisfy the integral equation

$$\frac{1}{2} v = \int_{\partial E} K_N(\cdot; y) v(y) d\sigma + \psi \tag{6.3}$$

where $K_N(\cdot;\cdot)$ is the Neumann kernel

$$K_N(x; y) = \frac{1}{\omega_N} \frac{(x - y) \cdot \mathbf{n}(x)}{|x - y|^N}. \tag{6.4}$$

Proposition 6.1 *Suppose that (6.3) has a solution $v \in C(\partial E)$. Then (6.2), for such a v, defines a solution to the Neumann problem (1.3) of Chapter 2.*

Proof. The function defined by (6.2) is harmonic in E and by the previous calculations, satisfies the Neumann data on ∂E. ■

Theorem 6.1. *A necessary and sufficient condition of solvability of (6.3) is that ψ be of zero-average over ∂E, i.e., that (1.4) of Chapter 2 holds.*

Proof (Necessity). Integrate (6.3) over ∂E in $d\sigma(x)$. By Corollary 3.1

$$\int_{\partial E} K_N(x; y) d\sigma(x) = -\int_{\partial E} K_D(y; x) d\sigma(x) = \frac{1}{2}. \qquad ■$$

The sufficient part of the theorem will be proved in the next Chapter.

For $v \in L^\infty(\partial E)$ set

$$A_N v = \int_{\partial E} K_N(\cdot; y) v(y) d\sigma. \tag{6.5}$$

Proposition 6.2 *The function $x \to A_N v(x)$ is continuous in ∂E.*

Corollary 6.1 *Let $\{v_n\}$ be equi-bounded in $L^\infty(\partial E)$. Then $\{A_N v_n\}$ is equi-bounded and equi-continuous in ∂E.*

7 The Green's Function for the Neumann Problem

Consider the family of Neumann problems

$$\begin{gathered} N(x;\cdot) \in C^2(E) \cap C^1(\bar{E}) \ \text{ for all } \ x \in E, \quad \Delta_y N(x;y) = k \ \text{ in } \ E \\ \frac{\partial}{\partial \mathbf{n}(y)} N(x;y) = \frac{\partial}{\partial \mathbf{n}(y)} F(x;y), \quad y \in \partial E \end{gathered} \tag{7.1}$$

where F is the fundamental solution of the Laplacean, introduced in (2.6) of Chapter 2, and k is a constant. Integrating the equation by parts and using (1.3) gives

$$k\,|E| = \int_{\partial E} \frac{\partial F(x;y)}{\partial \mathbf{n}(y)} d\sigma = -1.$$

Therefore, a necessary condition of solvability is $k = -|E|^{-1}$. Assuming for the moment that (7.1) has a solution set

$$(x,y) \to \mathcal{G}(x;y) = F(x;y) - N(x;y) \quad \text{for } \ x \neq y.$$

This is called Green's function for the Neumann problem, and it satisfies

$$\Delta_y \mathcal{G} = -k \ \text{ in } \ E, \quad \frac{\partial}{\partial \mathbf{n}(y)} \mathcal{G}(x;y) = 0 \ \text{ on } \ \partial E; \quad x \neq y. \tag{7.2}$$

Green's function is not unique; indeed if $\mathcal{G}(x;\cdot)$, is a Green's function, then

$$\mathcal{G}(x;\cdot) + v(x) \quad \text{for all } \ v \in C^2(E)$$

is still a Green's function for the Neumann problem. Having determined one such a function, say for example $\mathcal{G}_1(x;y)$, we let

$$v = -\fint_E \mathcal{G}_1(\cdot;y)dy \quad \text{and} \quad \mathcal{G}(x;y) \overset{\text{def}}{=} \mathcal{G}_1(x;y) + v(x).$$

In this way, among all the possible Green's functions for the Neumann problem, we have selected the one with zero-average for all $x \in E$. Such a selection implies that $\mathcal{G}(\cdot;\cdot)$ is symmetric. This is a particular case of the following

Lemma 7.1 *Let $\mathcal{G}(\cdot;\cdot)$ be a Green's function for the Neumann problem satisfying*

$$x \to \int_E \mathcal{G}(x;y)dy = \textit{const.}$$

Then $\mathcal{G}(x;y) = \mathcal{G}(y;x)$.

The proof is the same as in Lemma 3.1 of Chapter 2. From now on, we will select $\mathcal{G}$ satisfying the zero-average property. Therefore, by symmetry

$$\Delta_y \mathcal{G} = \Delta_x \mathcal{G} = -k \quad \text{in } \ E \ \text{ for } \ x \neq y.$$

Let u be a solution of the Neumann problem (1.3) of Chapter 2. From the Stokes identity (2.3)–(2.4) of Chapter 2, subtract the Green's identity (2.2), of the same Chapter, written for u and $N(x;\cdot)$. This gives

$$u = \int_{\partial E} \psi \mathcal{G}(\cdot;y) d\sigma - k \int_E u\, dy.$$

If u is a solution of the Neumann problem, then $u + C$ are also solutions of the same problem, for all constants C. Choosing

$$C = -\fint_E u dy$$

we select, among all solutions of the Neumann problem, the one with the zero-average property and satisfying the representation

$$u = \int_{\partial E} \psi \mathcal{G}(\cdot;y) d\sigma. \tag{7.3}$$

This representation is a candidate for a solution of the Neumann problem. By the symmetry of $\mathcal{G}(\cdot;\cdot)$

$$\Delta_x u = \int_{\partial E} \psi \Delta_x \mathcal{G}(\cdot;y)\, d\sigma = -k \int_{\partial E} \psi\, d\sigma = 0.$$

Thus the condition that ψ be of zero-average over ∂E is necessary for (7.3) to define a harmonic function. It would remain to establish that the boundary datum is taken in the sense of (6.1). This verification could be carried out if one had an explicit expression for $\mathcal{G}(\cdot;\cdot)$. This would be analogous to the Dirichlet problem for the ball, where a verification of the boundary data was possible via the explicit Poisson representation of Theorem 3.1 of Chapter 2.

Even though the method is elegant, the actual calculation of the Green's function $\mathcal{G}(\cdot;\cdot)$ can be effected explicitly only for domains with a simple geometry such as balls or cubes (see Section 7c of the Complements).

7.1 Finding $\mathcal{G}(\cdot;\cdot)$

One might look for $\mathcal{G}(\cdot;\cdot)$ of the form

$$\mathcal{G}(x;y) = F(x;y) - \gamma_o |y|^2 + h(x;y)$$

up to the addition of a function $x \to v(x)$. Here γ_o is a constant to be determined, and

$$h(x;\cdot) \in C^2(E) \cap C^1(\bar{E}) \text{ is harmonic for all } x \in E.$$

Such a $\mathcal{G}(\cdot;\cdot)$ satisfies the first of (7.2) for the choice $2N\gamma_o = k$. Imposing the boundary conditions on $\mathcal{G}(x;\cdot)$ implies that $h(x;\cdot)$ must satisfy

$$\Delta h(x;\cdot) = 0 \;\text{ in }\; E, \qquad \frac{\partial h(x;y)}{\partial \mathbf{n}(y)} = 2\gamma_o y \cdot \mathbf{n}(y) - \frac{\partial F(x;y)}{\partial \mathbf{n}(y)} \;\text{ on }\; \partial E.$$

This family of Neumann problems can be solved by the method of integral equations outlined in the previous section. Specifically, one looks for $h(x;\cdot)$ in the form of a single-layer potential

$$h(x;y) = \int_{\partial E} v(x;\eta) F(y;\eta)\, d\sigma(\eta), \qquad x, y \in E$$

where the unknown x-dependent density distribution $v(x;\cdot)$ satisfies the integral equation

$$\frac{1}{2} v(x;y) = 2\gamma_o y \cdot \mathbf{n}(y) - \frac{\partial F(x;y)}{\partial \mathbf{n}(y)} + \frac{1}{\omega_N} \int_{\partial E} v(x;\eta) \frac{(y-\eta)\cdot \mathbf{n}(y)}{|y-\eta|^N}\, d\sigma(\eta).$$

This integral equation is solvable if and only if

$$\int_{\partial E} \frac{\partial h(x;y)}{\partial \mathbf{n}(y)}\, d\sigma = 0 \qquad \text{for all } \; x \in E.$$

This is part of the existence theory for such integral equations that will be developed in the next chapter. To verify the zero-average condition, compute

$$\int_{\partial E} \frac{\partial F(x;y)}{\partial \mathbf{n}(y)}\, d\sigma = -W(\partial E, x; 1) = -1$$

by Proposition 1.2. On the other hand, by the divergence theorem and the indicated choices of γ_o and k

$$2\gamma_o \int_{\partial E} y \cdot \mathbf{n}(y) d\sigma = \gamma_o \int_E \operatorname{div} \nabla |y|^2 dx = 2N\gamma_o |E| = -1.$$

8 Eigenvalue Problems for the Laplacean

Consider the problem of finding $\lambda \in \mathbb{R} - \{0\}$ and a nontrivial $u \in C^2(E) \cap C^\eta(\bar{E})$, for some $\eta \in (0,1)$ satisfying

$$\Delta u = -\lambda u \;\text{ in }\; E, \qquad \text{and} \qquad u = 0 \;\text{ on }\; \partial E. \tag{8.1}$$

This is the eigenvalue problem for the Laplacean with homogeneous Dirichlet data on ∂E. If (8.1) has a nontrivial solution, by the results of Section 12 of Chapter 2, such a solution u satisfies

$$u = \lambda \int_E G(\cdot;y) u\, dy \tag{8.2}$$

where $G(\cdot;\cdot)$ is the Green's function for the Laplacean in E. The nontrivial pair (λ, u) represents an eigenvalue and an eigenfunction for the integral equation (8.2). Conversely, if (8.2) has a nontrivial solution pair (λ, u), such that $u \in C^\eta(\bar{E})$, for some $\eta \in (0,1)$, then by the same procedure of Section 12 of Chapter 2, such a u is also a solution of (8.1). We summarize

Lemma 8.1 *A nontrivial pair* (λ, u) *is a solution of (8.1) if and only if it solves (8.2).*

8.1 Compact Kernels Generated by Green's Function

Let E be a bounded open set in $\mathbb{R}^N$ with boundary ∂E of class C^1 and let $G(\cdot;\cdot)$ be the Green's function for the Laplacean in E. Set

$$L^p(E) \ni f \to A_G f = \int_E G(\cdot;y)f(y)dy \quad \text{for some } p \geq 1 \tag{8.3}$$

provided the right-hand side defines a function in $L^q(E)$ for some $q \geq 1$.

Theorem 8.1. A_G *is a compact mapping in* $L^p(E)$ *for all* $1 \leq p \leq \infty$, *i.e., it maps bounded sets in* $L^p(E)$, *into pre-compact sets in* $L^p(E)$.

Green's function G is defined in (3.3) of Chapter 2, where $F(\cdot;\cdot)$ is the fundamental solution of the Laplace equation in $\mathbb{R}^N$, for $N \geq 2$, defined in (2.6) of Chapter 2, and Φ is introduced in (3.1) of the same Chapter. Setting

$$L^p(E) \ni f \to \begin{cases} A_F f = \displaystyle\int_E F(\cdot;y)f(y)dy \\ \\ A_\Phi f = \displaystyle\int_E \Phi(\cdot;y)f(y)dy \end{cases} \quad \text{for some } p \geq 1 \tag{8.4}$$

the proof reduces to showing that both A_F and A_Φ are compact in $L^p(E)$.

9 Compactness of A_F in $L^p(E)$ for $1 \leq p \leq \infty$

The operator A_F maps bounded sets in $L^p(E)$ into bounded sets of $L^p(E)$, for all $1 \leq p \leq \infty$. This is content of Proposition 10.1 of Chapter 2. For $p > N$, the compactness of A_F follows from Corollary 10.1 of Chapter 2. Indeed, in such a case, A_F maps bounded sequences $\{f_n\} \subset L^p(E)$ into sequences $\{A_F f_n\}$ of equi-Lipschitz continuous functions in $\bar{E}$. Therefore compactness follows from the Ascoli–Arzelà theorem.

To establish compactness in $L^p(E)$ for $1 \leq p \leq N$, for a fixed vector $h \in \mathbb{R}^N$, introduce the translation operator

$$L^p(E) \ni v \to T_h v = \begin{cases} v(\cdot + h) & \text{if } \cdot + h \in E \\ 0 & \text{otherwise.} \end{cases}$$

Also for $\delta > 0$ set

$$E_\delta = \{x \in E \mid \operatorname{dist}(x, \partial E) > \delta\}. \tag{9.1}$$

The proof uses the following characterization of pre-compact subsets of $L^p(E)$ ([50] Chapter 5, Section 22).

Theorem 9.1. *A bounded subset $K \subset L^p(E)$, for $1 \le p < \infty$ is pre-compact in $L^p(E)$ if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all vectors $h \in \mathbb{R}^N$ of length $|h| < \delta$, and for all $v \in K$*

$$\|T_h v - v\|_{p,E} < \varepsilon \qquad \textit{and} \qquad \|v\|_{p,E-E_\delta} < \varepsilon. \tag{9.2}$$

To verify the assumptions of the theorem, set $v = A_F f$ and use Proposition 10.1 of Chapter 2 and (10.3) of the same Chapter, for $\alpha = 1$. Fix $\delta > 0$ so small that $E - E_\delta$ is not empty and let $h \in \mathbb{R}^N$ be such that $|h| < \delta$. Then

$$\begin{aligned}\int_{E_\delta} |v(x+h) - v(x)|dx &\le \int_{E_\delta}\int_0^1 \left|\frac{d}{dt}v(x+th)\right|dtdx \\ &\le |h|\int_0^1\int_{E_\delta} |\nabla v(x+th)|dxdt \\ &\le |h||E|^{1-\frac{1}{q}}\|\nabla A_F f\|_{q,E} \le \gamma|h|\|f\|_{p,E}\end{aligned}$$

where p and q are as in the indicated proposition, including possibly the limiting cases. Therefore for all $\sigma \in (0, \frac{1}{p})$

$$\begin{aligned}\int_{E_\delta} |T_h v - v|^p dx &= \int_{E_\delta} |T_h v - v|^{p\sigma + p(1-\sigma)}dx \\ &\le \left(\int_{E_\delta} |T_h v - v|dx\right)^{p\sigma}\left(\int_{E_\delta} |T_h v - v|^{\frac{p(1-\sigma)}{1-p\sigma}}dx\right)^{1-p\sigma}.\end{aligned}$$

Choose σ from

$$\frac{p(1-\sigma)}{1-p\sigma} = q \in \left(p, \frac{Np}{(N-p)_+}\right), \qquad \text{i.e.,} \qquad \sigma = \frac{q-p}{p(q-1)}.$$

One verifies that such a choice is possible if q is in the range (10.2) of Chapter 2, for $\alpha = 1$. Therefore

$$\int_{E_\delta} |T_h v - v|^p dx \le \gamma\delta^{p\sigma}\|f\|_{p,E}^{p(1-\sigma)}.$$

On the other hand

$$\begin{aligned}\int_{E-E_\delta} |T_h v - v|^p dx &\le 2^p|E - E_\delta|^{\frac{q-p}{q}}\left(\int_E |v|^q dx\right)^{p/q} \\ &\le \gamma|E - E_\delta|^{\frac{q-p}{q}}\|f\|_{p,E}^p.\end{aligned}$$

Since ∂E is of class C^1, there exists a constant γ depending on N and the structure of ∂E such that $|E - E_\delta| \le \gamma\delta$. Combining these estimates

$$\|T_h v - v\|_{p,E} = \|T_h A_F f - A_F f\|_{p,E} \le \gamma\delta^{\frac{q-p}{pq}}(1 + \|f\|_{p,E}). \qquad \blacksquare$$

10 Compactness of A_Φ in $L^p(E)$ for $1 \le p < \infty$

Since $\Phi(x;\cdot)$ is harmonic in E and equals $F(x;\cdot)$ on ∂E, possible singularities of $\Phi(x;y)$ occur for $x \in \partial E$. Therefore, for $\delta > 0$ fixed, $\Phi(\cdot;y) \in C^\infty(E_\delta)$ uniformly in $y \in \bar{E}$. If $\{f_n\}$ is a bounded sequence in $L^p(E)$, then the sequence $\{A_\Phi f_n\}$ is equi-bounded and equi-continuous in $\bar{E}_\delta$. By the Ascoli–Arzelà theorem, a subsequence can be selected, and relabeled with n, such that for every $\varepsilon > 0$, there exists a positive integer $n(\varepsilon)$ such that

$$\|A_\Phi f_n - A_\Phi f_m\|_{\infty,E_\delta} \le \varepsilon \qquad \text{for } n, m \ge n(\varepsilon). \tag{10.1}$$

The selection of the subsequence depends on δ. Let $\{\delta_j\}$ be a sequence of positive numbers decreasing to zero, and let $\{A_\Phi f_{n_j}\}$ be the subsequence, out of $\{A_\Phi f_n\}$, for which (10.1) holds within E_{δ_j}. By diagonalization, one may select a subsequence, and relabel it with n, such that $\{A_\Phi f_n\}$ is a Cauchy sequence in $C(\bar{E}_{\delta_j})$ for each fixed $j = 1, 2, \dots$. We claim that $\{A_\Phi f_n\}$ is a Cauchy sequence in $L^p(E)$. Fix $\varepsilon > 0$ arbitrarily small and $j \in \mathbb{N}$ arbitrarily large. There exists a positive integer $m(\varepsilon, j)$ such that

$$\|A_\Phi f_n - A_\Phi f_m\|_{\infty,E_{\delta_j}} \le \varepsilon \qquad \text{for } n,\, m \ge m(\varepsilon, j).$$

Next, for $n, m \ge m(\varepsilon, j)$

$$\begin{aligned}\|A_\Phi f_n - A_\Phi f_m\|_{p,E}^p &= \int_{E_{\delta_j}} |A_\Phi f_n - A_\Phi f_m|^p dx + \int_{E-E_{\delta_j}} |A_\Phi f_n - A_\Phi f_m|^p dx \\ &\le \varepsilon^p |E| + \int_{E-E_{\delta_j}} \Big| \int_E \Phi(x;y)|f_n(y) - f_m(y)|dy \Big|^p dx.\end{aligned}$$

Let us assume that $N \ge 3$, the proof for $N = 2$ being similar. By Proposition 12.1 of Chapter 2 the last integral is majorized by

$$\left(\int_E \Big| \int_E F(x;y)|f_n(y) - f_m(y)|dy \Big|^q dx \right)^{p/q} |E - E_{\delta_j}|^{1-p/q}$$

for some $q > p$. ■

11 Compactness of A_Φ in $L^\infty(E)$

Lemma 11.1 *Let $f \in L^\infty(E)$. Then $A_\Phi f$ is Hölder continuous in $\bar{E}$. Namely, there exist constants $\gamma > 1$ and $0 < \eta \le 1$, that can be determined a priori in terms of N and the structure of ∂E only such that*

$$|A_\Phi f(x_1) - A_\Phi f(x_2)| \le \gamma \|f\|_{\infty,E} |x_1 - x_2|^\eta \quad \textit{for all } x_1, x_2 \in \bar{E}.$$

Proof. The points $x_i \in \bar{E}$ being fixed, set $\delta = |x_1 - x_2|^\alpha$, where $\alpha > 0$ is to be chosen, and denote by $B_\delta(\bar{x})$ the ball of radius δ centered at $\bar{x} = \frac{1}{2}(x_1 + x_2)$. For such a δ, let E_δ be defined as in (9.1), where without loss of generality we may assume that δ is so small that $E_{4\delta} \neq \emptyset$. Assume first that $B_\delta(\bar{x}) \subset E_{2\delta}$. Then

$$\begin{aligned}
A_\Phi f(x_1) - A_\Phi f(x_2) &= \int_E [\Phi(x_1; y) - \Phi(x_2; y)] f(y) dy \\
&= \int_E \left(\int_0^1 \frac{d}{ds} \Phi(sx_1 + (1-s)x_2; y) ds \right) f(y) dy \\
&= (x_1 - x_2) \cdot \int_0^1 \nabla_x \left(\int_E \Phi(sx_1 + (1-s)x_2; y) f(y) dy \right) ds.
\end{aligned}$$

The function $A_\Phi f$ is harmonic in $B_{2\delta}(\bar{x})$, and $sx_1 + (1-s)x_2 \in B_\delta(\bar{x})$ for all $s \in [0,1]$. Therefore by Theorem 5.2 of Chapter 2

$$\begin{aligned}
\left| \nabla_x \int_E \Phi(x; y) f(y) dy \right| &\le \frac{\gamma}{\delta} \sup_{x \in B_{2\delta}(\bar{x})} \left| \int_E \Phi(x; y) f(y) dy \right| \\
&\le \frac{\gamma}{\delta} \|f\|_{\infty,E} \sup_{x \in E} \int_E F(x; y) dy \le \frac{\gamma}{\delta} \|f\|_{\infty,E}.
\end{aligned}$$

Combining these estimates

$$|A_\Phi f(x_1) - A_\Phi f(x_2)| \le \tilde{\gamma} \|f\|_{\infty,E} \frac{|x_1 - x_2|}{\delta}$$

provided $B_\delta(\bar{x}) \subset E_{2\delta}$. Assume now that $B_\delta(\bar{x}) \cap (E - E_{2\delta}) \neq \emptyset$, and write

$$\begin{aligned}
A_\Phi f(x_1) - A_\Phi f(x_2) &= \int_E [\Phi(x_1; y) - \Phi(x_2; y)] f(y) dy \\
&= \int_{E_{4\delta}} [\Phi(x_1; y) - \Phi(x_2; y)] f(y) dy \\
&\quad + \int_{E - E_{4\delta}} [\Phi(x_1; y) - \Phi(x_2; y)] f(y) dy = I_1 + I_2.
\end{aligned}$$

$$\begin{aligned}
I_1 &= (x_1 - x_2) \cdot \int_0^1 \nabla_x \left(\int_{E_{4\delta}} \Phi(sx_1 + (1-s)x_2; y) f(y) dy \right) ds \\
&= (x_1 - x_2) \cdot \int_0^1 \left(\int_{E_{4\delta}} \nabla_y \Phi(sx_1 + (1-s)x_2; y) f(y) dy \right) ds
\end{aligned}$$

by symmetry of $\Phi(\cdot;\cdot)$. Since $y \to \Phi(sx_1 + (1-s)x_2; y)$ is harmonic in $E_{4\delta}$, by Theorem 5.2 of Chapter 2

$$\begin{aligned}
|\nabla \Phi(sx_1 + (1-s)x_2; y)| &\le \frac{\gamma}{\delta} \sup_{z \in B_\delta(y)} \Phi(sx_1 + (1-s)x_2; z) \\
&\le \frac{\gamma}{\delta} \sup_{z \in B_\delta(y)} F(sx_1 + (1-s)x_2; z) \le \frac{\hat{\gamma}}{\delta^{N-1}}.
\end{aligned}$$

Therefore

$$|I_1| \le \tilde{\gamma}\|f\|_{\infty,E}\frac{|x_1 - x_2|}{\delta^{N-1}}.$$

Next

$$\begin{aligned}|I_2| &\le \int_{E-E_{4\delta}} [\Phi(x_1;y) + \Phi(x_2;y)]|f(y)|dy \\ &\le \int_{E-E_{4\delta}} [F(x_1;y) + F(x_2;y)]|f(y)|dy \\ &\le \gamma\|f\|_{\infty,E}|E - E_{4\delta}|^{1/N} \le \gamma\|f\|_{\infty,E}\delta^{1/N}.\end{aligned}$$

Therefore if $B_\delta(\bar{x}) \cap (E - E_{2\delta}) \neq \emptyset$

$$|A_\Phi f(x_1) - A_\Phi f(x_2)| \le \tilde{\gamma}\Big(\frac{|x_1 - x_2|}{\delta^{N-1}} + \delta^{1/N}\Big).$$

The remaining cases are treated by inserting between x_1 and x_2, finitely many points $\{\xi_1, \dots, \xi_n\}$, so that each of the pairs (x_1, ξ_1), (ξ_j, ξ_{j+1}), and (ξ_n, x_2) falls in one of the previous cases. The number n will depend on the structure of ∂E. ■

Corollary 11.1 *A_G is compact in $L^\infty(E)$.*

Corollary 11.2 *Let $u \in L^2(E)$ be a solution of (8.2). Then $u \in C^\eta(\bar{E})$ for some $\eta > 0$.*

Proof. Applying Proposition 10.1 of Chapter 2 a finite number of times implies that $u \in L^\infty(E)$. Then the conclusion follows from Lemma 11.1 and Corollary 10.1 of Chapter 2. ■

Remark 11.1 The corollary provides the necessary regularity for the eigenvalue problems (8.1) and (8.2) to be equivalent.

Problems and Complements

2c On the Integral Defining the Double-Layer Potential

2.1. Prove a sharper version of Lemma 2.1. In particular, find the optimal conditions on ∂E to ensure that

$$\frac{(x - y)\cdot \mathbf{n}(y)}{|x - y|^N} \in L^1(\partial E).$$

2.2. Consider the integral in (2.1). As $x_o \to x \in \partial E$, the integrand tends to

$$\frac{(x-y)\cdot \mathbf{n}(y)}{|x-y|^N} v(y) \qquad \text{for a.e. } \ y \in \partial E.$$

Moreover, such a function is in $L^{1+\varepsilon}(\partial E)$ for some $\varepsilon > 0$. This follows from Lemma 2.1. However, the limit cannot be carried under the integral. Explain.

5c The Dirichlet Problem by Integral Equations

5.1. Let $E \subset \mathbb{R}^N$ be open and bounded and with boundary ∂E of class C^1. Formulate the following exterior Dirichlet problem in terms of an integral equation:

$$\begin{gathered} u \in C^2(\mathbb{R}^N - \bar{E}) \cap C(\overline{\mathbb{R}^N - E}), \quad \Delta u = 0 \ \text{ in } \ \mathbb{R}^N - E \\ u\big|_{\partial E} = \varphi \in C(\partial E), \qquad \lim_{|x|\to\infty} u(x) = 0. \end{gathered}$$

Compare with Section 8.1c of the Problems and Complements of Chapter 2.

5.2. In the proof of Proposition 5.2, justify the passage of the limit under the integral. *Hint*:

$$\begin{aligned} &\int_{\partial E} |K_D(x_n;y) - K_D(x_o;y)||v(y)|d\sigma \\ &\qquad\qquad = \int_{\partial E \cap [|y-x_o|<\varepsilon]} |K_D(x_n;y) - K_D(x_o;y)||v(y)|d\sigma \\ &\qquad\qquad\quad + \int_{\partial E \cap [|y-x_o|\ge\varepsilon]} |K_D(x_n;y) - K_D(x_o;y)||v(y)|d\sigma. \end{aligned}$$

5.3. Prove a stronger version of Proposition 5.2. Namely, if $v \in L^\infty(\partial E)$, then $\{A_D v\}$ is Hölder continuous with exponent α/N, where α is the constant appearing in Lemma 2.1. *Hint*: For $x_1, x_2 \in \partial E$

$$\begin{aligned} &|A_D v(x_1) - A_D v(x_2)| \\ &\qquad \le \frac{\|v\|_{\infty,\partial E}}{\omega_N} \int_{\partial E} \left| \frac{(x_1-y)\cdot\mathbf{n}(y)}{|x_1-y|^N} - \frac{(x_2-y)\cdot\mathbf{n}(y)}{|x_2-y|^N} \right| d\sigma. \end{aligned}$$

May assume that $|x_1 - x_2| < 1$ and set

$$\partial_i = \{y \in \partial E \big| |x_i - y| < |x_1 - x_2|^{1/N}\}.$$

Divide the integral into one extended over $\partial_1 \cap \partial_2$ and another one extended over the complement.

6c The Neumann Problem by Integral Equations

6.1. Let $E \subset \mathbb{R}^N$ be open and bounded and with boundary ∂E of class C^1. Formulate the following exterior Neumann problem in terms of an integral equation:

$$u \in C^2(\mathbb{R}^N - \bar{E}) \cap C^1(\overline{\mathbb{R}^N - E}), \qquad \Delta u = 0 \text{ in } \mathbb{R}^N - E$$
$$\frac{\partial}{\partial \mathbf{n}} u\big|_{\partial E} = \varphi \in C(\partial E), \qquad \lim_{|x|\to\infty} u(x) = 0.$$

6.2. Prove a stronger version of Proposition 6.2. Namely, if $v \in L^\infty(\partial E)$, then $\{A_N v\}$ is Hölder continuous with exponent α/N, where α is the constant appearing in Lemma 2.1.

7c The Green's Function for the Neumann Problem

7.1c Constructing $\mathcal{G}(\cdot;\cdot)$ for a Ball in $\mathbb{R}^2$ and $\mathbb{R}^3$

Attempt finding $\mathcal{G}(\cdot;\cdot)$ of the form

$$\mathcal{G}(x;y) = F(x;y) + \underbrace{\Phi(x;y) + h(x;y) + \gamma|y|^2}_{-N(x;y)}$$

where F and Φ are defined in (2.6) and (3.7) of Chapter 2 respectively, γ is a constant, and $h(x;\cdot)$ is a suitable harmonic function in B_R to be chosen to satisfy the last of (7.2). As in Section 7.1 one computes $2N\gamma_o|E| = -1$. Using the explicit expression of Φ

$$\frac{\partial}{\partial \mathbf{n}(y)}(F + \Phi + \gamma|y|^2) = \frac{1}{\omega_N R|x-y|^{N-2}} + 2\gamma_o R.$$

Therefore the harmonic function $h(x;\cdot)$ has to be chosen to satisfy

$$\frac{\partial}{\partial \mathbf{n}(y)} h(x;y) = \frac{-1}{\omega_N R|x-y|^{N-2}} - 2N\gamma_o R.$$

7.1.1c The Case $N = 2$

Choose $h = 0$ and $\gamma_o = -1/4\pi R^2$ to conclude that Green's function for the Neumann problem for the disc $D_R \subset \mathbb{R}^2$ is

$$\mathcal{G}(x;y) = \frac{-1}{2\pi}\left(\ln|\xi - y|\frac{|x|}{R} + \ln|x-y|\right) - \frac{1}{4\pi R^2}|y|^2, \quad \xi = \frac{R^2}{|x|^2}x$$

up to an arbitrary additive smooth function of x.

7.1.2c The Case $N = 3$

The function $h(x;\cdot)$ is given by

$$h(x;y) = \frac{1}{4\pi}\ln \mathcal{H}, \quad \mathcal{H} = \frac{(\xi - y)\cdot x}{|x|} + |\xi - y|, \quad \xi = \frac{R^2}{|x|^2}x.$$

One computes

$$4\pi\nabla h = \frac{-1}{\mathcal{H}}\left(\frac{x}{|x|} + \frac{\xi - y}{|\xi - y|}\right) \tag{$*$}$$

and

$$\begin{aligned} 4\pi\Delta h &= \frac{2}{\mathcal{H}|\xi - y|} - \frac{1}{\mathcal{H}^2}\left(\frac{x}{|x|} + \frac{\xi - y}{|\xi - y|}\right)\cdot\left(\frac{x}{|x|} + \frac{\xi - y}{|\xi - y|}\right) \\ &= \frac{2}{\mathcal{H}|\xi - y|} - \frac{2}{\mathcal{H}^2}\left(1 + \frac{\xi - y}{|\xi - y|}\cdot\frac{x}{|x|}\right) = 0. \end{aligned}$$

From $(*)$, taking into account that $\xi = R^2 x/|x|^2$

$$\begin{aligned} 4\pi\frac{\partial h}{\partial \mathbf{n}(y)} &= 4\pi\nabla h\cdot\frac{y}{R} \\ &= \frac{-1}{R\mathcal{H}}\left(\frac{x\cdot y}{|x|} + \frac{(\xi - y)\cdot(y - \xi)}{|\xi - y|} + \frac{(\xi - y)\cdot\xi}{|\xi - y|}\right) \\ &= \frac{1}{R\mathcal{H}}\left((\xi - y)\cdot\frac{x}{|x|} + |\xi - y| - \frac{x}{|x|}\cdot\xi - \frac{(\xi - y)\cdot\xi}{|\xi - y|}\right) \\ &= \frac{1}{R\mathcal{H}}\left(\mathcal{H} - \frac{\left[(\xi - y)\cdot\frac{x}{|x|} + |\xi - y|\right]}{|\xi - y|}\frac{R^2}{|x|}\right) = \frac{1}{R} - \frac{1}{|x - y|}. \end{aligned}$$

The last equality follows from (3.6) of Chapter 2, since $y \in \partial B_R$. We conclude that Green's function for the Neumann problem for the ball $B_R \subset \mathbb{R}^3$ is

$$\begin{aligned} \mathcal{G}(x;y) =& \frac{1}{4\pi}\left(\frac{1}{|x - y|} + \frac{R}{|x|}\frac{1}{|\xi - y|}\right) \\ &+ \frac{1}{4\pi}\ln\left((\xi - y)\cdot\frac{x}{|x|} + |\xi - y|\right) - \frac{1}{8\pi R^3}|y|^2 \end{aligned}$$

up to an arbitrary additive smooth function of x.

8c Eigenvalue Problems

8.1. Formulate the homogeneous Neumann eigenvalue problem

$$\Delta u = -\lambda u \ \text{ in } \ E, \quad \tfrac{\partial}{\partial \mathbf{n}}u = 0 \ \text{ on } \ \partial E. \tag{8.1c}$$

8.2. Find eigenvalues and eigenfunctions of (8.1) and (8.1c), when E is a parallelepiped in $\mathbb{R}^3$ of sides a, b, c. Do the same for a disc $D_R \subset \mathbb{R}^2$.

8.3. Let $A_{\mathcal{G}}$ be defined as in (8.3) with G replaced by $\mathcal{G}$. Prove the analogue of Theorem 8.1.

4

INTEGRAL EQUATIONS AND EIGENVALUE PROBLEMS

1 Kernels in $L^2(E)$

Let E be a bounded open set in $\mathbb{R}^N$ with boundary ∂E of class C^1. For complex-valued f and g in $L^2(E)$, set

$$\langle f, g\rangle = \int_E f\bar{g}\,dx \qquad \text{and} \qquad \|f\|^2 = \langle f, f\rangle$$

and say that f and g are orthogonal if $\langle f, g\rangle = 0$. A complex-valued $dx \times dx$-measurable function $K(\cdot;\cdot)$ defined in $E \times E$ is a kernel acting in $L^2(E)$ if the two operators

$$Af = \int_E K(\cdot; y)f(y)dy, \qquad A^*f = \int_E \bar{K}(y;\cdot)f(y)dy$$

map bounded subsets of $L^2(E)$ into bounded subsets of $L^2(E)$, equivalently, if there is a constant γ depending only upon N and E such that

$$\|Af\| \le \gamma\|f\| \quad \text{and} \quad \|A^*f\| \le \gamma\|f\|, \quad \text{for all } f \in L^2(E).$$

It would be sufficient to require only one of these, since any of them implies the other. Indeed, assuming that the first holds

$$\begin{aligned}|\langle A^*f, g\rangle| &= \left|\int_E \left(\int_E \bar{K}(y;x)f(y)dy\right)\bar{g}(x)dx\right| \\ &= \left|\int_E f(y)\overline{\left(\int_E K(y;x)g(x)dx\right)}dy\right| = |\langle f, Ag\rangle| \le \gamma\|f\|\|g\|.\end{aligned}$$

The operators A and A^* are adjoint in the sense that

$$\langle Af, g\rangle = \langle f, A^*g\rangle \quad \text{for all } f, g \in L^2(E).$$

Their norm is defined by

© Springer Nature Switzerland AG 2023
E. DiBenedetto, U. Gianazza, *Partial Differential Equations*,
Cornerstones, https://doi.org/10.1007/978-3-031-46618-2_5

$$\|A\| = \sup_{\|f\|=1} \|Af\|, \qquad \|A^*\| = \sup_{\|f\|=1} \|A^* f\|.$$

By the characterization of the $L^2(E)$-norm ([50], Chapter V, Section 4)

$$\begin{aligned}
\|A\| &= \sup_{\|f\|=1} \sup_{\|g\|=1} \left| \int_E \left(\int_E K(x;y) f(y) dy \right) \bar{g}(x) dx \right| \\
&= \sup_{\|f\|=1} \sup_{\|g\|=1} \left| \int_E f(y) \overline{\left(\int_E \bar{K}(x;y) g(x) dx \right)} dy \right| \\
&\le \sup_{\|f\|=1} \sup_{\|g\|=1} \|f\| \left(\int_E \left| \int_E \bar{K}(x;y) g(x) dx \right|^2 dy \right)^{1/2} \\
&= \sup_{\|g\|=1} \left(\int_E \left| \int_E \bar{K}(x;y) g(x) dx \right|^2 dy \right)^{1/2} = \|A^*\|.
\end{aligned}$$

A similar calculation gives $\|A^*\| \le \|A\|$. Thus $\|A\| = \|A^*\|$.

A kernel $K(\cdot;\cdot)$ in $L^2(E)$ is *compact* if the resulting operators A and A^* are compact in $L^2(E)$, i.e., if they map bounded subsets of $L^2(E)$ into precompact subsets of $L^2(E)$. If A is compact, A^* is also compact.

A kernel $K(\cdot;\cdot)$ in $L^2(E)$ is *symmetric* if $K(x;y) = K(y,x)$ for a.e. $(x,y) \in E \times E$. If it is symmetric and real-valued, then $A = A^*$.

1.1 Examples of Kernels in $L^2(E)$

Given two n-tuples $\{\varphi_1, \dots, \varphi_n\}$ and $\{\psi_1, \dots, \psi_n\}$ of linearly independent, complex-valued functions in $L^2(E)$, set

$$K(x;y) = \sum_{i=1}^{n} \varphi_i(x) \bar{\psi}_i(y) \quad \text{for a.e. } (x,y) \in E \times E. \tag{1.1}$$

A kernel of this kind is called *separable*, or of *finite rank*, or *degenerate*. For such a kernel for any $f \in L^2(E)$

$$\int_E K(\cdot;y) f(y) dy = \sum_{i=1}^{n} \varphi_i \langle f, \psi_i \rangle.$$

Thus separable kernels are compact, but need not be symmetric. The Green function $G(\cdot;\cdot)$ for the Laplacean with homogeneous Dirichlet data is a real-valued, symmetric, compact kernel in $L^2(E)$ (see Theorem 8.1 of Chapter 3). This last example shows that a kernel $K(\cdot;\cdot)$ in $L^2(E)$ need not be in $L^2(E \times E)$.

1.1.1 Kernels in $L^2(\partial E)$

If ∂E is of class $C^{1,\alpha}$ for some $\alpha > 0$, one might consider complex-valued kernels defined and measurable in $\partial E \times \partial E$ and introduce in a similar manner integral operators A and A^* for such kernels, where the integrals are over ∂E for the Lebesgue surface measure on it. Examples of such kernels are the Dirichlet kernel K_D introduced in (5.3) and the Neumann kernel K_N introduced in (6.4) of Chapter 3. The corresponding operators A_D and A_N are introduced in (5.4) and (6.5) of the same Chapter. By Corollary 5.1 and Corollary 6.1 they map $L^2(\partial E)$ into $L^2(\partial E)$. The kernels K_D and K_N are real-valued but not symmetric.

2 Integral Equations in $L^2(E)$

A Fredholm integral equation of the second kind in $L^2(E)$ is an expression of the form ([80])

$$u = \lambda \int_E K(\cdot; y) u(y) dy + f \tag{2.1}$$

where λ is a complex parameter, f is a complex-valued function in $L^2(E)$, and $K(\cdot;\cdot)$ is a complex-valued kernel in $L^2(E)$. A solution of (2.1) is a complex-valued function $u \in L^2(E)$ for which (2.1) holds a.e. in E.

To the integral equation (2.1) associate the homogeneous, and the adjoint homogeneous, equations

$$\mathcal{U} = \lambda \int_E K(x; y) \mathcal{U}(y) dy, \qquad \mathcal{V} = \overline{\lambda} \int_E \bar{K}(y; x) \mathcal{V}(y) dy. \tag{2.2}$$

The general solution of (2.1) is the sum of a particular solution and a solution of the associated homogeneous equation.

Lemma 2.1 *The integral equation (2.1) has at most one solution if and only if $\mathcal{U} = 0$ is the only solution of the associated homogeneous equation (2.2).*

Denoting by $\mathbb{I}$ the identity operator, (2.1) can be rewritten in the operator form

$$(\mathbb{I} - \lambda A) u = f. \tag{2.3}$$

2.1 Existence of Solutions for Small $|\lambda|$

Theorem 2.1. *Let λ and $K(\cdot;\cdot)$ satisfy $|\lambda| \|A\| < 1$. Then for every $f \in L^2(E)$, there exists a solution to the integral equation (2.1). The solution is unique if the associated homogeneous equation (2.1) admits only the trivial solution.*

Proof. If $|\lambda|$ is small, a first approximation to a possible solution u is $u_o = f$. Then progressively improve the approximation by setting

$$u_n = \lambda A u_{n-1} + f \qquad n = 1, 2, \dots. \tag{2.4}$$

Set $A^o = \mathbb{I}$ and $A^n = AA^{n-1}$ for $n \in \mathbb{N}$, and estimate

$$\begin{aligned}\|A^n\| &= \sup_{\|f\|=1} \|A^n f\| = \sup_{\|f\|=1} \|A(A^{n-1}f)\| \\ &= \sup_{\|f\|=1} \frac{\|A(A^{n-1}f)\|}{\|A^{n-1}f\|}\|A^{n-1}f\| \le \|A\|\|A^{n-1}\|.\end{aligned}$$

By iteration, $\|A^n\| \le \|A\|^n$ for all $n \in \mathbb{N}$. With this symbolism, the approximating solutions u_n take the explicit form

$$u_n = \sum_{i=0}^{n} \lambda^i A^i f. \tag{2.5}$$

From this, for every pair of positive integers $n > m$

$$\|u_n - u_m\| \le \|f\| \sum_{i=m+1}^{n} |\lambda|^i \|A\|^i \to 0 \quad \text{as } m, n \to \infty.$$

Therefore $\{u_n\}$ is a Cauchy sequence in $L^2(E)$ and we let u denote its limit. Also

$$\|Au_n - Au\| \le \|A\|\|u_n - u\| \to 0 \quad \text{as } n \to \infty.$$

Therefore $\{Au_n\} \to Au$ in $L^2(E)$. To prove the theorem, we let $n \to \infty$ in (2.4), in the sense of $L^2(E)$. ■

Motivated by the convergence of the series $\sum |\lambda|^i \|A\|^i$ and by the formal symbolism of (2.5), write

$$u = \sum \lambda^i A^i f \overset{\text{def}}{=} (\mathbb{I} - \lambda A)^{-1} f. \tag{2.6}$$

The operator $(\mathbb{I} - \lambda A)^{-1} : L^2(E) \to L^2(E)$ is called the *resolvent*, and it satisfies

$$(\mathbb{I} - \lambda A)(\mathbb{I} - \lambda A)^{-1} = (\mathbb{I} - \lambda A)^{-1}(\mathbb{I} - \lambda A) = \mathbb{I}.$$

Since $\langle A^n f, g\rangle = \langle f, A^{*n} g\rangle$ for all $f, g \in L^2(E)$, there also hold

$$\langle (\mathbb{I} - \lambda A)^{-1} f, g\rangle = \langle f, (\mathbb{I} - \bar{\lambda} A^*)^{-1} g\rangle. \tag{2.7}$$

3 Separable Kernels

If the kernel $K(\cdot;\cdot)$ is of finite rank, as in (1.1), rewrite (2.1) in the form

$$u - f = \lambda \sum_{i=1}^{n} \varphi_i \int_E (u - f)\bar{\psi}_i \, dy + \lambda \sum_{i=1}^{n} \varphi_i \int_E f \bar{\psi}_i \, dy. \tag{3.1}$$

The associated homogeneous and adjoint homogeneous equations are

$$\mathcal{U} = \lambda \sum_{i=1}^{n} \varphi_i \int_E \mathcal{U} \bar{\psi}_i \, dy, \qquad \mathcal{V} = \bar{\lambda} \sum_{i=1}^{n} \psi_i \int_E \mathcal{V} \bar{\varphi}_i \, dy. \tag{3.2}$$

3.1 Solving the Homogeneous Equations

Solutions of (3.2) are of the form $\mathcal{U} = \sum_{i=1}^{n} w_i \varphi_i$ and $\mathcal{V} = \sum_{i=1}^{n} \tilde{w}_i \psi_i$ where the numbers $w_i = \langle \mathcal{U}, \psi_i \rangle$ and $\tilde{w}_i = \langle \mathcal{V}, \varphi_i \rangle$ are to be determined. Putting this form of $\mathcal{U}$ into the first of (3.2) gives

$$\begin{aligned} \sum_{i=1}^{n} w_i \varphi_i &= \lambda \sum_{i=1}^{n} \varphi_i \int_E \Bigg(\sum_{j=1}^{n} w_j \varphi_j(y) \Bigg) \bar{\psi}_i(y) dy \\ &= \lambda \sum_{i=1}^{n} \sum_{j=1}^{n} w_j \varphi_i \int_E \varphi_j \bar{\psi}_i dy = \lambda \sum_{i=1}^{n} \Bigg(\sum_{j=1}^{n} a_{ij} w_j \Bigg) \varphi_i \end{aligned}$$

where $a_{ij} = \langle \varphi_j, \psi_i \rangle$. Since the set of functions $\{\varphi_i\}_1^n$ is linearly independent, this leads to the linear system

$$[\mathbb{I} - \lambda(a_{ij})]\mathbf{w} = 0, \qquad \mathbf{w} = (w_1, \dots, w_n)^t. \tag{3.3}$$

Analogously, putting the form of $\mathcal{V}$ into the second of (3.2) and taking into account that the set $\{\psi_i\}_1^n$ is linearly independent leads to the linear system

$$[\mathbb{I} - \bar{\lambda}(\bar{a}_{ji})]\tilde{\mathbf{w}} = 0, \qquad \tilde{\mathbf{w}} = (\tilde{w}_1, \dots, \tilde{w}_n)^t. \tag{3.3*}$$

Let r be the rank of $[\mathbb{I} - \lambda(a_{ij})]$. If $r = n$ then $\det[\mathbb{I} - \lambda(a_{ij})] \neq 0$ and (3.3)–(3.3)* have only the trivial solution. Otherwise, the systems have $n-r$ linearly independent solutions, say $\mathbf{w}_j$ and $\tilde{\mathbf{w}}_j$ for $j = 1, \dots, n-r$, and (3.2) have, respectively, the $n-r$ solutions

$$\mathcal{U}_j = \sum_{i=1}^{n} w_{ij} \varphi_i, \qquad \mathcal{V}_j = \sum_{i=1}^{n} \tilde{w}_{ij} \psi_i, \qquad j = 1, \dots, n-r.$$

3.2 Solving the Inhomogeneous Equation

Solutions to (3.1) are of the form $u - f = \sum_{i=1}^{n} v_i \varphi_i$ where the complex numbers v_i are to be determined from (3.1). Putting this in (3.1), and setting $f_i = \langle f, \psi_i \rangle$, yields the linear system

$$[\mathbb{I} - \lambda(a_{ij})]\mathbf{v} = \lambda \mathbf{f}.$$

If $\det[\mathbb{I} - \lambda(a_{ij})] \neq 0$, then for every $\mathbf{f} \in \mathbb{R}^N$, this system admits a unique solution. Otherwise, the system is solvable if and only if $\mathbf{f}$ is orthogonal to the $(n-r)$-dimensional subspace spanned by the solutions of (3.3)*, that is, if and only if

$$\begin{aligned} 0 = \mathbf{f} \cdot \tilde{\mathbf{w}}_j &= \sum_{i=1}^{n} \int_E f \overline{\tilde{w}}_{ij} \bar{\psi}_i \, dy = \int_E f \overline{\sum_{i=1}^{n} \tilde{w}_{ij} \psi_i} \, dy \\ &= \int_E f \bar{\mathcal{V}}_j dy = \langle f, \mathcal{V}_j \rangle, \qquad j = 1, \dots, n-r. \end{aligned}$$

Theorem 3.1. *Let* $K(\cdot;\cdot)$ *be separable. Then the integral equation (2.1) is solvable if and only if* f *is orthogonal to all the solutions of the adjoint homogeneous equation (2.2). In particular, if* λ *is not an eigenvalue of the matrix* (a_{ij})*, then (2.1) is uniquely solvable for every* $f \in L^2(E)$*.*

More generally, one might consider separable kernels of the type

$$K_{\text{sep}}(\cdot;\cdot) = \sum_{i=1}^{n} \varphi_i(\cdot;\lambda)\overline{\psi}_i(\cdot;\lambda) \tag{3.4}$$

where

$$\lambda \to a_{ij}(\lambda) = \int_E \varphi_i(y;\lambda)\bar{\psi}_j(y;\lambda)dy \tag{3.5}$$

are analytic functions of λ in the complex plane $\mathbb{C}$. Then $\det[\mathbb{I} - \lambda\big(a_{ij}(\lambda)\big)]$ can vanish only at isolated points of $\mathbb{C}$. Therefore, for such kernels, (2.1) is uniquely solvable for every $f \in L^2(E)$ except for isolated values of λ in $\mathbb{C}$.

Remark 3.1 The theorem, due to Fredholm, discriminates between those values of λ that are eigenvalues of (a_{ij}) and the remaining ones. For this reason it is referred to as the Fredholm alternative ([79, 80], see also Mikhlin [183] and Tricomi [260]).

4 Small Perturbations of Separable Kernels

Consider the integral equation (2.1) for kernels of the form

$$K(\cdot;\cdot) = K_{\text{sep}}(\cdot;\cdot) + K_o(\cdot;\cdot) \tag{4.1}$$

where $K_{\text{sep}}(\cdot;\cdot)$ is separable and $K_o(\cdot;\cdot)$ is a kernel in $L^2(E)$. Setting

$$A_o f = \int_E K_o(\cdot;y)f(y)dy, \qquad A_o^* f = \int_E \bar{K}_o(y;\cdot)f(y)dy$$

the *perturbation* $K_o(\cdot;\cdot)$ is said to be small if

$$|\lambda|\|A_o\| < 1 \quad \text{and} \quad |\lambda|\|A_o^*\| < 1. \tag{4.2}$$

This implies that the resolvent $(\mathbb{I} - \lambda A_o)^{-1}$ is well defined in the sense of (2.6) and permits one to rewrite (2.1) in the form

$$\begin{aligned}(\mathbb{I} - \lambda A_o)u &= \lambda \int_E K_{\text{sep}}(\cdot;y)u\,dy + f \\ &= \lambda \sum_{i=1}^{n} \varphi_i \int_E \bar{\psi}_i(\mathbb{I} - \lambda A_o)^{-1}(\mathbb{I} - \lambda A_o)u\,dy + f.\end{aligned}$$

Set $z = (\mathbb{I} - \lambda A_o)u$ and observe that by (2.7)

$$\int_E \bar{\psi}_i (\mathbb{I} - \lambda A_o)^{-1} (\mathbb{I} - \lambda A_o) u \, dy = \int_E \overline{(\mathbb{I} - \bar{\lambda} A_o^*)^{-1} \psi_i} \, z \, dy.$$

Therefore, solving the integral equation is equivalent to solving

$$z = \lambda \sum_{i=1}^n \varphi_i \int_E \overline{(\mathbb{I} - \bar{\lambda} A_o^*)^{-1} \psi_i} \, z \, dy + f. \tag{4.3}$$

This, in turn, has the associated adjoint homogeneous equation

$$\mathcal{V} = \bar{\lambda} \sum_{i=1}^n (\mathbb{I} - \bar{\lambda} A_o^*)^{-1} \psi_i \int_E \bar{\varphi}_i \mathcal{V} \, dy \tag{4.4}$$

which can be rewritten as

$$(\mathbb{I} - \bar{\lambda} A_o^*) \mathcal{V} = \bar{\lambda} \sum_{i=1}^n \psi_i \int_E \bar{\varphi}_i \mathcal{V} \, dy$$

or equivalently

$$\mathcal{V} = \bar{\lambda} \sum_{i=1}^n \psi_i \int_E \bar{\varphi}_i \mathcal{V} \, dy + \bar{\lambda} \int_E \bar{K}_o(y; \cdot) \mathcal{V} \, dy = \bar{\lambda} \int_E \bar{K}(y; \cdot) \mathcal{V} \, dy.$$

This is precisely the adjoint homogeneous equation in (2.2) associated with the original kernel $K(\cdot;\cdot)$. Thus $\mathcal{V} \in L^2(E)$ is a solution of (4.4) if and only if it is a solution of the adjoint homogeneous equation

$$\mathcal{V} = \bar{\lambda} A^* \mathcal{V}, \qquad Af = \int_E K(\cdot; y) f \, dy \tag{4.5}$$

associated with the original kernel.

4.1 Existence and Uniqueness of Solutions

By Theorem 3.1, the integral equation (4.3) is solvable if and only if f is orthogonal to all the solutions of the adjoint homogeneous equation (4.4), that is, if and only if f is orthogonal to all the solutions of the adjoint homogeneous equation (4.5). The solution of (4.3) is unique if the associated homogeneous equation

$$v = \lambda \sum_{i=1}^n \varphi_i \int_E v \bar{\psi}_i \, dy + \lambda A_o v \tag{4.6}$$

admits only the trivial solution. We may regard this as an integral equation with separable kernel and with *forcing term* $f = \lambda A_o v$. By Theorem 3.1, such an equation has at most one solution if λ is not an eigenvalue of the matrix (a_{ij}). In such a case, since $v = 0$ solves (4.6), it must be the only solution. Recalling that λ is restricted by the condition (4.2), we conclude that if λ is not an eigenvalue of (a_{ij}) satisfying $|\lambda| < \|A_o\|^{-1}$, then (4.6) has only the trivial solution, and consequently (4.3) has at most one solution.

Analogously, if λ is not an eigenvalue of (a_{ij}) satisfying $|\lambda| < \|A_o\|^{-1}$, then (4.5) has only the trivial solution. Therefore any $f \in L^2(E)$ would be orthogonal to all the solutions of the adjoint homogeneous equation (4.5), and therefore (4.3) is uniquely solvable for every $f \in L^2(E)$.

Theorem 4.1. *If the kernel $K(\cdot;\cdot)$ is a small perturbation of a separable kernel, in the sense of (4.1)–(4.2), the integral equation (2.1) is solvable if and only if f is orthogonal to all the solutions of the adjoint homogeneous equation (4.5). In particular, if λ is not an eigenvalue of (a_{ij}) such that $|\lambda| < \|A_o\|^{-1}$, then (2.1) is uniquely solvable for every $f \in L^2(E)$.*

More generally, one might consider kernels that are small perturbations of a separable kernel of the type of (3.4), with the corresponding functions in (3.5) analytic in the disc $D = \{|\lambda| < \|A_o\|^{-1}\} \subset \mathbb{C}$. Then $\det[\mathbb{I} - \lambda\big(a_{ij}(\lambda)\big)]$ can vanish only at isolated points of D. Therefore, for such kernels, (2.1) is uniquely solvable for every $f \in L^2(E)$ except for isolated values of λ in D.

5 Almost Separable Kernels and Compactness

A kernel $K(\cdot;\cdot)$ in $L^2(E)$ is almost separable if for all $\varepsilon > 0$ it can be decomposed as in (4.1) with $K_{\text{sep}}(\cdot;\cdot)$ separable and a *small* perturbation $K_o(\cdot;\cdot)$ such that

$$\|A_o\| = \sup_{\|f\|=1} \left\| \int_E K_o(\cdot;y) f\, dy \right\| \le \varepsilon. \tag{5.1}$$

The perturbation kernel $K_o(\cdot;\cdot)$ is not required to be compact. It turns out however that the original kernel $K(\cdot;\cdot)$ is compact.

Proposition 5.1 (F. Riesz [213]) *An almost separable kernel $K(\cdot;\cdot)$ in $L^2(E)$ is compact.*

Proof. Let A be the operator generated by $K(\cdot;\cdot)$. It will suffice to prove that every bounded sequence $\{f_n\} \subset L^2(E)$ contains a subsequence $\{f_{n'}\} \subset \{f_n\}$ such that $\{Af_{n'}\}$ is Cauchy in $L^2(E)$. For $j \in \mathbb{N}$, let $A_{\text{sep},j}$ and $A_{o,j}$ be the operators corresponding to the decomposition (4.1) and (5.1) for the choice $\varepsilon = 1/j$. Since $A_{\text{sep},j}$ are compact, select by diagonalization, a subsequence $\{f_{n'}\} \subset \{f_n\}$ such that $\{A_{\text{sep},j} f_{n'}\}$ is convergent for all $j \in \mathbb{N}$. Then

$$\|Af_{n'} - Af_{m'}\| < \|A_{\text{sep},j} f_{n'} - A_{\text{sep},j} f_{m'}\| + 2\|A_{o,j}\| \sup_n \|f_n\|. \qquad \blacksquare$$

Corollary 5.1 *Let $A : L^2(E) \to L^2(E)$ be such that for all $\varepsilon > 0$ it can be decomposed into the sum of a compact operator and a "small perturbation" A_ε of norm $\|A_\varepsilon\| < \varepsilon$. Then A is compact.*

5.1 Solving Integral Equations for Almost Separable Kernels

It follows from Theorem 4.1, and the remarks following it, that (2.1) for an almost separable kernel can always be solved for every $f \in L^2(E)$ except at most finitely many values of λ within the disc $[|z| < \varepsilon^{-1}] \subset \mathbb{C}$. Since $\varepsilon > 0$ is arbitrary, we conclude that solvability is ensured for all $f \in L^2(E)$, except for countably many complex numbers $\{\lambda_n\}$. Every disc $|z| < \varepsilon^{-1}$ of the complex plane contains at most finitely many such exceptional values of λ. Therefore if the sequence $\{\lambda_n\}$ is infinite, then $\{|\lambda_n|\} \to \infty$. Equivalently, for all $\varepsilon > 0$, the adjoint homogeneous equation associated to such almost separable kernels has only the trivial solution, except for finitely many values of λ within the disc $[|z| < \varepsilon^{-1}] \subset \mathbb{C}$. These are the *eigenvalues* of the adjoint homogeneous equation. Since $\varepsilon > 0$ is arbitrary, we conclude:

Theorem 5.1. *The integral equation (2.1) with almost separable kernel $K(\cdot;\cdot)$ is solvable if and only if f is orthogonal to all the solutions of the associated adjoint homogeneous equation in (2.2). Moreover, it is always uniquely solvable for every $f \in L^2(E)$ except for countably many values of λ. These are the eigenvalues of the associated adjoint homogeneous equation (2.2). If the sequence $\{\lambda_n\}$ is infinite, then $\{|\lambda_n|\} \to \infty$.*

5.2 Potential Kernels Are Almost Separable

A *potential* kernel is a measurable function $K(\cdot;\cdot) : E \times E \to \mathbb{R}$ such that

$$|K(x;y)| \le C|x-y|^{-N+\alpha} \qquad \text{for almost all } (x,y) \in E \times E \tag{5.2}$$

for some positive constants C and α. For $\delta > 0$ let

$$K_\delta(x;y) = \begin{cases} \delta^{-N+\alpha} & \text{if } K(x,y) \ge \delta^{-N+\alpha} \\ K(x;y) & \text{if } |K(x;y)| < \delta^{-N+\alpha} \\ -\delta^{-N+\alpha} & \text{if } K(x,y) \le -\delta^{-N+\alpha}. \end{cases}$$

Since $K_\delta(\cdot;\cdot)$ is uniformly continuous in $\bar{E}\times\bar{E}$, by the Weierstrass theorem, for each $\varepsilon > 0$ there exists a polynomial $P_n(x;y)$ in the $2N$ variables $(x,y) \in \bar{E}\times\bar{E}$ such that

$$\|K_\delta - P_n\|_{\infty, E\times E} \le \frac{\varepsilon}{2\sqrt{|E|}}.$$

Writing $K = P_n + (K - P_n)$, the perturbation $K_o = K - P_n$ satisfies (5.1). Indeed, for all $f \in L^2(E)$ of norm $\|f\| = 1$

$$\begin{aligned} \int_E |K_o(\cdot;y)f|dy &\le \int_E |K_\delta(\cdot;y) - P_n(\cdot;y)||f|dy + \int_E |K(\cdot;y) - K_\delta(\cdot;y)||f|dy \\ &\le \frac{\varepsilon}{2} + C\int_{|x-y|<C^{\frac{1}{N-\alpha}}\delta} |x-y|^{-N+\alpha}|f|dy. \end{aligned}$$

It remains to choose δ so small that the $L^2(E)$ norm of the last integral is less than $\varepsilon/2$. This is possible by virtue of Proposition 10.1 of Chapter 2.

Remark 5.1 Analogous considerations can be carried almost verbatim for integral equations set on ∂E such as (5.2) and (6.3) of Chapter 3. By virtue of Lemma 2.1 of that Chapter, the Dirichlet and Neumann kernels K_D and K_N, introduced in (5.3) and (6.4) of Chapter 3 respectively, are potential, almost separable kernels in $\partial E\times\partial E$. For these integral equations, Theorem 5.1 continues to hold with the proper modifications.

The idea of approximating potential kernels by separable ones, appears in E. Schmidt ([227]) and J. Radon ([210]).

6 Applications to the Neumann Problem

Solving the Neumann problem (1.3) of Chapter 2 is equivalent to solving the integral equation (6.3) of Chapter 3. The latter is in turn solvable if and only if the Neumann datum ψ is orthogonal, in the sense of $L^2(\partial E)$, to all the solutions of the adjoint, homogeneous equation

$$\begin{aligned}\frac{1}{2}\mathcal{V}(x) &= \int_{\partial E} K_N(y;x)\mathcal{V}\,d\sigma = -\frac{1}{\omega_N}\int_{\partial E}\frac{(x-y)\cdot\mathbf{n}(y)}{|x-y|^N}\mathcal{V}\,d\sigma \\ &= -\int_{\partial E} K_D(x;y)\mathcal{V}\,d\sigma = -\int_{\partial E}\frac{\partial F(x;y)}{\partial\mathbf{n}}\mathcal{V}\,d\sigma.\end{aligned}\tag{6.1}$$

By Corollary 3.2 of Chapter 3, this is solved by $\mathcal{V}=\text{const}$. If the constants are the only solutions, then the zero-average condition (1.4) of Chapter 2 on the Neumann datum ψ would imply that such a ψ is orthogonal to all the solutions of the homogeneous, adjoint equation (6.1) and thus would provide the characteristic condition of solvability of the Neumann problem. Therefore the sufficient part of Theorem 6.1 of Chapter 3 will be a consequence of the following

Proposition 6.1 *The integral equation (6.1) admits only the constant solutions.*

Proof. Solutions $\mathcal{V}\in L^2(\partial E)$ of (6.1) are continuous in ∂E. Indeed, applying Corollary 10.2 of Chapter 2 a finite number of times, one finds $\mathcal{V}\in L^\infty(\partial E)$. Then $\mathcal{V}\in C(\partial E)$, by Proposition 5.2 of Chapter 3. Let v be the harmonic extension of $\mathcal{V}$ in E and denote by φ the resulting normal derivative of such an extension on ∂E, that is

$$\Delta v = 0 \ \text{ in } \ E, \quad v\big| = \mathcal{V}, \quad \text{and} \quad \varphi \overset{\text{def}}{=} \frac{\partial v}{\partial\mathbf{n}} \ \text{ on } \ \partial E.$$

By the Stokes representation formula (2.7) of Chapter 2

$$E\ni x\to v(x) = -\int_{\partial E}\mathcal{V}\frac{\partial F(x;y)}{\partial\mathbf{n}}\,d\sigma + \int_{\partial E}\varphi F(x;y)\,d\sigma.$$

Letting $x \to \partial E$ and using Proposition 3.1 of Chapter 3, gives

$$\mathcal{V} = \frac{1}{2}\mathcal{V} - \int_{\partial E} K_D(\cdot;y)\mathcal{V}\,d\sigma + \int_{\partial E} \varphi F(\cdot;y)\,d\sigma.$$

Therefore if $\mathcal{V}$ is a solution of (6.1), then

$$\int_{\partial E} \varphi F(x;y)\,d\sigma = 0 \quad \text{for all } x \in \partial E. \tag{6.2}$$

To prove the proposition it suffices to establish that (6.2) implies $\varphi = 0$. Indeed, in such a case, v would be harmonic in E and with zero flux on ∂E; thus $v = \mathcal{V} = \text{const}$. Assume $N \ge 3$ and consider the function

$$H = \int_{\partial E} \varphi F(\cdot;y) d\sigma.$$

It is harmonic in E and it vanishes on ∂E. Therefore it vanishes identically in E. Likewise, it is harmonic in $\mathbb{R}^N - E$, it vanishes on ∂E, and $|H(x)| \to 0$ as $|x| \to \infty$. Therefore $H = 0$ in $\mathbb{R}^N - E$. Fix $x \in \partial E$ and let $\{x^i\}$ and $\{x^e\}$ be sequences of points approaching x respectively from the interior and the exterior of E. Let also $\tilde{W}(\partial E, x_o; \varphi)$ be defined as in (4.1) of Chapter 3. By the previous remarks

$$\tilde{W}(\partial E, x_o; \varphi) = -\nabla H(x_o) \cdot \mathbf{n}(x) = 0 \quad \text{in } x_o \in \mathbb{R}^N - \partial E.$$

On the other hand by the jump condition of Corollary 4.1 of Chapter 3

$$0 = \lim_{x^i \to x} \tilde{W}(\partial E, x^i; \varphi) - \lim_{x^e \to x} \tilde{W}(\partial E, x^e; \varphi) = \varphi(x).$$

For $N = 2$ see Section 6c of the Complements. ∎

7 The Eigenvalue Problem

In what follows we will assume that A is generated by a real-valued, compact, symmetric, almost separable kernel $K(\cdot;\cdot)$ in $L^2(E)$, so that $A = A^*$. Consider the problem of finding nontrivial pairs (λ, u), solutions of

$$u = \lambda A u, \qquad \lambda \in \mathbb{C}, \quad u \in L^2(E). \tag{7.1}$$

The numbers λ are called the *eigenvalues* of the operator A, and the functions u are its corresponding *eigenfunctions.*

Proposition 7.1 *Any two distinct eigenfunctions corresponding to two distinct eigenvalues are orthogonal in $L^2(E)$. Moreover, the eigenvalues of A are real and the eigenfunctions of A are real-valued.*

Remark 7.1 If (λ, u) is a solution pair to (7.1), then also $(\lambda, \mu u)$, for all $\mu \in \mathbb{C}$, are solution pairs. Therefore a more precise statement of the proposition would be that the eigenvalues of A are real, and the eigenfunctions can be taken to be real-valued.

Proof (of Proposition 7.1). Let (λ_i, u_i), $i = 1, 2$, be distinct solution pairs, say

$$u_1 = \lambda_1 A u_1, \quad u_2 = \lambda_2 A u_2, \quad \lambda_1 \neq \lambda_2.$$

Multiply the first by u_2, and integrate over E to obtain

$$\frac{1}{\lambda_1} \int_E u_1 u_2 \, dx = \int_E A u_1 u_2 \, dy = \int_E u_1 A u_2 \, dy = \frac{1}{\lambda_2} \int_E u_1 u_2 \, dy.$$

Therefore $\langle u_1, \bar{u}_2 \rangle = 0$. Let now (λ, u) be a nontrivial solution of (7.1). Then

$$\bar{u} = \overline{\lambda A u} = \bar{\lambda} A \bar{u}.$$

Therefore the pair $(\bar{\lambda}, \bar{u})$ is also a solution of (7.1). If $\lambda \neq \bar{\lambda}$, then $\langle u, u \rangle = \|u\|^2 = 0$. Thus $\lambda = \bar{\lambda}$. Since both u and $\bar{u}$ are eigenfunctions for the same eigenvalue λ, the functions

$$\frac{u + \bar{u}}{2} = \mathrm{Re}(u), \qquad \frac{u - \bar{u}}{2i} = \mathrm{Im}(u)$$

are also eigenfunctions for the same eigenvalue λ. Thus u can be taken to be real. ■

Proposition 7.2 *The operator A admits at most countably many distinct eigenvalues $\{\lambda_n\}$. If the sequence $\{\lambda_n\}$ is infinite, then $\{|\lambda_n|\} \to \infty$. Moreover, to each eigenvalue λ there correspond finitely many, linearly independent eigenfunctions $\{u_{\lambda,1}, \dots, u_{\lambda,n_\lambda}\}$, for some $n_\lambda \in \mathbb{N}$.*

Proof. Regard (7.1) as an integral equation of the type of (2.1) with $f = 0$. According to Theorem 5.1 this is uniquely solvable except for countably many numbers $\{\lambda_n\}$. If $\lambda \neq \lambda_n$, then $u = 0$ is the only solution. Therefore nontrivial pairs (λ, u) occur for at most countably many values of λ. To prove the second statement, let λ be a fixed eigenvalue of (7.1). Since $K(\cdot;\cdot)$ is real-valued and almost separable, it can be written as in (4.1) with $|\lambda| \|A_o\| < 1$, and the integral equation (7.1) can be rewritten as an analogue of (4.3) for $z = (\mathbb{I} - \lambda A_o) u$, that is

$$z = \lambda \sum_{i=1}^{n} \varphi_i \int_E (\mathbb{I} - \lambda A_o^*)^{-1} \psi_i z \, dy. \tag{7.2}$$

Solutions of this are of the form

$$z = \lambda \sum_{i=1}^{n} z_i \varphi_i, \qquad \text{where} \qquad z_i = \int_E (\mathbb{I} - \lambda A_o^*)^{-1} \psi_i z \, dy.$$

Multiply (7.2) by $(\mathbb{I}-\lambda A_o^*)^{-1}\psi_j$ and integrate over E to arrive at the algebraic system

$$z_j = \lambda a_{ij} z_i, \qquad a_{ij} = \int_E \varphi_i(\mathbb{I}-\lambda A_o^*)^{-1}\psi_j dy.$$

This has at most n linearly independent vector solutions $(z_{1,j},\dots,z_{n,j})$ for $j=1,\dots,n$. Accordingly, (7.2) has finitely many solutions, and (7.1) has finitely many linearly independent solutions $u=(\mathbb{I}-\lambda A_o)z$ for a given λ. ■

An eigenvalue λ of A is *simple* if to λ there corresponds only one eigenfunction u up to a multiplicative constant μ. The eigenvalues of (7.1) need not be simple. To an eigenvalue λ there corresponds a maximal set of linearly independent eigenfunctions $\{v_{\lambda,1},\dots,v_{\lambda,n_\lambda}\}$. Any linear combination of these is an eigenfunction for the same eigenvalue λ. We let

$$\mathcal{E}_\lambda = \{\text{the linear span of the eigenvectors of } \lambda\}.$$

By the Gram–Schmidt orthonormalization procedure we may arrange for $\mathcal{E}_\lambda$ to be spanned by an orthonormal system of eigenvectors $\{u_{\lambda,1},\dots,u_{\lambda,n_\lambda}\}$.

Corollary 7.1 *The set of eigenfunctions of (7.1) can be chosen to be orthonormal in* $L^2(E)$.

8 Finding a First Eigenvalue and Its Eigenfunctions

Let S_1 be the unit sphere of $L^2(E)$. If (λ,u) is a a nontrivial solution pair of (7.1), by possibly replacing u with $u\|u\|^{-1}$, one may assume that $u\in S_1$. Thus

$$\|A\varphi\|^2 \le \|A\|^2 \text{ for all } \varphi\in S_1 \text{ and } \|Au\|^2=\lambda^{-2}.$$

This suggests that the eigenvalue λ and the eigenfunction u satisfy

$$\sup_{\varphi\in S_1}\|A\varphi\|^2 = \|Au\|^2 = \lambda^{-2} \tag{8.1}$$

and that they can be found by solving such an extremal problem.[1]

Theorem 8.1. *The eigenvalue problem (7.1) admits a nontrivial solution.*

Proof. Select $\varphi_1\in S_1$. If $\|A\varphi_1\|\ge\|A\varphi\|$ for all $\varphi\in S_1$, then the supremum in (8.1) is achieved at φ_1. If not, there exists $\varphi_2\in S_1$ such that $\|A\varphi_2\|>\|A\varphi_1\|$. Proceeding in this manner, we generate a *maximizing* sequence $\{\varphi_n\}\subset S_1$. Since A is compact, one may select out of $\{\varphi_n\}$ a subsequence, relabeled with n, such that $\{\varphi_n\}\to u$ weakly in $L^2(E)$ and $\{A\varphi_n\}\to w$, strongly in $L^2(E)$, and in addition

$$\lim_{n\to\infty}\|A\varphi_n\|^2 = \sup_{\varphi\in S_1}\|A\varphi\|^2 \overset{\text{def}}{=} \lambda^{-2}.$$

---

[1] This idea, due to Hilbert [119], applies to general, linear, symmetric, compact operators in $L^2(E)$; also in F. Reillich [216].

Lemma 8.1 *$u = \lambda^2 A^2 u$.*

Proof. Observe first $A^2\varphi_n \to Aw$ strongly in $L^2(E)$. Indeed

$$\|A^2\varphi_n - Aw\| = \|A[A\varphi_n - w]\| \le \|A\|\|A\varphi_n - w\|.$$

For the assertion, it will suffice to show that

$$\|\varphi_n - \lambda^2 A^2 \varphi_n\| \to 0 \quad \text{as } n \to \infty. \tag{*}$$

Indeed, since $\{A^2\varphi_n\}$ converges to Aw strongly in $L^2(E)$, this would imply that $\{\varphi_n\} \to u$ strongly in $L^2(E)$, and as a consequence $w = Au$ and $Aw = A^2u$. To prove (*) write

$$\|\varphi_n - \lambda^2 A^2\varphi_n\|^2 = \|\varphi_n\|^2 - 2\lambda^2\langle A^2\varphi_n, \bar{\varphi}_n\rangle + \lambda^4\|A^2\varphi_n\|^2.$$

Since A is symmetric $\langle A^2\varphi_n, \bar{\varphi}_n\rangle = \|A\varphi_n\|^2$. Moreover $\|A\varphi_n\|^2 \le \lambda^{-2}$, and

$$\|A^2\varphi_n\|^2 = \frac{\|A(A\varphi_n)\|^2}{\|A\varphi_n\|^2}\|A\varphi_n\|^2 \le \sup_{\varphi\in S_1}\|A\varphi\|^4 \le \frac{1}{\lambda^4}.$$

Combining these estimates

$$\|\varphi_n - \lambda^2 A^2\varphi_n\|^2 \le 2 - 2\lambda^2\|A\varphi_n\|^2 \to 0.$$ ■

To prove the theorem, rewrite the conclusion of Lemma 8.1 as

$$(\mathbb{I} - \lambda^2 A^2)u = (\mathbb{I} + \lambda A)(\mathbb{I} - \lambda A)u = 0.$$

If $(\mathbb{I} - \lambda A)u = 0$, then the pair (λ, u) is a nontrivial solution of the eigenvalue problem (7.1). Otherwise, setting $(\mathbb{I} - \lambda A)u = \psi$, the pair $(-\lambda, \psi)$ solves $(\mathbb{I} - (-\lambda)A)\psi = 0$, and therefore is a nontrivial solution of (7.1). ■

9 The Sequence of Eigenvalues

Let λ_1 be the eigenvalue claimed by Theorem 8.1, denote by $\mathcal{E}_1$ the linear span of all the eigenvectors of λ_1, and let $\mathcal{E}_1^\perp$ be its orthogonal complement. Motivated by the previous maximization procedure, we construct another eigenvalue λ_2 by the formula

$$\sup_{\varphi\in S_1\cap\mathcal{E}_1^\perp}\|A\varphi\|^2 = \lambda_2^{-2}. \tag{9.1}$$

If $A\varphi = 0$ for all $\varphi \in S_1 \cap \mathcal{E}_1^\perp$, then $\lambda_2 = \infty$. Otherwise, we proceed as before and find a nontrivial pair (λ_2, u_2) such that $\lambda_1^2 < \lambda_2^2$ and $u_2 \perp \mathcal{E}_1$. Having constructed the first n eigenvalues $\{\lambda_1, \dots, \lambda_n\}$ and the corresponding eigenspaces $\{\mathcal{E}_1, \dots, \mathcal{E}_n\}$, construct λ_{n+1} by the maximization problem

$$\sup_{\varphi \in S_1 \cap [\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_n]^\perp} \|A\varphi\| = \lambda_{n+1}^{-2}. \tag{9.2}$$

If for some $n \in \mathbb{N}$, $A\varphi = 0$ for all $\varphi \in S_1 \cap [\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_n]^\perp$, then $\lambda_{n+1} = \infty$ and the process terminates. Otherwise, proceeding inductively, we construct a sequence $\{\lambda_n\}$ of eigenvalues such that $\lambda_n^2 < \lambda_{n+1}^2$, and a sequence $\{u_m\}$ of eigenfunctions that can be chosen to form an orthonormal sequence in $L^2(E)$. Such a sequence in general need not be complete. Necessary and sufficient conditions on the kernel $K(\cdot;\cdot)$ to ensure completeness will be given in the next section.

9.1 An Alternative Construction Procedure of the Sequence of Eigenvalues

Having determined λ_1, let $\{u_{\lambda_1,1}, \dots, u_{\lambda_1,n_1}\}$ be a set of real-valued linearly independent orthonormal eigenfunctions spanning the eigenspace $\mathcal{E}_1$, and set

$$K_1(x;y) = \frac{1}{\lambda_1} \sum_{i=1}^{n_1} u_{\lambda_1,i}(x) u_{\lambda_1,i}(y).$$

The kernel $K_1(\cdot;\cdot)$ is symmetric, and the corresponding operator

$$L^2(E) \ni f \to A_1 f = \int_E K_1(\cdot;y) f(y) dy = \frac{1}{\lambda_1} \sum_{i=1}^{n_1} u_{\lambda_1,i} \langle u_{\lambda_1,i}, \bar{f} \rangle$$

is compact and symmetric. Then compute an eigenvalue, λ_2, for the problem

$$u = \lambda(A - A_1)u \tag{9.3}$$

by the previous procedure, that is

$$\sup_{\varphi \in S_1} \|(A - A_1)\varphi\|^2 = \lambda_2^{-2}.$$

Unlike the maximization in (9.1) and (9.2), here the supremum is taken over the entire unit sphere S_1 of $L^2(E)$. However, the operator $(A - A_1)$ is, roughly speaking, *inactive* on the eigenspace $\mathcal{E}_1$. In particular, if $A = A_1$, then $\lambda_2 = \infty$, and the process terminates.

Lemma 9.1 *A pair (λ, u) different from $(\lambda_1, u_{\lambda_1,i})$ for all $i = 1, \dots, n_1$, is a solution of the eigenvalue problem (7.1) if and only if it is a solution of the eigenvalue problem (9.3).*

Proof. If (λ, u) solves (7.1) and $\lambda \neq \lambda_1$, then $u \perp \mathcal{E}_1$, and therefore it is a solution of (9.3). Now let (λ, u) solve (9.3). Multiply both sides by $u_{\lambda_1,i}$ and integrate over E. Since $A - A_1$ is symmetric

$$\begin{aligned} \langle u, u_{\lambda_1,i} \rangle &= \lambda \langle (A - A_1)u, u_{\lambda_1,i} \rangle = \lambda \langle u, (A - A_1)u_{\lambda_1,i} \rangle \\ &= \lambda \Big(\langle u, A u_{\lambda_1,i} \rangle - \frac{\langle u, u_{\lambda_1,i} \rangle}{\lambda_1} \Big) = 0 \end{aligned}$$

since $u_{\lambda_1,i}$ is an eigenfunction for A corresponding to λ_1. This implies that

$$u = \lambda(A - A_1)u = \lambda Au - \lambda \sum_{i=1}^{n_1} u_{\lambda_1,i}\langle u_{\lambda_1,i}, u\rangle = \lambda Au. \qquad \blacksquare$$

Proceeding inductively, construct λ_{n+1} from

$$\sup_{\varphi\in S_1} \Big\|(A - \sum_{j=1}^{n} A_j)\varphi\Big\|^2 = \lambda_{n+1}^{-2} \tag{9.4}$$

where A_j is the compact symmetric operator in $L^2(E)$ corresponding to the kernel

$$K_j(x;y) = \frac{1}{\lambda_j}\sum_{i=1}^{n_j} u_{\lambda_j,i}(x)u_{\lambda_j,i}(y)$$

and $\{u_{\lambda_j,1}, \dots, u_{\lambda_j,n_j}\}$ are real-valued linearly independent orthonormal eigenfunctions corresponding to the eigenvalue λ_j. If, for some $n \in \mathbb{N}$

$$K(x;y) = \sum_{j=1}^{n} K_j(x;y) = \sum_{j=1}^{n} \frac{1}{\lambda_j}\sum_{i=1}^{n_j} u_{\lambda_j,i}(x)u_{\lambda_j,i}(y)$$

set $\lambda_{n+1} = \infty$, and the process terminates. If not, recall that $\{\lambda_n^2\} \to \infty$ and deduce from (9.4) that

$$\limsup_{n\to\infty} \Big\|\big(A - \sum_{j=1}^{n} A_j\big)f\Big\| = 0 \qquad \text{for all } \ f \in L^2(E).$$

Therefore, letting $n \to \infty$ in (9.4) gives a *formal* expansion of the kernel $K(\cdot;\cdot)$, in terms of its eigenvalues and eigenfunctions. Since the right-hand side of (9.4) tends to zero as $n \to \infty$, such an expansion can be rigorously interpreted in the sense of

$$\langle K(x;\cdot), f\rangle = \sum_{j=1}^{\infty} \frac{1}{\lambda_j}\sum_{i=1}^{n_j} u_{\lambda_j,i}(x)\langle u_{\lambda_j,i}, f\rangle \tag{9.5}$$

for all $f \in L^2(E)$. Equivalently

$$\sum_{j=1}^{n} A_j f \to Af \qquad \text{in} \qquad L^2(E). \tag{9.6}$$

10 Questions of Completeness and the Hilbert–Schmidt Theorem

Let $\{\lambda_n\}$ and $\{u_{\lambda_j,i}\}$ be the sequences of eigenvalues and corresponding eigenfunctions of A. If $\{\lambda_n\}$ is finite, say $\{\lambda_1, \dots, \lambda_m\}$ for some $m \in \mathbb{N}$, set $\lambda_j = \infty$ for $j > m$ and $u_{\lambda_j,i} = 0$ for $i = 1, \dots, n_j$ for all $j > m$. Reorder $\{u_{\lambda_j,i}\}$ into a sequence $\{v_n\}$ of real-valued linearly independent orthonormal eigenfunctions, and rewrite (9.5) in the form

$$\langle K(x;\cdot), f\rangle = \sum_{i=1}^{\infty} \frac{1}{\lambda_i} v_i(x)\langle v_i, f\rangle \tag{10.1}$$

where λ_i remains the same as $\{v_i\}$, for $i = 1, \dots, n_{\lambda_i}$ spans the eigenspace $\mathcal{E}_j$. The system $\{v_n\}$ is *complete* if it spans the whole of $L^2(E)$. Equivalently if $[\text{span}\{v_n\}]^{\perp} = \{0\}$, that is if $\langle f, v_i\rangle = 0$ for all i implies $f = 0$. If $\{v_n\}$ is complete then every $f \in L^2(E)$ can be represented as

$$f = \sum_{i=1}^{\infty} \langle f, v_i\rangle v_i \quad \text{in the sense } \Big\| f - \sum_{i=1}^{n} \langle f, v_i\rangle v_i \Big\| \to 0 \text{ as } n \to \infty.$$

The series on the right-hand side is the Fourier series of f.

Proposition 10.1 $Af = 0 \Longleftrightarrow \langle f, v_i\rangle = 0$ *for all* $i \in \mathbb{N}$.

Proof. For fixed (λ_i, v_i), $\langle f, v_i\rangle = \lambda_i\langle Af, v_i\rangle$. This proves the implication $\Longrightarrow$. The converse statement follows from (10.1). ■

Corollary 10.1 *The orthonormal system* $\{v_n\}$ *is complete in* $L^2(E)$ *if and only if* $f \neq 0$ *implies* $Af \neq 0$.

Remark 10.1 If the kernel $K(\cdot;\cdot)$ is of finite rank, then $\{v_n\}$ cannot be complete in $L^2(E)$.

The corollary gives a simple criterion to check the completeness of $\{v_n\}$. We will apply it to the case when $K(\cdot;\cdot)$ is the Green's function $G(\cdot;\cdot)$ for the Laplacean with homogeneous Dirichlet data on $\partial\Omega$.

Proposition 10.2 $L^2(E) \ni f \neq 0 \Longrightarrow \int_E G(\cdot; y) f dy \neq 0$.

Proof. Let $\varphi \in C_o^{\infty}(E)$ and recall the representation of Corollary 3.1 of Chapter 2. For $f \in L^2(E)$

$$\begin{aligned}
\langle Af, \Delta\varphi\rangle &= \int_E \int_E G(x;y) f(y) \Delta\bar{\varphi}(x)\, dydx \\
&= \int_E f(y) \int_E G(x;y)\Delta\bar{\varphi}(x)\, dxdy \\
&= -\int_E f\bar{\varphi} dy = -\langle f, \varphi\rangle.
\end{aligned}$$

■

10.1 The Case of $K(x;\cdot) \in L^2(E)$ Uniformly in x

Assume that $K(\cdot;\cdot)$ is a real-valued compact symmetric kernel acting on $L^2(E)$ that generates an orthonormal system $\{v_n\}$ complete in $L^2(E)$. It is natural to ask whether, or under what conditions, the Fourier series of a function $f \in$

$L^2(E)$ converges to f in some stronger topology, for example in the topology of the uniform convergence in $\bar{E}$. This requires more stringent assumptions on f and on the kernel $K(\cdot;\cdot)$.

Assume that $\|K(x;\cdot)\| \le C$ for some positive constant C uniformly in x. In such a case, in (10.1), we may take $f(x;y) = K(x;y)$ to obtain

$$\sum_{i=1}^{\infty} \frac{1}{\lambda_i^2} v_i^2(x) \le C^2 \qquad \text{for all } \; x \in E. \tag{10.2}$$

Theorem 10.1 (Hilbert–Schmidt). *Let $K(\cdot;\cdot)$ be a real-valued compact symmetric kernel in $L^2(E)$, which generates an orthonormal system $\{v_n\}$ complete in $L^2(E)$ and such that $\|K(x;\cdot)\| \le C$ for some $C > 0$, uniformly in x. Then every function $f \in L^2(E)$ that can be represented as*

$$f = \int_E K(\cdot;y) g\, dy \;\; \textit{for some} \;\; g \in L^2(E) \tag{10.3}$$

has a Fourier series $\sum_{i=1}^{\infty} \langle f, v_i \rangle v_i$, absolutely and uniformly convergent to f in E, that is, $\big\| f - \sum_{i=1}^{n} \langle f, v_i \rangle v_i \big\|_{\infty} \to 0$ as $n \to \infty$.

Proof. It suffices to show that $\big\| \sum_{i=m}^{\infty} |\langle f, v_i \rangle| |v_i| \big\|_{\infty} \to 0$ as $m \to \infty$. This is a consequence of (10.2) and the representation (10.3). Indeed, for all $x \in E$

$$\begin{aligned}\left[\sum_{i=m}^{\infty} |\langle f, v_i \rangle| |v_i(x)| \right]^2 &= \left[\sum_{i=m}^{\infty} |\langle g, v_i \rangle| \frac{|v_i(x)|}{|\lambda_i|} \right]^2 \\ &\le \left[\sum_{i=m}^{\infty} |\langle g, v_i \rangle|^2 \right] \left[\sum_{i=m}^{\infty} \frac{1}{\lambda_i^2} v_i^2(x) \right]. \end{aligned}$$ ■

Remark 10.2 The Green's function $G(\cdot;\cdot)$ satisfies the assumptions of the Hilbert–Schmidt theorem only for $N = 2, 3$.

Corollary 10.2 *Let $N = 2, 3$. Then a function $f \in C_o^1(E) \cap C^2(\bar{E})$ has a Fourier series that converges absolutely and uniformly to f in E.*

Proof. It follows from the Hilbert–Schmidt theorem and the representation of Corollary 3.1 of Chapter 2. ■

11 The Eigenvalue Problem for the Laplacean

Eingenvalues and eigenfunctions for the Laplacean with homogeneous Dirichlet data are those related to (8.1)–(8.2) of the previous Chapter, which were shown to be equivalent.

Theorem 11.1. *The eigenvalues of the Laplacean on a bounded open set* $E \subset \mathbb{R}^N$*, with homogeneous Dirichlet data on* ∂E*, are positive and form a monotone increasing sequence* $\{\lambda_n\} \to \infty$ *as* $n \to \infty$*. Moreover, the corresponding orthonormal system of eigenfunctions* $\{v_n\}$ *is complete in* $L^2(E)$*. Finally, the first eigenfunction can be taken to be positive, and* λ_1 *is simple.*

Proof. If the pair (λ, u) solves (8.1) of Chapter 3, is nontrivial, and $\lambda < 0$, then u cannot take a positive maximum in E. Indeed, if a positive maximum were taken at some $x_o \in E$

$$-\Delta u(x_o) = \lambda u(x_o) < 0.$$

By the same argument, u cannot attain a negative minimum in E. Therefore $u = 0$. The statement about completeness follows from Proposition 10.2.

The maximization process (8.1) implies that if the supremum is achieved for some u_1 then it is achieved also for $|u_1|$. Indeed, since $G(\cdot;\cdot) \ge 0$

$$\lambda_1^{-2} = \|Au_1\|^2 = \left\| \int_E G(\cdot;y)u_1 dy \right\|^2 \le \left\| \int_E G(\cdot;y)|u_1| dy \right\|^2 = \lambda_1^{-2}.$$

Thus u_1 and $|u_1|$ are both eigenfunctions for the same eigenvalue λ_1. In particular

$$-\Delta |u_1| = \lambda_1 |u_1|.$$

This in turn implies that the function

$$E \times \mathbb{R} \ni (x,t) \to w(x,t) = |u_1(x)| e^{\sqrt{\lambda_1} t}$$

is a non-negative harmonic function in the $(N+1)$-dimensional strip $E \times \mathbb{R}$. By the Harnack estimate of Corollary 5.1 of Chapter 2, $|u_1| > 0$ in E, and therefore $u_1 = |u_1|$. We conclude that all the eigenfunctions corresponding to λ_1 can be taken to be positive. In particular, no two of them can be orthogonal. Thus λ_1 is simple. ■

11.1 An Expansion of the Green's Function

Formula (9.5) provides an expansion of the Green's function $G(\cdot;\cdot)$ in terms of its eigenvalues and eigenfunctions. Namely, for all $f \in L^2(E)$ and for a.e. $x \in E$

$$\langle G(x;\cdot), f \rangle = \frac{1}{\lambda_1} u_{\lambda_1}(x) \langle u_{\lambda_1}, f \rangle + \sum_{j=2}^{\infty} \frac{1}{\lambda_j} \sum_{i=1}^{n_j} u_{\lambda_j,i}(x) \langle u_{\lambda_j,i}, f \rangle.$$

Problems and Complements

2c Integral Equations

2.1c Integral Equations of the First Kind

Equation (2.1) was also called an integral equation of the *second* kind. An integral equation of the *first* kind is of the form

$$\int_E K(\cdot;y)u\,dy = f. \tag{2.1c}$$

Here $f \in L^2(E)$ is given, $K(\cdot;\cdot)$ is a kernel in $L^2(E)$, and u is the unknown function. Below we give an example of an integral equation of the first kind.

2.2c Abel Equations ([2, 3])

A particle constrained on a vertical plane falls from rest under the action of gravity along a trajectory γ. On the vertical plane introduce a Cartesian system originating at ground level, and with $\mathbf{j}$ directed along the ascending vertical. If the particle is initially at level x from the ground, we seek the trajectory γ such that it will hit the ground after a time $t = \bar{f}(x)$, where f is a given function. Parametrize γ by the angle θ that the tangent line at points of γ forms with the horizontal axis, taken counterclockwise starting from the positive direction of the horizontal axis.

The speed of the falling particle at level $y \in [0, x]$ is $\sqrt{2g(x-y)}$, where g is the acceleration of gravity. The velocity along $\mathbf{j}$ is

$$\frac{dy}{dt} = -\sqrt{2g(x-y)}\sin\theta$$

or, separating the variables

$$\frac{dy}{\sqrt{2g(x-y)}\sin\theta} = -dt.$$

Integrate the left-hand side from the initial level x to the final level 0, and the right-hand side from the initial time 0 to the final time $\bar{f}(x)$. This gives the Abel integral equation of the first kind

$$\int_0^x \frac{v(y)dy}{\sqrt{x-y}} = f(x) \quad \text{where} \quad v(y) = \frac{1}{\sin\theta} \quad \text{and} \quad f = -\sqrt{2g}\bar{f}. \tag{2.2c}$$

When $f =$const, this is the problem of the *tautochrone* trajectory. More generally, an Abel integral equation takes the form

$$\int_0^x \frac{v(y)dy}{(x-y)^\alpha} = f(x) \tag{2.3c}$$

for some $\alpha > 0$ and $f \in C^1[0,\infty)$. This can be recast in the form (2.1c) as follows. First limit x not to exceed some fixed positive number a. Then set

$$K(x;y) = \begin{cases} (x-y)^{-\alpha} & \text{if } 0 \le y < x \\ 0 & \text{if } x \le y \le a. \end{cases}$$

and rewrite (2.3c) as

$$\int_0^a K(\cdot;y)v dy = f.$$

Kernels of this kind are said to be of Volterra type ([266, 267]).

2.3c Solving Abel Integral Equations

In (2.3c) replace x by a running variable η, multiply both sides by $(x-\eta)^{\alpha-1}$ and integrate in $d\eta$ over $(0,x)$. Interchanging the order of integration gives

$$\int_0^x v(y)\left[\int_y^x \frac{d\eta}{(x-\eta)^{1-\alpha}(\eta-y)^\alpha}\right]dy = \int_0^x \frac{f(\eta)}{(x-\eta)^{1-\alpha}}d\eta.$$

Compute the integral in braces by the change of variables

$$\eta = y + \frac{1}{s+1}(x-y), \qquad s \in [0,\infty).$$

This gives

$$\int_y^x \frac{d\eta}{(x-\eta)^{1-\alpha}(\eta-y)^\alpha}d\eta = \int_0^\infty \frac{ds}{s^{1-\alpha}(1+s)} = \frac{\pi}{\sin\alpha\pi}$$

where the last integral has been computed by the method of residues ([31] page 107). Combining these calculations

$$\begin{aligned} \frac{\pi}{\sin\alpha\pi}\int_0^x v(y)dy &= \int_0^x \frac{f(y)}{(x-y)^{1-\alpha}}dy \\ &= \frac{x^\alpha}{\alpha}f(0) + \frac{1}{\alpha}\int_0^x (x-y)^\alpha f'(y)dy. \end{aligned}$$

Taking the derivative gives an explicit representation, of the solution of the Abel integral equation (2.3c), in the form

$$v(x) = \frac{\pi}{\sin\alpha\pi}\left[\frac{f(0)}{x^{1-\alpha}} + \int_0^x \frac{f'(y)}{(x-y)^{1-\alpha}}dy\right]. \tag{2.4c}$$

2.4c The Cycloid ([3])

To find the parametric equations of the tautochrone trajectory, in (2.4c) take $\alpha = \frac{1}{2}$ and $f = C$. Using (2.2c)

$$v(x) = \frac{C}{\pi\sqrt{x}}, \qquad \sin\theta = \frac{\pi\sqrt{x}}{C}.$$

Denoting by $x = x(\theta)$ the vertical component of the parametrization of γ

$$x(\theta) = \frac{C^2}{\pi^2}\sin^2\theta = \frac{C^2}{2\pi^2}(1-\cos 2\theta). \tag{2.5c}$$

Let $y = y(\theta)$ denote the horizontal component of the parametrization of γ, and let γ have the local representation $y = y(x)$. Then

$$dy = \frac{dx}{\tan\theta} = \frac{C^2}{\pi^2}\frac{2\sin\theta\cos\theta}{\tan\theta}d\theta = \frac{C^2}{\pi^2}(1+\cos 2\theta)d\theta$$

and by integration

$$y(\theta) = \frac{C^2}{\pi^2}\left(\theta + \frac{1}{2}\cos 2\theta\right) + c_o \tag{2.6c}$$

for a constant c_o. The equations (2.5c) and (2.6c) are the parametric equations of a cycloid.

2.5c Volterra Integral Equations ([266, 267])

Let f be bounded and continuous in $\mathbb{R}^+$, and consider the Volterra equation

$$u(x) = \lambda\int_0^x K(x;y)u(y)dy + f(x)$$

where $K(\cdot;\cdot)$ is bounded and continuous in $\mathbb{R}^+\times\mathbb{R}^+$. Assuming that $K(x;y) = 0$ for $y > x$, rewrite this as

$$u = \lambda\int_0^\infty K(\cdot;y)u(y)dy + f.$$

Prove that for all $x \in \mathbb{R}^+$

$$|(A^n f)| \le \sup_{\mathbb{R}^+}\|f\|\frac{K^n x^n}{n!}, \qquad \text{where } \ (Af)(x) = \int_0^x K(x;y)f(y)dy$$

and where K is an upper bound for $|K(\cdot;\cdot)|$. Conclude that a solution must be continuous and locally bounded in $\mathbb{R}^+$.

2.1. Say in what sense the Dirichlet and Neumann problems for the Laplacean in a bounded domain are mutually adjoint. *Hint:* See Sections 5–6 of Chapter 3 and the arguments of Section 6 of this Chapter.

2.2. Find A^2 if $K(x;y) = e^{|x-y|}$ and $E = (0,1)$.

2.3. Find A^2 and A^3 if $K(x;y) = x - y$ and $E = (0,1)$.

One might ask whether these integral equations set in $\mathbb{R}^+$, have a solution if $K(\cdot;\cdot)$ does not vanish for $y > x$. It turns out that some decay has to imposed on $K(\cdot;\cdot)$. For kernels of the type $K(x;y) = K(x-y)$ and $K(s) \to 0$ exponentially fast as $s \to \infty$ a theory is developed by N. Wiener and E. Hopf ([279]). See also G. Talenti [250].

3c Separable Kernels

3.1c Hammerstein Integral Equations ([114])

Consider the nonlinear integral equation of Hammerstein type

$$u = \int_E K(x;y) f(y, u(y)) dy.$$

If the kernel is separable, set

$$\gamma_i = \int_E \psi_i f(y, u(y)) dy \qquad \big(K(\cdot;\cdot) = \sum_{i=1}^n \varphi_i \psi_i\big)$$

where the numbers γ_i are to be determined from

$$u = \sum_{i=1}^n \gamma_i \varphi_i = \sum_{i=1}^n \varphi_i \int_E \psi_i f\big(y, \sum_{i=1}^n \gamma_i \varphi_i(y)\big) dy.$$

Therefore, the numbers γ_i are the possible solutions (real or complex) of the system

$$\gamma_i = \int_E \psi_i f\big(y, \sum_{i=1}^n \gamma_i \varphi_i(y)\big) dy.$$

3.1. Solve the Hammerstein equations

$$u(x) = \lambda \int_0^1 xyu^2(y) dy, \qquad u(x) = \lambda \int_{-1}^1 \frac{|x-y|}{1+u^2(y)} dy.$$

3.2. Let $\varphi \in L^2[0,1]$ be non-negative. Show that if $\|\varphi\|_{2,[0,1]} > 1$, there are no real-valued solutions of the Hammerstein equation

$$u(x) = \frac{1}{2} \int_0^1 \varphi(x)\varphi(y)\big(1 + u^2(y)\big) dy.$$

6c Applications to the Neumann Problem

Prove Proposition 6.1 for $N = 2$ by the following steps.

Step 1: Consider the double-layer potential

$$\mathbb{R}^2 - \partial E \ni x \to W(\partial E, x; \varphi) = \frac{1}{2\pi} \int_{\partial E} \varphi(y) \frac{\partial \ln |x-y|}{\partial \mathbf{n}(y)} d\sigma.$$

By the first part of the proof of Proposition 6.1, such a function is identically zero in E. Prove that it vanishes on ∂E. Thus $W(\partial E, \cdot; \varphi)$ is harmonic in $\mathbb{R}^2 - E$, vanishes as $|x| \to \infty$ and has zero normal derivative on ∂E.

Step 2: Prove that $\forall N \ge 2$, there exists at most one solution to the problem

$$\begin{gathered} u \in C^2(\mathbb{R}^N - \bar{E}) \cap C^1(\mathbb{R}^N - E), \quad \Delta u = 0 \text{ in } \mathbb{R}^N - \bar{E} \\ \frac{\partial u}{\partial \mathbf{n}} = 0 \text{ on } \partial E, \qquad \text{and} \qquad \lim_{|x| \to \infty} u(x) = 0. \end{gathered}$$

Prove that positive maxima or negative minima cannot occur on ∂E.

9c The Sequence of Eigenvalues

9.1. Let A be generated by the Green's function for the Laplacean with homogeneous Dirichlet data. Prove that the maximization process (8.1) is *formally* equivalent to

$$\min_{\varphi \in \mathcal{C}_o \cap S_1} \|\nabla u\|^2 = \lambda^2 \qquad \text{where}\, \mathcal{C}_o = \{u \in C_o(E) \big| |\nabla u| \in L^2(E)\}.$$

9.2. Let $f \in L^2(E)$. Prove that the minimum

$$\min_{\{f_1,\dots,f_n\} \in \mathbb{C}^n\}} \big\| f - \sum_{i=1}^n f_i u_i \big\|$$

is achieved for $f_i = \langle f, u_i \rangle$. *Hint:* Compute the derivatives

$$\frac{\partial}{\partial f_i} \big\| f - \sum_{i=1}^n f_i u_i \big\|^2.$$

9.3. Prove Bessel's inequality $\sum_{i=1}^n f_i^2 \le \|f\|^2$.

9.4. Prove Parseval's identity $\sum_{i=1}^\infty f_i^2 = \|f\|^2$.

10c Questions of Completeness

10.1. If $K(x; \cdot) \in L^2(E)$ uniformly in $x \in E$, then (10.2) gives another proof that to each eigenvalue λ_i there correspond only finitely many linearly independent eigenfunctions. *Hint:* If n_i is the number of linearly independent eigenfunctions corresponding to the eigenvalue λ_i, then $n_i \le C|\lambda_i|E|$.

10.2. Prove that $\left\{\sqrt{\frac{2}{L}}\sin\frac{n\pi}{L}x\right\}$ is a complete orthonormal system in the space $L^2(0,L)$. *Hint:* Compute the eigenvalues and eigenfunctions of the Laplacean in one dimension, over $(0,L)$ with homogeneous Dirichlet data on $x=0$ and $x=L$.

10.3. Let m be an even positive integer, and let $C^m_{\text{odd}}(0,L)$ denote the space of functions in $C^m(0,L)$ whose even order derivatives vanish at $x=0$ and $x=L$, i.e.,

$$\frac{\partial^j}{\partial x^j}\varphi(0^+)=\frac{\partial^j}{\partial x^j}\varphi(L^-)=0 \quad \text{for all even integers} \quad 0\le j\le m.$$

Denoting by $\{v_n\}$ the complete orthonormal system in $L^2(0,L)$, of the previous problem prove that if $\varphi\in C^m_{\text{odd}}(0,L)$

$$|\langle\varphi,v_n\rangle|\le\frac{2L^j}{(n\pi)^j}\Big\|\frac{\partial^j}{\partial x^j}\varphi\Big\|_{\infty,[0,L]} \quad \text{for all } 0\le j\le m.$$

As a consequence

$$\Big\|\varphi-\sum_{n=1}^{j-1}\langle\varphi,v_n\rangle v_n\Big\|_{\infty,[0,L]}\le\frac{\text{const}}{j^m}\Big\|\frac{\partial^j}{\partial x^j}\varphi\Big\|_{\infty,[0,L]}.$$

10.1c Periodic Functions in $\mathbb{R}^N$

A function $f:\mathbb{R}^N\to\mathbb{R}$ is periodic of period 1 if $f(x+n)=f(x)$ for all $x\in\mathbb{R}^N$ and every N-tuple of integers $n\in\mathbb{Z}^N$.

Let $\mathbf{Q}=(0,1)^N$ denote the unit cube in $\mathbb{R}^N$. Every $f\in L^2(\mathbf{Q})$ can be regarded as the restriction to $\mathbf{Q}$ of a periodic function in $\mathbb{R}^N$ of period 1. If f is periodic of period 1, there exists a constant γ such that $f+\gamma$ is periodic of period 1 and has zero average over $\mathbf{Q}$. Consider the space

$$L^2_p(\mathbf{Q})=\Big\{f\in L^2(\mathbf{Q})\Big|\int_{\mathbf{Q}}f\,dx=0\Big\}$$

where the subscript p denotes "periodic function". An orthogonal basis for $L^2_p(\mathbf{Q})$ is found by solving the eigenvalue problem

$$-\Delta u=\lambda u \text{ in } \mathbf{Q}, \qquad \text{and} \qquad \frac{\partial u}{\partial\mathbf{n}}=0 \text{ on } \partial\mathbf{Q}.$$

Verify that

$$\prod_{j=1}^N\cos n_j\pi x_j, \qquad \prod_{j=1}^N\sin n_j\pi x_j; \qquad n=(n_1,\dots,n_N)\in\mathbb{Z}^N,\ |n|^2=\sum_{j=1}^N n_j^2$$

are eigenfunctions for the eigenvalues $\lambda=(\pi|n|)^2$. Any complex linear combination of these is still an eigenfunction. Prove that the system $\{e^{i\pi\langle n,x\rangle}\}$ for $n\in\mathbb{Z}^N$ is a complete orthogonal basis for $L^2_p(\mathbf{Q})$.

10.2c The Poisson Equation with Periodic Boundary Conditions

Consider the Neumann problem

$$u \in C^2(\mathbf{Q}) \cap C^1(\bar{\mathbf{Q}}), \quad \Delta u = f \in C^1(\bar{\mathbf{Q}}), \quad \frac{\partial u}{\partial \mathbf{n}} = 0 \text{ on } \partial\mathbf{Q}.$$

The necessary and sufficient condition for solvability is that f has zero average over $\mathbf{Q}$, which we assume. Write

$$f = \sum_{n \in \mathbb{Z}^N} \hat{f}_n e^{i\pi\langle n,x\rangle}, \qquad \text{where} \qquad \hat{f}_n = \langle f, e^{i\pi\langle n,x\rangle}\rangle$$

and seek a solution of the type

$$u = \sum_{n \in \mathbb{Z}^N} \hat{u}_n e^{i\pi\langle n,x\rangle}, \qquad \hat{u}_n = \langle u, e^{i\pi\langle n,x\rangle}\rangle.$$

Prove that

$$\hat{u}_n = \frac{\hat{f}_n}{-\pi|n|^2} \qquad \text{for all} \qquad n \in \mathbb{Z}^N - \{0\}.$$

11c The Eigenvalue Problem for the Laplacean

11.1. A linear operator $A : L^2(E) \to L^2(E)$ is *positive* if

$$\langle Af - Ag, f - g\rangle \geq 0 \quad \text{for all } f, g \in L^2(E).$$

The operator A generated by the Green's function for the Laplacean with homogeneous Dirichlet data on ∂E, is positive in the sense that $\langle Af, f\rangle > 0$ for all $f \in L^2(E)$, $f \neq 0$. Assume first that $f \in C_o^\eta(E)$ for some $\eta \in (0,1)$. Then the function Af is the unique solution of the problem

$$u \in C^2(E) \cap C(\bar{E}), \quad -\Delta u = f, \quad \text{in } E, \quad u\big|_{\partial E} = 0.$$

Therefore

$$\langle Af, f\rangle = \langle u, -\Delta u\rangle = \|\nabla u\|^2.$$

Prove the positivity of A for general $f \in L^2(E)$.

11.2. Prove that if A is a symmetric, positive, compact operator in $L^2(E)$, then its eigenvalues are positive.

5

THE HEAT EQUATION

1 Preliminaries

Consider a material homogeneous body occupying a region $E \subset \mathbb{R}^N$ with boundary ∂E of class C^1 and outward unit normal $\mathbf{n}$. Identify the body with E and denote by $k > 0$ its dimensionless conductivity. The temperature distribution $(x, t) \to u(x, t)$ satisfies the second-order parabolic equation

$$u_t = k\Delta u \quad \text{for} \quad x \in E \quad \text{and} \quad t \in (t_1, t_2) \tag{1.1}$$

where $(t_1, t_2) \subset \mathbb{R}$ is some time interval of observation. By changing the time scale, we may assume that $k = 1$. Set formally

$$H(\cdot) = \frac{\partial}{\partial t} - \Delta, \qquad H^*(\cdot) = \frac{\partial}{\partial t} + \Delta.$$

The formal operators $H(\cdot)$ and $H^*(\cdot)$ are called the heat operator and the adjoint heat operator respectively. If $0 < T < \infty$ denote by E_T the cylindrical domain $E \times (0, T]$, and if $E = \mathbb{R}^N$, let S_T denote the *strip* $\mathbb{R}^N \times (0, T]$. The heat operator and its adjoint are well defined for functions in the class

$$\mathcal{H}(E_T) = \{u : E_T \to \mathbb{R} \,\big|\, u_t, u_{x_i x_j} \in C(E_T),\ i, j = 1, \dots, N\}.$$

Information on the thermal status of the body is gathered at the boundary of E over an interval of time $(0, T]$. That is, one might be given the temperature or the heat flux at $\partial E \times (0, T)$. Physically relevant problems consist in finding the temperature distribution in E for $t \ge 0$, from information on $\partial E \times (0, T)$ and the knowledge of the temperature $x \to u_o(x)$ at time $t = 0$. This leads to the following boundary value problems:

1.1 The Dirichlet Problem

Find $u \in \mathcal{H}(E_T) \cap C(\bar{E}_T)$ satisfying

© Springer Nature Switzerland AG 2023

E. DiBenedetto, U. Gianazza, *Partial Differential Equations*,
Cornerstones, https://doi.org/10.1007/978-3-031-46618-2_6

$$\begin{aligned} H(u) &= 0 \quad \text{ in } \ E_T \\ u\big|_{\partial E\times[0,T]} &= g \in C(\partial E \times (0,T]) \\ u(\cdot,0) &= u_o \in C(\bar{E}). \end{aligned} \tag{1.2}$$

1.2 The Neumann Problem

Find $u \in \mathcal{H}(E_T) \cap C^1(\bar{E}_T)$ satisfying

$$\begin{aligned} H(u) &= 0 \ \text{ in } \ E_T \\ Du \cdot \mathbf{n} &= g \in C(\partial E \times (0,T)) \\ u(\cdot,0) &= u_o \in C(\bar{E}). \end{aligned} \tag{1.3}$$

where D denotes the gradient with respect to the space variables only.

1.3 The Characteristic Cauchy Problem

Find $u \in \mathcal{H}(S_T) \cap C(\bar{S}_T)$ satisfying

$$\begin{aligned} H(u) &= 0 \quad \text{ in } \ S_T \\ u(\cdot,0) &= u_o \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N). \end{aligned} \tag{1.4}$$

The initial datum in (1.4) is taken in the topology of the uniform convergence over compact sets $K \subset \mathbb{R}^N$, that is, $\|u(\cdot,t) - u_o\|_{\infty,K} \to 0$, as $t \to 0$, for all such K. In (1.4) the data are assigned on the characteristic surface $t = 0$. The Cauchy–Kowalewski theorem fails to hold in such a circumstance. Even if u_o is analytic, a solution of (1.4) near $t = 0$, that is for small positive and negative times, in general cannot be found. Indeed, changing t into $-t$ does not preserve (1.1) and the PDE distinguishes between solutions forward and backward in time. This corresponds to the physical fact that heat conduction is, in general, irreversible, i.e., given $x \to u_o(x)$, we may predict future temperatures, but we cannot in general determine the thermal status that generated that particular temperature distribution.

2 The Cauchy Problem by Similarity Solutions

The PDE $H(u) = 0$ is invariant by linear transformations $\bar{x} = hx$, $\bar{t} = h^2 t$ for $h \neq 0$. These are transformations that leave invariant the ratio $\xi = |x|^2/t$. This suggests looking for solutions u that are "separable" in the variables t and ξ, that is, solutions of the form $u(x,t) = h(t)f(\xi)$. Substituting this in the PDE $H(u) = 0$ gives

$$th'f - 2Nhf' = h\xi[4f'' + f'].$$

Setting each side equal to zero yields

$$f(\xi) = \exp(-\xi/4), \qquad h(t) = t^{-N/2}$$

up to multiplicative constants. These remarks imply that a solution of $H(u) = 0$ in $\mathbb{R}^N \times (0, \infty)$ is given by

$$\Gamma(x,t) = \frac{1}{(4\pi t)^{N/2}} e^{-|x|^2/4t} \tag{2.1}$$

where the multiplicative constant $(4\pi)^{-N/2}$ has been chosen to satisfy the normalization (Section 2.1c of the Complements)

$$\frac{1}{[4\pi(t-s)]^{N/2}} \int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4(t-s)}} dy = 1 \tag{2.2}$$

for all $x \in \mathbb{R}^N$ and all $s < t$.

Remark 2.1 The function Γ is called the *heat kernel* or the *fundamental solution* of the heat equation. It satisfies

$$\begin{aligned} (x,t) \to \Gamma(x,t) &\in C^\infty(\mathbb{R}^N \times \mathbb{R}^+) \\ x \to \Gamma(x,t) &\text{ is analytic for } t > 0. \end{aligned}$$

Let $H_{(\eta,\tau)}$ and $H^*_{(\eta,\tau)}$ denote respectively the heat operator and its adjoint with respect to the variables $\eta \in \mathbb{R}^N$ and $\tau \in \mathbb{R}$. By direct calculation

$$\begin{aligned} H_{(x,t)}\Gamma(x-y;t-s) &= 0 \\ H^*_{(y,s)}\Gamma(x-y;t-s) &= 0 \end{aligned} \qquad \text{for } s < t < \infty. \tag{2.3}$$

Assume that $u \in \mathcal{H}(S_T)$ is a solution of the Cauchy problem (1.4) satisfying

$$\int_{\mathbb{R}^N} |u(x,t)| dx < \infty \qquad \text{for all } 0 \le t \le T \tag{2.4}$$

and the asymptotic decay

$$\left.\begin{aligned} \limsup_{r\to\infty} \left| \int_{|y|=r} \Gamma(x-y;t) Du \cdot \frac{y}{r} d\sigma \right| &= 0 \\ \limsup_{r\to\infty} \left| \int_{|y|=r} u D\Gamma(x-y;t) \cdot \frac{y}{r} d\sigma \right| &= 0 \end{aligned}\right. \qquad \text{for all } 0 \le t < T \tag{2.5}$$

where $d\sigma$ denotes the surface measure on the sphere $|y| = r$. Multiply the first of (1.4) viewed in the variables (y,s) by $\Gamma(x-y;t-s)$, and integrate by parts in $dyds$ over the cylindrical domain $B_r \times (0, t-\varepsilon)$ for $\varepsilon \in (0,t)$. Letting $r \to \infty$ with the aid of (2.4) and (2.5), we arrive at

$$\int_{\mathbb{R}^N} u(y,t-\varepsilon)\varepsilon^{-N/2} e^{-\frac{|x-y|^2}{4\varepsilon}} dy = \int_{\mathbb{R}^N} u_o(y) t^{-N/2} e^{-\frac{|x-y|^2}{4t}} dy. \tag{2.6}$$

We let $\varepsilon \to 0$ as follows. Fix $\sigma > 0$ and write

$$\begin{aligned}\int_{\mathbb{R}^N} u(y,t-\varepsilon)\varepsilon^{-N/2} e^{-\frac{|x-y|^2}{4\varepsilon}} dy &= \int_{|y-x|>\sigma} u(y,t-\varepsilon)\varepsilon^{-N/2} e^{-\frac{|x-y|^2}{4\varepsilon}} dy \\ &\quad + \int_{|y-x|\le\sigma} u(y,t-\varepsilon)\varepsilon^{-N/2} e^{-\frac{|x-y|^2}{4\varepsilon}} dy \\ &= I_\varepsilon^{(1)} + I_\varepsilon^{(2)}.\end{aligned}$$

As $\varepsilon \to 0$, the first integral on the right-hand side tends to zero. We rewrite the second integral as

$$\begin{aligned}I_\varepsilon^{(2)} &= \int_{|y-x|<\sigma} [u(y,t-\varepsilon) - u(x,t)]\varepsilon^{-N/2} e^{-\frac{|x-y|^2}{4\varepsilon}} dy \\ &\quad + u(x,t) \int_{|y-x|<\sigma} \varepsilon^{-N/2} e^{-\frac{|x-y|^2}{4\varepsilon}} dy \\ &= [u(x,t) + O(\sigma+\varepsilon)] \int_{|y|<\sigma} \varepsilon^{-N/2} e^{-\frac{|y|^2}{4\varepsilon}} dy\end{aligned}$$

where $O(\sigma+\varepsilon)$ denotes a quantity that tends to zero as $(\sigma+\varepsilon) \to 0$. Write

$$\varepsilon^{-N/2} \int_{|y|<\sigma} e^{-\frac{|y|^2}{4\varepsilon}} dy = \varepsilon^{-N/2} \int_{\mathbb{R}^N} e^{-\frac{|y|^2}{4\varepsilon}} dy - \varepsilon^{-N/2} \int_{|y|>\sigma} e^{-\frac{|y|^2}{4\varepsilon}} dy.$$

The first integral can be computed from (2.2) with $x = 0$ and $(t-s) = \varepsilon$, i.e.,

$$\varepsilon^{-N/2} \int_{\mathbb{R}^N} e^{-\frac{|y|^2}{4\varepsilon}} dy = (4\pi)^{N/2}.$$

To estimate the second integral, introduce the change of variables $y = 2\sqrt{\varepsilon}\eta$, whose Jacobian is $(4\varepsilon)^{N/2}$. This gives

$$\varepsilon^{-N/2} \int_{|y|>\sigma} e^{-\frac{|y|^2}{4\varepsilon}} dy = \int_{|\eta|>\frac{\sigma}{2\sqrt{\varepsilon}}} e^{-|\eta|^2} d\eta.$$

This integrals tends to zero as $\varepsilon \to 0$, for $\sigma > 0$ fixed. Combine these calculations in (2.6), and let $\varepsilon \to 0$, while $\sigma > 0$ remains fixed, to obtain

$$(4\pi)^{N/2} u(x,t) = t^{-N/2} \int_{\mathbb{R}^N} u_o(y) e^{-\frac{|x-y|^2}{4t}} dy + O(\sigma).$$

Letting $\sigma \to 0$ gives the representation formula

$$u(x,t) = \frac{1}{(4\pi t)^{N/2}} \int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4t}} u_o(y) dy = \int_{\mathbb{R}^N} \Gamma(x-y;t) u_o(y) dy. \tag{2.7}$$

Therefore every solution of the Cauchy problem satisfying the decay conditions (2.4)–(2.5) must be represented as in (2.7). Now consider (2.7), regardless of its derivation process. If $u_o \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, the integral on the right-hand side is convergent and defines a function u that satisfies the decay conditions (2.4)–(2.5). Moreover, by Remark 2.1, $u(x,t) \in C^\infty(S_T)$ and

$$x \to u(x,t) \text{ is locally analytic in } \mathbb{R}^N \text{ for all } 0 < t \le T. \tag{2.8}$$

Theorem 2.1. *Let $u_o \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Then u defined by (2.7) is a solution to the Cauchy problem (1.4). Moreover, u is bounded in $\mathbb{R}^N \times \mathbb{R}^+$, and it is the only bounded solution to the Cauchy problem (1.4).*

Proof (existence). By construction $H(u) = 0$ in S_T. Moreover

$$\|u(\cdot,t)\|_{\infty,\mathbb{R}^N} \le \|u_o\|_{\infty,\mathbb{R}^N} \int_{\mathbb{R}^N} \Gamma(x-y;t)dy = \|u_o\|_{\infty,\mathbb{R}^N}. \tag{2.9}$$

Therefore u defined by (2.7) is bounded in $\mathbb{R}^N \times \mathbb{R}^+$. It remains to show that the initial datum is taken in the topology of uniform convergence over compact subsets of $\mathbb{R}^N$. Fix a compact set $K \subset \mathbb{R}^N$, recall the normalization (2.2), and write for $x \in K$

$$u(x,t) - u_o(x) = \frac{1}{(4\pi t)^{N/2}} \int_{\mathbb{R}^N} [u_o(y) - u_o(x)] e^{-\frac{|x-y|^2}{4t}} dy.$$

Divide the domain of integration on the right-hand side into $|x-y| < \sigma$ and $|x-y| \ge \sigma$ where $\sigma > 0$ is arbitrary but fixed. As $t \to 0$, the integral extended over $|x-y| > \sigma$ tends to zero and the one extended over $|x-y| < \sigma$ is majorized by

$$\sup_{\substack{x\in K\\ |x-y|<\sigma}} |u_o(y) - u_o(x)| \int_{\mathbb{R}^N} \Gamma(x-y;t)dy.$$

Therefore, for arbitrary $\sigma > 0$

$$\lim_{t\to 0} \|u(x,t) - u_o(x)\|_{\infty,K} \le \sup_{\substack{x\in K\\ |x-y|<\sigma}} |u_o(y) - u_o(x)|. \qquad \blacksquare$$

The proof of uniqueness will make use of the maximum principle discussed in the next sections. A first form of such a principle can be read from (2.9), that is, the supremum of $|u(\cdot,t)|$ at all instants $t > 0$ is no larger than the supremum of $|u_o|$.

Remark 2.2 Suppose that in (2.7), u_o is non-negative, not identically zero, and supported in the ball B_ε of radius $\varepsilon > 0$ centered at some point in $\mathbb{R}^N$. Then $u(x,t)$ is strictly positive for all $(x,t) \in S_T$. In particular, the *initial disturbance*, confined in B_ε, for however small ε, is felt by the solution at any $|x|$ however large, and any positive t, however small. Thus the initial disturbance propagates with infinite speed.

2.1 The Backward Cauchy Problem

Let $S^T = \mathbb{R}^N \times (-T, 0)$, and consider the problem of finding $u \in \mathcal{H}(S^T) \cap C(\bar{S}^T)$ satisfying

$$\begin{aligned} H(u) &= 0 \quad \text{in } S^T \\ u(\cdot, 0) &= u_o \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N). \end{aligned} \tag{2.10}$$

The backward problem (2.10) is ill posed in the sense that unlike the forward problem (1.4), it is not solvable in general within the class of bounded solutions. Indeed, if a bounded, continuous solution did exist for every choice of data $u_o \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, we would have by (2.7) and Theorem 2.1

$$u_o(x) = \int_{\mathbb{R}^N} \Gamma(x - y; T) u(y, -T)\, dy \tag{2.11}$$

and this would contradict (2.8), if for example, u_o is merely continuous.

3 The Maximum Principle and Uniqueness (Bounded Domains)

Let E be a bounded open subset of $\mathbb{R}^N$ and let $\partial_* E_T = \partial E_T - E \times \{T\}$ denote the parabolic boundary of E_T.

Theorem 3.1. *Let $u \in \mathcal{H}(E_T) \cap C(\bar{E}_T)$ satisfy $H(u) \le 0 (\ge 0)$ in E_T. Then*

$$\sup_{E_T} u = \sup_{\partial_* E_T} u \qquad \big(\inf_{E_T} u = \inf_{\partial_* E_T} u\big).$$

Proof. We prove the statement only for $H(u) \le 0$. Let $\varepsilon \in (0, T)$ be arbitrary but fixed, and consider the function

$$\bar{E}_{T-\varepsilon} \ni (x, t) \to v(x, t) = u(x, t) - \varepsilon t$$

which satisfies $H(v) < -\varepsilon < 0$ in $\bar{E}_{T-\varepsilon}$. Since v is continuous in $\bar{E}_{T-\varepsilon}$, it achieves its maximum at some $(x_o, t_o) \in \bar{E}_{T-\varepsilon}$. If $(x_o, t_o) \notin \partial_* E_{T-\varepsilon}$, then $H(v)(x_o, t_o) \ge 0$, contradicting $H(v) < 0$. Thus $(x_o, t_o) \in \partial_* E_{T-\varepsilon}$ and

$$u(x, t) \le 2\varepsilon T + \sup_{\partial_* E_T} u \qquad \text{for all } (x, t) \in \bar{E}_{T-\varepsilon} \text{ for all } \varepsilon > 0. \qquad \blacksquare$$

Corollary 3.1 *Let $u \in \mathcal{H}(E_T) \cap C(\bar{E}_T)$ satisfy $H(u) = 0$ in E_T. Then*

$$\|u\|_{\infty, E_T} = \|u\|_{\infty, \partial_* E_T}.$$

Remark 3.1 Theorem 3.1 is a weak maximum principle since it does not exclude that u might obtain its extremal values also at some other points in $\bar{E}_T$. For example, u could be identically constant in E_T. A strong maximum principle would assert that this is the only other possibility.

3.1 A Priori Estimates

Denote by λ the diameter of E. After a rotation and translation, we may, if necessary, arrange the coordinate axes so that

$$\text{for all } x = (x_1, \dots, x_N) \in \bar{E}, \quad x_1^o - \lambda \le x_1 \le x_1^o$$

for some $x^o = (x_1^o, \dots, x_N^o) \in \partial E$. This is possible since the heat operator is invariant under rotations and translations of the space variables. Let $u \in \mathcal{H}(E_T) \cap C(\bar{E}_T)$ be such that $\|H(u)\|_{\infty, E_T} < \infty$ and construct the two functions

$$w_\pm(x,t) = \|u\|_{\infty, \partial_* E_T} + e^\lambda [1 - e^{(x_1 - x_1^o)}] \|H(u)\|_{\infty, E_T} \pm u.$$

One verifies that $H(w_\pm) \ge 0$ in E_T and that

$$w_\pm \Big|_{\partial_* E_T} \ge \|u\|_{\infty, \partial_* E_T} \pm u \Big|_{\partial_* E_T} .$$

Therefore $w_\pm \ge 0$, by Theorem 3.1. This gives the following a priori estimate.

Corollary 3.2 *Let* $u \in \mathcal{H}(E_T) \cap C(\bar{E}_T)$. *Then*

$$\|u\|_{\infty, E_T} \le \|u\|_{\infty, \partial_* E_T} + (e^{\text{diam}(E)} - 1) \|H(u)\|_{\infty, E_T}.$$

3.2 Ill Posed Problems

A boundary value problem for $H(u) = 0$ with data prescribed on the whole boundary of E_T in general is not well posed. For example, consider the rectangle $R = [0 < x < 1] \times [0 < t < 1]$, and let $\varphi \in C(\partial R)$ be non-constant and such that it takes an absolute maximum on the open line segment $[0 < x < 1] \times [t = 1]$. Then the problem

$$u \in \mathcal{H}(R), \quad u_t - u_{xx} = 0 \text{ in } R, \quad u\big|_{\partial R} = \varphi$$

cannot have a solution, for it would violate Theorem 3.1.

3.3 Uniqueness (Bounded Domains)

Corollary 3.3 *There exists at most one solution* $u \in \mathcal{H}(E_T) \cap C(\bar{E}_T)$ *of the boundary value problem*

$$u_t - \Delta u = f \in C(\bar{E}_T), \quad u\big|_{\partial_* E_T} = g \in C(\partial_* E_T).$$

Proof. If u and v are solutions, $w = u - v$ solves

$$w_t - \Delta w = 0 \text{ in } E_T, \quad w\big|_{\partial_* E_T} = 0$$

and hence $w \equiv 0$ by Theorem 3.1. ∎

4 The Maximum Principle in $\mathbb{R}^N$

Results analogous to Theorem 3.1 are possible in $\mathbb{R}^N$ if one imposes some conditions on the behavior of $x \to u(x,t)$ as $|x| \to \infty$. Such conditions are dictated by the solution formula (2.7). For such a formula to have a meaning, u_o does not have to be regular or bounded. It would suffice to require the convergence of the integral on the right-hand side for $0 < t \le T$. The next proposition gives some sufficient conditions for this to occur.

Proposition 4.1 *Assume that $u_o \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ and satisfies the growth condition*

$$\begin{cases} \text{there exist positive constants } C_o, \alpha_o, r_o \text{ such that} \\ |u_o(x)| \le C_o e^{\alpha_o |x|^2} \text{ for almost all } |x| \ge r_o. \end{cases} \tag{4.1}$$

Then (2.7) defines a function $u \in C^\infty(S_T)$ for every $T \in \big(0, \frac{1}{4\alpha_o}\big)$. Moreover, $H(u) = 0$ in S_T, and for every $\varepsilon \in \big(0, \frac{1}{4\alpha_o}\big)$, there exists positive constants α, C, and r depending upon α_o, C_o, r_o, N, and ε such that

$$\begin{aligned} &|u(x,t)| \le \Gamma\big(\varepsilon; \frac{\varepsilon^2}{2N}\big)\|u_o\|_{1,B_{r_o}} + Ce^{\alpha|x|^2} \\ &\text{for all } |x| > r \text{ and for all } 0 < t < \frac{1}{4\alpha_o} - \varepsilon. \end{aligned} \tag{4.2}$$

Proof. Fix $\varepsilon \in \big(0, \frac{1}{4\alpha_o}\big)$ and $|x| > r_o + \varepsilon$, and write the integral in (2.7) as

$$\int_{|y|\le r_o} \Gamma(x-y;t)u_o(y)dy + \int_{|y|>r_o} \Gamma(x-y;t)u_o(y)dy = J_1 + J_2.$$

For $|x-y| > \varepsilon$

$$|J_1| \le \sup_{t\ge 0} \Gamma(\varepsilon;t)\|u_o\|_{1,B_{r_o}} = \Gamma\big(\varepsilon; \frac{\varepsilon^2}{2N}\big)\|u_o\|_{1,B_{r_o}}.$$

In estimating J_2 we perform the change of variables $y - x = 2\sqrt{t}\eta$, of Jacobian $(4t)^{N/2}$, and use (4.1) to estimate $|u_o(y)|$ from above. This gives

$$|J_2| \le C_o \pi^{-N/2} \int_{|y|>r_o} e^{-|\eta|^2} e^{\alpha_o |x+2\sqrt{t}\eta|^2} d\eta.$$

By the Schwarz inequality, for all $\delta > 0$

$$|x + 2\sqrt{t}\eta|^2 \le \big(1 + \frac{1}{\delta}\big)|x|^2 + 4(1+\delta)t|\eta|^2.$$

Therefore, for all $|x| > r = r_o + \varepsilon$

$$|J_2| \le C_o \pi^{-N/2} e^{\alpha_o(1+1/\delta)|x|^2} \int_{\mathbb{R}^N} e^{-(1-4\alpha_o(1+\delta)t)|\eta|^2} d\eta.$$

The integral on the right-hand side is convergent if

$$t < \frac{1}{4\alpha_o(1+\delta)} = \frac{1}{4\alpha_o} - \varepsilon.$$

This defines the choice of δ. Therefore $|J_2| \le Ce^{\alpha|x|^2}$, where

$$C = C_o \pi^{-N/2} \int_{\mathbb{R}^N} e^{-(1-4\alpha_o(1+\delta)t)|\eta|^2} d\eta \quad \text{and} \quad \alpha = \alpha_o\big(1 + \frac{1}{\delta}\big). \qquad \blacksquare$$

In deriving a maximum principle for solutions of the heat equation in S_T, we require that such solutions satisfy a behavior of the type (4.2) as $|x| \to \infty$, but we make no further reference to the representation formula (2.7).

Theorem 4.1. *Let* $u \in \mathcal{H}(S_T) \cap C(\bar{S}_T)$ *satisfy* $H(u) \ge 0$ *in* S_T *and* $u(\cdot, 0) \ge 0$. *Assume moreover that*

$$\begin{cases} \textit{there exist positive constants } C, \alpha, r \textit{ such that} \\ u(x,t) \ge -Ce^{\alpha|x|^2} \textit{ for all } |x| \ge r \textit{ and all } 0 \le t \le T. \end{cases} \tag{4.3}$$

Then $u \ge 0$ *in* S_T.

Proof. Choose $\beta > \alpha$ so large that $T > \frac{1}{8\beta} \overset{\text{def}}{=} T_1$. We first prove that $u \ge 0$ in the strip S_{T_1}. The function

$$v(x,t) = \frac{1}{(1-4\beta t)^{N/2}} e^{\beta|x|^2/(1-4\beta t)}$$

satisfies $H(v) = 0$, and $v(x,t) \ge e^{\beta|x|^2}$, in S_{T_1}. Let $\varepsilon > 0$ be arbitrary but fixed and set $w = u + \varepsilon v$. In view of the arbitrariness of ε, it will suffice to show that $w \ge 0$ in S_{T_1}. The function w satisfies $H(w) \ge 0$ in S_{T_1}, $w(\cdot, 0) \ge 0$, and

$$\liminf_{|x| \to \infty} w(x,t) \ge 0, \qquad \text{uniformly in } t \in [0, T_1].$$

Therefore, having fixed $(x_o, t_o) \in S_{T_1}$ and $\sigma > 0$, there exists $\rho > |x_o|$ such that $w(x,t) \ge -\sigma$ for $|x| \ge \rho$ for all $t \in [0, T_1]$. On the (bounded) cylinder $Q = [|x| < \rho] \times (0, T_1)$ the function $\bar{w} = w + \sigma$ satisfies $H(\bar{w}) \ge 0$ in Q and $\bar{w} \ge 0$ on the parabolic boundary $\partial_* Q$ of Q. Therefore $\bar{w} \ge 0$ in Q, by Theorem 3.1. In particular, $w(x_o, t_o) \ge -\sigma$ for all $\sigma > 0$. Therefore $w \ge 0$ in S_{T_1}, since $(x_o, t_o) \in S_{T_1}$ is arbitrary. To conclude the proof we repeat the argument in adjacent non-overlapping strips of width not exceeding $\frac{1}{8\beta}$, up to cover the whole of S_T. $\blacksquare$

Theorem 4.2. *Let* $u \in \mathcal{H}(S_T) \cap C(\bar{S}_T)$ *satisfy* $H(u) \le 0$ *in* S_T *and*

$$\begin{cases} \textit{there exist positive constants } C, \alpha, r \textit{ such that} \\ u(x,t) \le Ce^{\alpha|x|^2} \textit{ for all } |x| \ge r \textit{ and all } 0 \le t \le T. \end{cases} \tag{4.4}$$

Then

$$u(x,t) \le \sup_{\mathbb{R}^N} u(\cdot, 0) \qquad \textit{for all } (x,t) \in S_T.$$

Proof. We may assume that $u(\cdot,0) \in L^\infty(\mathbb{R}^N)$; otherwise, the statement is trivial. Assume first that T is so small that $4\alpha T < 1$ and consider the (bounded) cylinder $Q = [|x| < \rho] \times (0,T)$. The function

$$w = u - \frac{\varepsilon}{[4\pi(T-t)]^{N/2}} e^{|x|^2/4(T-t)}, \qquad \varepsilon > 0$$

satisfies $H(w) \le 0$ in S_T, and $w(x,0) \le \sup_{\mathbb{R}^N} u(\cdot,0)$ for $|x| < \rho$. Moreover, for $|x| = \rho$,

$$w \Big|_{|x|=\rho} \le Ce^{\alpha\rho^2} - \varepsilon(4\pi T)^{-N/2} e^{\rho^2/4T}.$$

Therefore, since $4T < 1/\alpha$, having fixed $\varepsilon > 0$, the parameter ρ can be chosen so large that $w|_{|x|=\rho} \le 0$. The conclusion now follows from Theorem 3.1 and the arbitrariness of ε. If $4\alpha T \ge 1$, subdivide S_T into finitely many strips of width less than $\frac{1}{4\alpha}$. ■

4.1 A Priori Estimates

Proposition 4.2 *Let* $u \in \mathcal{H}(S_T) \cap C(\bar{S}_T)$ *satisfy (4.3) and (4.4). Then*

$$\|u\|_{\infty,S_T} \le \|u(\cdot,0)\|_{\infty,\mathbb{R}^N} + T\|H(u)\|_{\infty,S_T}.$$

Proof. Assume that $\|u_o\|_{\infty,\mathbb{R}^N}$ and $\|H(u)\|_{\infty,S_T}$ are finite; otherwise, the statement is trivial. The functions

$$w_\pm = \|u(\cdot,0)\|_{\infty,\mathbb{R}^N} + t\|H(u)\|_{\infty,S_T} \pm u$$

satisfy $H(w_\pm) \ge 0$ in S_T and $w_\pm(\cdot,0) \ge 0$. Moreover, both $\bar{w}_\pm$ satisfy the asymptotic behavior (4.3). Therefore $w_\pm \ge 0$ in S_T, by Theorem 4.1. ■

Remark 4.1 The functional dependence of this estimate is optimal. Indeed, the estimate holds with equality for the function $u = 1 + t$.

4.2 About the Growth Conditions (4.3) and (4.4)

The conclusion of Theorem 4.1 fails if (4.3) is replaced by

$$u(x,t) \ge -ke^{\beta|x|^{2+\varepsilon}} \qquad \text{for any } \varepsilon > 0.$$

However Theorem 4.2 continues to hold for a growth slightly faster than (4.4). Precisely (S. Tacklind [249])

$$u(x,t) \le Ce^{\alpha|x|h(|x|)} \qquad \text{as } |x| \to \infty$$

where $h(\cdot)$ is positive nondecreasing and satisfies the optimal condition

$$\int^\infty \frac{ds}{h(s)} = +\infty.$$

5 Uniqueness of Solutions to the Cauchy Problem

Consider the class of functions $w \in \mathcal{H}(S_T)$ satisfying the growth condition

$$\begin{cases} \text{there exist positive constants } \ C, \alpha, r \ \text{ such that} \\ |u(x,t)| \le Ce^{\alpha|x|^2} \ \text{ for all } \ |x| \ge r \ \text{ and all } \ 0 \le t \le T. \end{cases} \tag{5.1}$$

Let $u, v \in \mathcal{H}(S_T) \cap C(\bar{S}_T)$ be solutions of the Cauchy problem (1.4) with initial data $u_o, v_o \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. If both u and v satisfy (5.1), then by Proposition 4.2,

$$\|u - v\|_{\infty, S_T} \le \|u_o - v_o\|_{\infty, \mathbb{R}^N}.$$

This inequality represents both a uniqueness and a stability result. Namely:

- **(i).** *Uniqueness*: solutions of the Cauchy problem (1.4) are unique within the class (5.1).
- **(ii).** *Stability*: within such a class, small variations on the data, measured in the norm of $L^\infty(\mathbb{R}^N)$, yield small variations on the solution measured in the same norm.

Proof (of Theorem 2.1 (Uniqueness)). If $u_o \in L^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N)$, the function u defined by the representation formula (2.7) is bounded by virtue of (2.9). It solves the heat equation in S_T, and it satisfies (5.1) by virtue of Proposition 4.1. Therefore, it is the only *bounded* solution of the Cauchy problem (1.4). ■

5.1 A Counterexample of Tychonov ([**263**])

The growth condition (5.1) is essential for uniqueness, as shown by the following counterexample due to Tychonov.

Proposition 5.1 *There exists a non-identically zero solution to the Cauchy problem*

$$u_t = u_{xx} \ \text{ in } \ \mathbb{R} \times (0, \infty), \quad u(x, 0) = 0.$$

Proof. For $z \in \mathbb{C}$, let

$$\varphi(z) = \begin{cases} e^{-1/z^2} & \text{for } \ z \ne 0 \\ 0 & \text{for } \ z = 0 \end{cases}$$

and define

$$u(x,t) = \begin{cases} \displaystyle\sum_{n=0}^{\infty} \frac{d^n}{dt^n} \varphi(t) \frac{x^{2n}}{(2n)!} & \text{for } \ t > 0 \\ 0 & \text{for } \ t = 0. \end{cases} \tag{5.2}$$

Proceeding formally

$$\lim_{t\to 0} u(x,t) = \sum_{n=0}^{\infty} \frac{d^n}{dt^n}\varphi(t)\,\Big|_{t=0}\, \frac{x^{2n}}{(2n)!} = 0 \tag{i}$$

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \sum_{n=0}^{\infty} \frac{d^n}{dt^n}\varphi(t) 2n(2n-1)\frac{x^{2n-2}}{(2n)!} \\ &= \sum_{n=1}^{\infty} \frac{d^n}{dt^n}\varphi(t)\frac{x^{2(n-1)}}{(2(n-1))!} \qquad \text{(ii)} \\ &= \sum_{n=0}^{\infty} \frac{d^{n+1}}{dt^{n+1}}\varphi(t)\frac{x^{2n}}{(2n)!} = \frac{\partial u}{\partial t}. \end{aligned}$$

These calculations become rigorous after we prove the following

Lemma 5.1 *The series in (5.2) and (i)–(ii) are uniformly convergent in a neighborhood of every point of* $\mathbb{R}\times\mathbb{R}^+$.

Proof. The function $z \to \varphi(z)$ is holomorphic in $\mathbb{C}-\{0\}$. We identify the t-axis as the real axis of the complex plane. If $t>0$ is fixed, the circle

$$\gamma = \big\{z\in\mathbb{C} \mid z = t + \frac{t}{2}e^{i\theta}\big\}, \quad 0<\theta\le 2\pi$$

does not meet the origin, and by the Cauchy formula ([31] page 72)

$$\frac{d^n}{dt^n}\varphi(t) = \frac{n!}{2\pi i}\int_\gamma \frac{\varphi(z)}{(z-t)^{n+1}}dz \qquad \text{for all } n\in\mathbb{N}.$$

From this

$$\Big|\frac{d^n}{dt^n}\varphi(t)\Big| \le \frac{n!}{2\pi}\int_\gamma \frac{e^{-Re(z^{-2})}}{|z-t|^{n+1}}|dz| = \frac{n!}{2\pi}\left(\frac{2}{t}\right)^n \int_0^{2\pi} e^{-Re(z^{-2})}d\theta.$$

For $z\in\gamma$

$$z^2 = t^2\left(1+\frac{1}{2}e^{i\theta}\right)^2 \qquad \text{and} \qquad \frac{1}{z^2} = \frac{1}{t^2}\frac{\left(1+\frac{1}{4}e^{-2i\theta}+e^{-i\theta}\right)}{\left|\left(1+\frac{1}{2}e^{i\theta}\right)^2\right|^2}.$$

From this $\mathrm{Re}(z^{-2}) \ge (2t)^{-2}$ and

$$\Big|\frac{d^n}{dt^n}\varphi(t)\Big| \le n!\left(\frac{2}{t}\right)^n e^{-1/4t^2}, \qquad n\in\mathbb{N}.$$

Fix $a>0$. For all $|x|<a$, the series in (5.2) is majorized, term by term, by the uniformly convergent series

$$e^{-1/4t^2}\sum_{n=0}^{\infty}\left(\frac{1}{t}\right)^n \frac{(a^2)^n}{n!} = e^{-1/4t^2}e^{a^2/t}.$$

Here we have used the Stirling inequality

$$\frac{2^n n!}{(2n)!} \le \frac{1}{n!}. \qquad \blacksquare$$

Remark 5.1 The function in (5.2) can also be defined for $t < 0$. Therefore the *backward* Cauchy problem

$$u_t - \Delta u = 0 \text{ in } \mathbb{R}^N \times (-\infty, 0), \quad u(x, 0) = 0$$

fails in general to have a unique solution.

6 Initial Data in $L^1_{\rm loc}(\mathbb{R}^N)$

The Cauchy problem for the heat equation can be solved uniquely for rather coarse initial data, for example $u_o \in L^1_{\rm loc}(\mathbb{R}^N)$, provided they satisfy the growth condition (4.1). The solution will exist only within the strip S_T for $0 < T < \frac{1}{4\alpha_o}$, and the initial datum is taken in the sense of $L^1_{\rm loc}(\mathbb{R}^N)$, i.e.,

$$\|u(\cdot, t) - u_o\|_{1,K} \to 0 \text{ as } t \to 0, \text{ for all compact } K \subset \mathbb{R}^N. \tag{6.1}$$

Theorem 6.1. *Let $u_o \in L^1_{\rm loc}(\mathbb{R}^N)$ satisfy (4.1). Then (2.7) defines a solution of the Cauchy problem*

$$\begin{aligned} &H(u) = 0 && \text{in } S_T \text{ for } 0 < T < \frac{1}{4\alpha_o} \\ &u(\cdot, 0) = u_o && \text{in the sense of } L^1_{\rm loc}(\mathbb{R}^N). \end{aligned} \tag{6.2}$$

Such a solution is unique within the class (5.1).

Proof. Fix $\rho \ge r_o$, where r_o is the constant in the growth condition (4.1). For almost all $x \in B_\rho$, write

$$|u(x, t) - u_o(x)| \le \int_{\mathbb{R}^N} \Gamma(x - y; t)|u_o(y) - u_o(x)|dy.$$

Integrating in dx over B_ρ

$$\begin{aligned} \int_{B_\rho} |u(x, t) - u_o(x)|dx &\le \int_{B_\rho} \int_{\mathbb{R}^N} \Gamma(x - y; t)|u_o(y) - u_o(x)|dydx \\ &= \int_{B_\rho} \int_{|x-y|\le\sigma} \Gamma(x - y; t)|u_o(y) - u_o(x)|dydx \\ &\quad + \int_{B_\rho} \int_{|x-y|>\sigma} \Gamma(x - y; t)|u_o(y) - u_o(x)|dydx \\ &= I_1 + I_2. \end{aligned}$$

Let h be a vector in $\mathbb{R}^N$ of size $|h| \le \sigma$. Then the first integral is estimated by

$$I_1 \le \sup_{\substack{h\in\mathbb{R}^N \\ |h|\le\sigma}} \int_{B_\rho} |u_o(x + h) - u_o(x)|dx.$$

The second integral is estimated by

$$I_2 \le \pi^{-N/2} \int_{|\eta|>\sigma/2\sqrt{t}} e^{-|\eta|^2} \int_{B_\rho} |u_o(x+2\sqrt{t}\eta) - u_o(x)| dx d\eta.$$

For all $t > 0$ and $\eta \in \mathbb{R}^N$ such that $2\sqrt{t}|\eta| < 2\rho$, estimate

$$\int_{B_\rho} |u_o(x+2\sqrt{t}\eta) - u_o(x)| dx \le 2\|u_o\|_{1,B_{2\rho}}.$$

If $2\sqrt{t}|\eta| \ge 2\rho$, by the growth condition (4.1)

$$\int_{B_\rho} |u_o(x+2\sqrt{t}\eta) - u_o(x)| dx \le \|u_o\|_{1,B_{2\rho}} + C|B_\rho| \sup_{x\in B_\rho} e^{2\alpha_o|x|^2+8\alpha_o t|\eta|^2}.$$

Therefore for $\rho > r_o$ fixed

$$\begin{aligned}\|u(\cdot,t) - u_o\|_{1,B_\rho} \le \text{const}(\rho) &\int_{|\eta|>\sigma/2\sqrt{t}} e^{-(1-8\alpha_o t)|\eta|^2} d\eta \\ &+ \sup_{\substack{h\in\mathbb{R}^N \\ |h|\le\sigma}} \|u_o(x+h) - u_o(x)\|_{1,B_\rho}.\end{aligned}$$

The proof is concluded by recalling that the translation $T_h u_o = u_o(\cdot + h)$ is continuous in $L^1_{\text{loc}}(\mathbb{R}^N)$ ([50], Chapter IV, Section 20). ■

6.1 Initial Data in the Sense of $L^1_{\text{loc}}(\mathbb{R}^N)$

Part of the definition of a solution to the Cauchy problem (1.4) or (6.2) is to make precise in what sense the initial data are taken. The notion (6.1) is the weakest unambiguous requirement for data $u_o \in L^1_{\text{loc}}(\mathbb{R}^N)$. If (6.1) holds, then there exists a sequence of times $\{t_n\} \to 0$ such that

$$u(x,t_n) \to u_o(x) \quad \text{for almost all } x \in \mathbb{R}^N.$$

and one might be tempted to take this as the sense in which $u(\cdot,t)$ takes its datum at $t = 0$. Such a definition might, however, generate ambiguity. Indeed, the uniqueness may be lost, as shown by the following examples. The function Γ satisfies the heat equation in S_∞ and the growth condition (5.1). Moreover, $\Gamma(\cdot,t) \to 0$ a.e. in $\mathbb{R}^N$, as $t \to 0$, and yet $\Gamma \not\equiv 0$. For such a "solution" the identically zero initial datum is not taken in the sense of $L^1_{\text{loc}}(\mathbb{R}^N)$. Indeed, for all $\rho > 0$

$$\int_{B_\rho} \Gamma(x,t) dx = \pi^{-N/2} \int_{|\eta|<\rho/2\sqrt{t}} e^{-|\eta|^2} d\eta \to 1 \quad \text{as } t \to 0.$$

Even more striking is the following example in one space dimension. The function

$$v(x,t) = \frac{x}{4\sqrt{\pi}t^{3/2}} e^{-\frac{x^2}{4t}}$$

is a solution of the heat equation in $\mathbb{R}\times\mathbb{R}^+$ satisfies all the previous properties, and in addition, $v(x,t) \to 0$ as $t \to 0$ for *all* $x \in \mathbb{R}$. And yet $v \not\equiv 0$. One checks that for all $\rho > 0$

$$\sqrt{t} \int_{-\rho}^{\rho} |v(x,t)|dx \to \frac{1}{\sqrt{\pi}} \qquad \text{as } t \to 0$$

that is, the initial datum $u_o = 0$ is not taken in the sense of $L^1_{\text{loc}}(\mathbb{R}^N)$.

7 Remarks on the Cauchy Problem

7.1 About Regularity

Let u_o be locally analytic in $\mathbb{R}^N$ and assume that it satisfies the growth condition (4.1). Then formula (2.7) defines the unique solution, within the class (5.1), to the Cauchy problem (1.4) in S_T for $0 < T < \frac{1}{4\alpha_o}$. Such a solution is locally analytic in the space variables. It is also analytic in the time variable within $\mathbb{R}^N \times (\varepsilon, T)$ for all $\varepsilon \in (0,T)$. Having in mind the Cauchy–Kowalewski theorem, it is natural to ask whether u is analytic in the x and t variables up to $t = 0$. This is in general false, as shown by the following argument.

If u were analytic in t up to $t = 0$, then u, u_t, Δu would have, in a right neighborhood of the origin, the absolutely convergent series representations

$$\begin{aligned} u(x,t) &= \sum_{n=0}^{\infty} \varphi_n(x) t^n \qquad (7.1) \\ u_t(x,t) &= \sum_{n=1}^{\infty} \varphi_n(x) n t^{n-1} \\ \Delta u(x,t) &= \sum_{n=0}^{\infty} \Delta\varphi_n(x) t^n \end{aligned}$$

with analytic coefficients φ_n. This in the equation $H(u) = 0$ gives

$$\varphi_{n+1} = \frac{1}{(n+1)} \Delta\varphi_n, \qquad n = 0, 1, \dots$$

From this, by iteration, starting from $\varphi_o = u_o$

$$\varphi_n = \frac{\Delta^n u_o}{n!}, \qquad n = 0, 1, 2, \dots$$

where $\Delta^0 = \mathbb{I}$ and $\Delta^n = \Delta^{n-1}\Delta$, for $n \in \mathbb{N}$. Putting this in (7.1) gives a representation of u in the form

$$u(x,t) = \sum_{n=0}^{\infty} \frac{\Delta^n u_o}{n!} t^n.$$

From the uniform convergence, it follows that

$$\frac{\Delta^n u_o(x)}{n!} t^n \to 0 \quad \text{as } n \to \infty \tag{7.2}$$

for (x,t) fixed in the domain of uniform convergence. Let $D^n u_o$ denote the generic derivative of u_o, of order n . Expanding u_o about x within a ball of radius t we must have

$$\frac{|D^n u_o(x)|}{n!} t^n \to 0 \quad \text{as } n \to \infty. \tag{7.3}$$

Now there exist locally analytic initial data u_o satisfying (7.3) but not (7.2).[1]

7.2 Instability of the Backward Problem

We have already remarked that the backward Cauchy problem (2.10) in general does not have a solution. If it does, the datum u_o must by analytic by Remark 2.1 and the representation formula (2.11). Stability however might be lost, as shown by the following example, due to Hadamard ([111]):

$$u(x,t) = \varepsilon e^{-t/\varepsilon^2} \sin\left(\frac{x}{\varepsilon}\right), \quad \varepsilon > 0$$

solves (2.10) with $N = 1$ and $u_o(x) = \varepsilon \sin(x/\varepsilon)$. As $\varepsilon \to 0$, $u_o \to 0$ in the $L^\infty(\mathbb{R})$-norm. Yet for all $t < 0$, for all intervals $(-\rho, \rho)$, and for all $0 < p \le \infty$

$$\|u(\cdot,t)\|_{p,(-\rho,\rho)} \to \infty \quad \text{as } \varepsilon \to 0.$$

8 Estimates Near $t = 0$

Let $u_o \in L^\infty_{\text{loc}}(\mathbb{R}^N)$ satisfy the growth condition (4.1), and let u be defined by (2.7) in the strip S_T for $0 < T < \frac{1}{4\alpha_o}$. We will study the behavior of $u(\cdot,t)$ and $|Du(\cdot,t)|$ as $t \to 0$. Since u_o is locally bounded, in (4.1) we may take $r_o = 0$, by possibly modifying the constant C_o. We will also investigate the behavior of $u_t(\cdot,t)$ and $u_{x_i x_j}(\cdot,t)$ as $t \to 0$, under the more stringent assumption that $u_o \in C^\delta_{\text{loc}}(\mathbb{R}^N)$ for some $\delta \in (0,1)$.

Proposition 8.1 *Let $u_o \in L^\infty_{\text{loc}}(\mathbb{R}^N)$ satisfy (4.1) with $r_o = 0$. For all $\rho > 0$, there exist constants A_ℓ, for $\ell = 0, 1$, depending only on ρ, N, α_o, and C_o, such that*

$$|u(x,t)| \le A_o, \qquad |Du(x,t)| \le A_1 t^{-1/2} \tag{8.1}$$

for $(x,t) \in B_\rho \times (0,T]$.

---

[1] Give examples of such functions in $\mathbb{R}$. *Hint:* Attempt e^{x^2} or $\ln(1+x^2)$, or a variant of these.

Proposition 8.2 *Let $u_o \in C^\delta_{\text{loc}}(\mathbb{R}^N)$, for some $\delta > 0$, satisfy (4.1) with $r_o = 0$. For all $\rho > 0$ there exists a constant A depending only on ρ, N, α_o, C_o, δ, and the Hölder constant of u_o over B_ρ such that*

$$|u_t(x,t)| + |u_{x_i x_j}(x,t)| \le A t^{\delta/2-1} \tag{8.2}$$

for $(x,t) \in B_\rho \times (0,T]$ and for all $i,j = 1,\dots,N$.

Proof (Proposition 8.1). Both estimates will follow from estimating

$$J_\ell = \int_{\mathbb{R}^N} \left(\frac{|x-y|}{2t}\right)^\ell \Gamma(x-y;t)|u_o(y)|dy \quad \text{for } \ell = 0,1.$$

The change of variable $y - x = 2\sqrt{t}\eta$ yields

$$\begin{aligned} J_\ell &= \frac{1}{\pi^{N/2}(\sqrt{t})^\ell} \int_{\mathbb{R}^N} |\eta|^\ell e^{-|\eta|^2} |u_o(x + 2\sqrt{t}\eta)| d\eta \\ &\le \frac{C_o e^{2\alpha_o |x|^2}}{\pi^{N/2}(\sqrt{t})^\ell} \int_{\mathbb{R}^N} |\eta|^\ell e^{-(1-8\alpha_o t)|\eta|^2} d\eta. \end{aligned}$$

Thus if $|x| < \rho$ and t is so small that $(1 - 8\alpha_o t) \ge \frac{1}{2}$

$$J_\ell \le \frac{\text{const}(C_o, N, \alpha_o)}{t^{\ell/2}} \int_0^\infty r^{N-1+\ell} e^{-\frac{1}{2}r^2} dr. \qquad \blacksquare$$

Proof (Proposition 8.2). First one computes

$$\begin{aligned} 0 = \frac{\partial^2}{\partial x_i \partial x_j} \int_{\mathbb{R}^N} \Gamma(x-y;t)dy &= \int_{\mathbb{R}^N} \frac{\partial^2}{\partial x_i \partial x_j} \Gamma(x-y;t)dy \\ &= \int_{\mathbb{R}^N} \frac{\partial^2}{\partial y_i \partial y_j} \Gamma(x-y;t)dy. \end{aligned}$$

Then for $|x| < \rho$ and $0 < t \le T$

$$\begin{aligned} |u_t(x,t)| + |u_{x_i x_j}(x,t)| &\le 2 \sum_{h,k=1}^N \left| \int_{\mathbb{R}^N} \frac{\partial^2}{\partial y_h \partial y_k} \Gamma(x-y;t)(u_o(y) - u_o(x))dy \right| \\ &\le 2N \int_{\mathbb{R}^N} \left(\frac{|x-y|^2}{4t^2} + \frac{1}{2t}\right) \Gamma(x-y;t)|u_o(y) - u_o(x)|dy \\ &= N \int_{|y|<2\rho} \left(\frac{|x-y|^2}{2t^2} + \frac{1}{t}\right) \Gamma(x-y;t)|u_o(y) - u_o(x)|dy \\ &\quad + N \int_{|y|>2\rho} \left(\frac{|x-y|^2}{2t^2} + \frac{1}{t}\right) \Gamma(x-y;t)|u_o(y) - u_o(x)|dy \\ &= H_1 + H_2. \end{aligned}$$

Since $u_o \in C^\delta_{\text{loc}}(\mathbb{R}^N)$

$$H_1 \le 2Nh_o(4\pi t)^{-N/2} \int_{|y|<2\rho} \Big(\frac{|x-y|^{2+\delta}}{4t^2} + \frac{|x-y|^{\delta}}{2t}\Big) e^{-\frac{|x-y|^2}{4t}} dy$$

where h_o is the Hölder constant of u_o over $B_{2\rho}$. Perform the change of variables $y-x = 2\sqrt{t}\eta$, and majorize the resulting integral by extending it to the whole of $\mathbb{R}^N$ to get

$$H_1 \le \frac{\tilde{A}}{t^{1-\delta/2}} \int_0^\infty |\eta|^{N-1}(|\eta|^{2+\delta} + |\eta|^\delta) e^{-|\eta|^2} d|\eta|$$

where $\tilde{A} = 4Nh_o\pi^{-N/2}\omega_N 2^\delta$. To estimate H_2, perform the same change of variables to get

$$H_2 \le 4C_oN\pi^{-N/2}e^{2\alpha_o|x|^2} \int_{|\eta|>\rho/2\sqrt{t}} \frac{(|\eta|^2+1)}{t} e^{-(1-8\alpha_o t)|\eta|^2} d\eta.$$

If t is so small that $(1-8\alpha_o t) \ge \frac{1}{2}$, this gives $H_2 \le \hat{A}t^{\delta/2-1}$, where

$$\hat{A} = \sup_{t\in(0,1/4\alpha_o)} 4C_oN\pi^{-N/2}e^{2\alpha_o\rho^2} \int_{|\eta|>\rho/2\sqrt{t}} \frac{(|\eta|^2+1)}{t^{\delta/2}} e^{-\frac{1}{2}|\eta|^2} d\eta. \qquad \blacksquare$$

9 The Inhomogeneous Cauchy Problem

Consider the problem of finding $u \in \mathcal{H}(S_T)$ satisfying

$$H(u) = f \text{ in } S_T, \quad u(\cdot,0) = u_o. \tag{9.1}$$

Assume that u_o is in $L^1_{\text{loc}}(\mathbb{R}^N)$ and satisfies the growth condition (4.1). The initial datum in (9.1) is taken in the sense of $L^1_{\text{loc}}(\mathbb{R}^N)$. On the forcing term f, we assume

$$f(\cdot,t) \in C^\delta_{\text{loc}}(\mathbb{R}^N) \text{ for some } \delta > 0 \text{ uniformly in } t > 0. \tag{9.2}$$

Moreover $f(\cdot,t)$ is required to satisfy the same growth condition as (4.1), uniformly in t, i.e.,

$$|f(x,t)| \le C_o e^{\alpha_o|x|^2} \quad \text{for } |x| > r_o \text{ uniformly in } t \ge 0. \tag{9.3}$$

Theorem 9.1. *Let (9.2) and (9.3) hold. Then there exists a solution to the inhomogeneous Cauchy problem (9.1) in the strip* S_T *for* $0 < T < \frac{1}{4\alpha_o}$. *Moreover, the solution is unique within the class (5.1) and is represented by*

$$u(x,t) = \int_{\mathbb{R}^N} \Gamma(x-y;t)u_o(y)dy + \int_0^t \int_{\mathbb{R}^N} \Gamma(x-y;t-s)f(y,s)dyds. \tag{9.4}$$

Proof. Since the heat operator is linear, u can be constructed as the sum of the solution of the homogeneous Cauchy problem ($f = 0$) and the solution of (9.1) with $u_o = 0$. Thus it suffices to take $u_o = 0$ in (9.1). The family of homogeneous Cauchy problems

$$\begin{aligned}&(x,t;s) \to v(x,t;s) \in \mathcal{H}(\mathbb{R}^N \times [0 < s < t \le T])\\ &v_t - \Delta v = 0 \ \text{ in } \ \mathbb{R}^N \times (s,T)\\ &v(\cdot,s;s) = f(\cdot,s)\end{aligned}$$

has, for all $0 < s < t \le T$, the unique bounded solution

$$v(x,t;s) = \int_{\mathbb{R}^N} \Gamma(x-y;t-s) f(y,s)dy$$

valid for $0 < t - s < T$. We claim that the function

$$u(x,t) = \int_0^t v(x,t;s)ds$$

solves (9.1), with $u_o = 0$. To show this, first observe that by virtue of the estimates of the previous section and assumption (9.2) and (9.2), the integrals

$$\int_0^t v(x,t;s)dx, \quad \int_0^t v_t(x,t;s)ds, \quad \int_0^t v_{x_i x_j}(x,t;s)ds \tag{9.5}$$

are uniformly convergent over compact subsets of $\mathbb{R}^N$. The convergence of the first integral implies that $u(\cdot,t) \to 0$ as $t \to 0$, in the sense of $L^1_{\text{loc}}(\mathbb{R}^N)$. Moreover, by direct calculation

$$\begin{aligned} u_t &= v(x,t;t) + \int_0^t v_t(x,t;s)ds\\ &= f(x,t) + \int_0^t \Delta v(x,t;s)ds = f(x,t) + \Delta u(x,t)\end{aligned}$$

where the calculation of the derivatives under the integral is justified by the uniform convergence of the integrals in (9.5). Thus u is a solution of (9.1) with $u_o = 0$. Such a solution is unique in view of (9.3). ■

This method is a particular case of the Duhamel principle. See Section 3.1c of the Complements of Chapter 6.

10 Problems in Bounded Domains

Let E be a bounded region of $\mathbb{R}^N$ with boundary ∂E of class C^1 and consider the Dirichlet problem (1.2). If $g = 0$, the problem is referred to as the homogeneous Dirichlet problem. We may solve such a homogeneous problem by separation of variables, i.e., by seeking solutions of the form $u(x,t) = X(x)T(t)$. Using the PDE, we find

$$T'(t) = -\lambda T(t), \quad t > 0; \qquad \Delta X = -\lambda X, \quad X \big|_{\partial E} = 0. \tag{10.1}$$

The second of these is solved by an infinite sequence of pairs (λ_n, v_n), where $\{\lambda_n\}$ is an increasing sequence of positive numbers and $\{v_n\}$ is a sequence of functions that form a complete orthonormal set in $L^2(E)$ (Section 11 of Chapter 4). In particular, the initial datum u_o, regarded as an element of $L^2(E)$, can be expanded as

$$u_o(x) = \sum_{i=1}^{\infty} \langle u_o, v_i \rangle v_i(x) \quad \text{with} \quad \|u_o\|_{2,E}^2 = \sum |\langle u_o, v_i \rangle|^2.$$

Then with λ_n determined by (10.1), one has $T_n(t) = T_{o,n} e^{-\lambda_n t}$, where $T_{o,n}$ are selected to satisfy the initial condition u_o. This gives approximate solutions of the form

$$u_n(x,t) = \sum_{i=1}^{n} T_{o,i} e^{-\lambda_i t} v_i(x), \quad T_{o,i} = \langle u_o, v_i \rangle.$$

Lemma 10.1 *The sequence $\{u_n(\cdot,t)\}$, is Cauchy in $L^2(E)$, uniformly in t.*

Proof. Fix $\varepsilon > 0$ and let $n_o = n_o(\varepsilon)$ be such that

$$\sum_{i>n_o}^{\infty} |\langle u_o, v_i \rangle|^2 < \varepsilon. \tag{10.2}$$

Next for all $m > n > n_o$ and all $0 \le t \le T$

$$\|u_m(\cdot,t) - u_n(\cdot,t)\|_{2,E}^2 \le \Big\| \sum_{i=n}^{m} \langle u_o, v_i \rangle e^{-\lambda_i t} v_i(x) \Big\|_{2,E}^2 \le \sum_{i>n_o}^{\infty} |\langle u_o, v_i \rangle|^2 < \varepsilon. \ \blacksquare$$

Thus, formally, a solution to the homogeneous Dirichlet problem (1.2) is

$$u(x,t) = \sum_{i=1}^{\infty} \langle u_o, v_i \rangle e^{-\lambda_i t} v_i(x) \tag{10.3}$$

where the convergence of the series is meant in the sense of $L^2(E)$, uniformly in $t \in [0,T]$. It remains to interpret in what sense the PDE is satisfied and in what sense u takes the boundary data.

Lemma 10.2 *Let u be defined by (10.3). Then $t \to u(\cdot,t)$ is continuous in $L^2(E)$, Moreover, $u(\cdot,t)$ takes the initial datum u_o in the sense of $L^2(E)$*

$$\|u(\cdot,t) - u_o\|_{2,E} \to 0 \quad \text{as}\, t \to 0. \tag{10.4}$$

Finally, $u(\cdot,t)$ satisfies the decay estimate

$$\|u(\cdot,t)\|_{2,E} \le e^{-\lambda_1 t} \|u_o\|_{2,E} \tag{10.5}$$

where λ_1 is the first eigenvalue of the Laplacean in E.

Proof. From the definitions

$$u(x,t) - u_o(x) = \sum_{i=1}^{\infty} T_{o,i}(e^{-\lambda_i t} - 1)v_i(x).$$

Fix $\varepsilon > 0$ and choose n_o as in (10.2). Then

$$\begin{aligned}
\|u(\cdot,t) - u_o\|_{2,E}^2 &= \sum_{i=1}^{\infty} |\langle u_o, v_i\rangle|^2 (e^{-\lambda_i t} - 1)^2 \\
&\le \sum_{i=1}^{n_o} |\langle u_o, v_i\rangle|^2 (e^{-\lambda_i t} - 1)^2 + \varepsilon \\
&\le (1 - e^{-\lambda_{n_o} t})^2 \|u_o\|_{2,E}^2 + \varepsilon.
\end{aligned}$$

Therefore, letting $t \to 0$ gives

$$\limsup_{t\to 0} \|u(\cdot,t) - u_o\|_{2,E} \le \sqrt{\varepsilon}.$$

This proves (10.4) and also that $t \to u(\cdot,t)$ is continuous at $t = 0$, in the topology of $L^2(E)$. The continuity at every $t \in [0,T]$ is proved in a similar fashion. The decay estimate (10.5) follows from the representation (10.3), Parseval's identity, and the fact that $\{\lambda_n\}$ is an increasing sequence. ■

Remark 10.1 This construction procedure as well as Lemmas 10.1 and 10.2 require only that the initial datum u_o be in $L^2(E)$.

10.1 The Strong Solution

Assume that $N \le 3$ and that the initial datum u_o is in $C^2(\bar{E})$ and satisfies $u_o = 0$ and $Du_o = 0$ on ∂E. Then, by Corollary 10.2 of Chapter 4, the series in (10.3) is absolutely and uniformly convergent. This implies that u satisfies the homogeneous boundary data on ∂E, in the sense of continuous functions. Also, the series

$$\sum_{i=0}^{\infty} \langle u_o, v_i\rangle \frac{d}{dt} e^{-\lambda_i t} v_i(x) \quad \text{and} \quad \sum_{i=0}^{\infty} \langle u_o, v_i\rangle e^{-\lambda_i t} \Delta v_i(x)$$

are absolutely and uniformly convergent. Therefore, the heat operator $H(\cdot)$ can be applied term by term in (10.3) to give

$$H(u) = \sum_{i=0}^{\infty} \langle u_o, v_i\rangle H[e^{-\lambda_i t} v_i(x)] = 0.$$

We conclude that if $1 \le N \le 3$, and if u_o satisfies the indicated regularity properties, then u as defined by (10.3) is a solution of the homogeneous Dirichlet problem (1.2).

10.2 The Weak Solution and Energy Inequalities

If $N > 3$, or if $u_o \in L^2(E)$, we will interpret the PDE in a weak sense. By construction, each u_n satisfies

$$\begin{aligned} u_{n,t} - \Delta u_n &= 0 \quad \text{in } E_T \\ u_n(\cdot,t)\big|_{\partial E} &= 0 \\ u_n(\cdot,0) &= \sum_{i=0}^{n} \langle u_o, v_i\rangle v_i. \end{aligned} \tag{10.6}$$

Let $\varphi \in C^2(\bar{E}_T)$ vanish on ∂E for all t. Multiply the PDE satisfied by u_n by φ and integrate by parts over E_t to obtain

$$\int_E (u_n\varphi)(t)dx - \int_0^t \int_E [u_n\varphi_t + u_n\Delta\varphi]\, dx\, dt = \int_E (u_{o,n}\varphi)(x,0)dx.$$

Letting $n \to \infty$ gives

$$\int_E (u\varphi)(t)dx - \int_0^t \int_E uH^*(\varphi)\, dx\, dt = \int_E u_o\varphi(x,0)dx. \tag{10.7}$$

In this limiting process we use Lemma 10.1, which is valid for all $N \ge 1$. We regard (10.7), as a weak notion of a solution of the homogeneous Dirichlet problem (1.2), and we call u a *weak solution.*

Lemma 10.3 *Weak solutions in the sense of (10.5), (10.7) are unique.*

Proof. The difference $w = u_1 - u_2$ of any two solutions satisfies (10.3), and in particular

$$\int_E (w\varphi)(t)dx - \int_0^t \int_E w\Delta\varphi\, dx\, d\tau = 0 \tag{10.8}$$

for all $\varphi \in C_o^1(E)$ independent of t. Since $w \in L^2(E_T)$, it must have for a.e. $t \in (0,T)$ a representation in terms of the eigenfunctions $\{v_n\}$, i.e.,

$$w(x,t) = \lim_{n\to\infty} \sum_{i=0}^{n} a_i(t)v_i(x) \qquad \text{for a.e. } t \in (0,T).$$

In (10.8) choose $\varphi = v_i$ to get

$$a_i(t) + \lambda_i \int_0^t a_i(s)ds = 0 \qquad \text{for all } i \in \mathbb{N}.$$

Thus $a_i(\cdot) = 0$ for all $i \in \mathbb{N}$, and $w = 0$. ■

Remark 10.2 The choice $\varphi = v_i$ is admissible if $v_i \in C_o^1(E)$. By Corollary 11.2 of Chapter 3, the eigenfunctions v_i are Hölder continuous in $\bar{E}$. By the Schauder estimates of Section 9 of Chapter 2, $v_i \in C^{2+\eta}(E)$, and by a bootstrap argument, $v_i \in C^\infty(E)$. Actually v_i are of class $C_o^{1+\eta}$ up to ∂E. Such an estimate up to the boundary, has been indicated in Section 9c of the Complements of Chapter 2.

Remark 10.3 If $u_o \in C^1_o(E)$, and u is smooth enough, we may take $u = \varphi$ in (10.7) to obtain the energy identity

$$\frac{1}{2}\|u(t)\|^2_{2,E} - \frac{1}{2}\|u_o\|^2_{2,E} + \int_0^t \int_E |Du|^2 dx\, dt = 0.$$

This identity also contains a statement of uniqueness since the PDE is linear. Indeed, $u_o = 0$ implies $u(\cdot, t) = 0$.

11 Energy and Logarithmic Convexity

Let E be a bounded open set in $\mathbb{R}^N$ with boundary ∂E of class C^1 and let $u \in \mathcal{H}(E_T) \cap C(\bar{E}_T)$ be a solution of the homogeneous Dirichlet problem (1.2). The quantity

$$\mathcal{E}(t) = \|u(\cdot, t)\|^2_{2,E}$$

is the thermal energy of the body E at time t.

Proposition 11.1 *For every* $0 \le t_1 < t < t_2 \le T$

$$\mathcal{E}(t) \le [\mathcal{E}(t_1)]^{\frac{t_2 - t}{t_2 - t_1}} \, [\mathcal{E}(t_2)]^{\frac{t - t_1}{t_2 - t_1}} . \tag{11.1}$$

Proof. Assume first that u is sufficiently regular as to justify the formal calculations below. Multiply the first of (1.2) by $u(\cdot, t)$ and integrate by parts over E, taking into account that $u(\cdot, t)$ vanishes on ∂E. This gives

$$\mathcal{E}' = 2\int_E u \Delta u \, dx = -2 \int_E Du \cdot Du \, dx.$$

From this

$$\mathcal{E}'' = -4 \int_E Du_t \cdot Du \, dx = 4 \int_E u_t \Delta u \, dx = 4 \int_E u_t^2 dx.$$

From this and Hölder's inequality

$$\mathcal{E}'^2 = \left(2 \int_E u u_t \, dx \right)^2 \le \mathcal{E}\mathcal{E}''.$$

First assume that $\mathcal{E}(t) > 0$ for all $t \in [t_1, t_2]$. Then the function $t \to \ln \mathcal{E}(t)$ is well defined and convex in such an interval, since

$$\frac{d^2}{dt^2} \ln \mathcal{E} = \frac{\mathcal{E}''\mathcal{E} - \mathcal{E}'^2}{\mathcal{E}^2} \ge 0.$$

Therefore, for all $t_1 < t < t_2$

$$\ln \mathcal{E}(t) \le \frac{t_2 - t}{t_2 - t_1} \ln \mathcal{E}(t_1) + \frac{t - t_1}{t_2 - t_1} \ln \mathcal{E}(t_2).$$

If $\mathcal{E}(t) \ge 0$, replacing it with $\mathcal{E}_\varepsilon = \mathcal{E} + \varepsilon$ for $\varepsilon > 0$ proves (11.1) for $\mathcal{E}_\varepsilon$. Then let $\varepsilon \to 0$. These calculations can be made rigorous by working with the approximate solutions $\{u_n\}$ of (10.5) and then by letting $n \to \infty$. ■

Remark 11.1 The energy $\mathcal{E}(\cdot)$ can also be defined for solutions of the homogeneous Neumann problem (1.2), and (11.1) holds for it ([200]).

11.1 Uniqueness for Some Ill Posed Problems

Corollary 11.1 *There exists at most one solution to the homogeneous backward Dirichlet problem*

$$\begin{gathered} u \in \mathcal{H}(E_T) \cap C(\bar{E}_T), \quad H(u) = 0 \ \textit{ in } \ E_T \\ u(\cdot, T) = u_T \in C(\bar{E}), \quad u(\cdot, t) \big|_{\partial E} = 0. \end{gathered}$$

Proof. It suffices to show that $u_T = 0$ implies $u(\cdot, t) = 0$. This follows from (11.1) with $t_2 = T$. ■

12 Local Solutions

We have observed that solutions of the Cauchy problem representable by (2.7) are analytic in the space variables and C^∞ in time for $t > 0$. It turns out that this is also the case for every local solution of the heat equation in a space-time cylindrical domain E_T. Let $Q_\rho = B_\rho \times (-\rho^2, 0)$ denote the cylinder with "vertex" at the origin, height ρ^2, and transversal cross section the ball B_ρ. For $(x_o, t_o) \in \mathbb{R}^{N+1}$, we let $(x_o, t_o) + Q_\rho$ denote the box congruent to Q_ρ and with "vertex" at (x_o, t_o), i.e.

$$(x_o, t_o) + Q_\rho = [|x - x_o| < \rho] \times [t_o - \rho^2, t_o].$$

If $(x_o, t_o) \in E_T$, we let $\rho > 0$ be so small that $(x_o, t_o) + Q_{4\rho} \subset E_T$. We also denote the integral average of $|u|$ over $(x_o, t_o) + Q_{4\rho}$ by

$$\fint_{(x_o,t_o)+Q_{4\rho}} |u| \, dy \, ds = \frac{1}{|Q_{4\rho}|} \int_{(x_o,t_o)+Q_{4\rho}} |u| \, dy \, ds.$$

Proposition 12.1 (Gevrey [97]) *Let $u \in \mathcal{H}(E_T)$ be a solution of the heat equation in E_T. There exist constants γ and C depending only on N such that for every box $(x_o, t_o) + Q_{4\rho} \subset E_T$*

$$\sup_{(x_o,t_o)+Q_\rho} \big| D^\alpha u \big| \le \gamma \frac{C^{|\alpha|} |\alpha|!}{\rho^{|\alpha|}} \fint_{(x_o,t_o)+Q_{4\rho}} |u| \, dy \, ds \tag{12.1}$$

for all multi-indices α. Moreover

$$\sup_{(x_o,t_o)+Q_\rho} \Big| \frac{\partial^k u}{\partial t^k} \Big| \le \gamma \frac{C^{2k} (2k)!}{\rho^{2k}} \fint_{(x_o,t_o)+Q_{4\rho}} |u| \, dy \, ds \tag{12.2}$$

for all positive integers k.

Proof. It suffices to prove only (12.1), since

$$\frac{\partial^k}{\partial t^k} u = \Delta^k u.$$

After a translation, we may assume that (x_o, t_o) coincides with the origin and u is a solution of the heat equation in $Q_{4\rho}$. Construct a non-negative smooth cutoff function ζ in $Q_{4\rho}$ satisfying

$$\begin{gathered}\zeta = 1 \text{ in } Q_{2\rho}, \quad |D\zeta| \le \frac{1}{4\rho}, \quad |\zeta_{y_i y_j}| \le \frac{1}{\rho^2}\, i,j = 1,\dots,N \\ 0 \le \zeta_t \le \frac{1}{\rho^2}, \quad \zeta(y,s) = 0 \text{ for } |y| \ge 4\rho \text{ and } s \le -(4\rho)^2.\end{gathered}$$

The function

$$w = \begin{cases} u\zeta & \text{in } [|y| \le 4\rho] \times (-(4\rho)^2, 0) \\ 0 & \text{otherwise} \end{cases}$$

coincides with u within $Q_{2\rho}$ and satisfies

$$H(w) = uH(\zeta) - 2Du \cdot D\zeta \overset{\text{def}}{=} f \quad \text{in } \mathbb{R}^N \times (-(4\rho)^2, 0].$$

Therefore, it can be viewed as the unique solution of the inhomogeneous Cauchy problem

$$H(w) = f \text{ in } \mathbb{R}^N \times (-(4\rho)^2, 0], \quad w(\cdot, -(4\rho)^2) = 0.$$

By Theorem 9.1

$$w(x,t) = \int_{-(4\rho)^2}^{t} \int_{\mathbb{R}^N} \Gamma(x-y; t-s) f(y,s)\, dy\, ds.$$

From this, after an integration by parts

$$\begin{aligned} w(x,t) = \int_{-(4\rho)^2}^{t} \int_{\mathbb{R}^N} u(y,s) \big[&\Gamma(x-y;t-s) H^*(\zeta) \\ &+ 2D\Gamma(x-y;t-s) \cdot D\zeta \big]\, dy\, ds \\ = \int_{-(4\rho)^2}^{-(2\rho)^2} \int_{|y|<4\rho} &\Gamma(x-y;t-s)\zeta_t\, u\, dy\, ds \\ + \int_{-(4\rho)^2}^{t} \int_{2\rho<|y|<4\rho} &\Gamma(x-y;t-s)\Delta\zeta\, u\, dy\, ds \\ + 2\int_{-(4\rho)^2}^{t} \int_{2\rho<|y|<4\rho} &D\Gamma(x-y;t-s) \cdot D\zeta\, u\, dy\, ds. \end{aligned}$$

Observe that in these integrals, if $|x| < \rho$ and $-\rho^2 \le t \le 0$, the kernel is not singular. Take the space derivatives of any order of both sides, for x and t in such a range, and use the properties of the cutoff function ζ to obtain

$$\begin{aligned}\sup_{Q_\rho}|D^\alpha u| &\le \frac{1}{\rho^2}\int_{-(4\rho)^2}^{-(2\rho)^2}\int_{|y|<4\rho}|D^\alpha\Gamma(x-y;t-s)||u|\,dy\,ds\\ &\quad+\frac{1}{\rho^2}\int_{-(4\rho)^2}^{t}\int_{2\rho<|y|<4\rho}|D^\alpha\Gamma(x-y;t-s)||u|\,dy\,ds\\ &\quad+\frac{1}{\rho}\int_{-(4\rho)^2}^{t}\int_{2\rho<|y|<4\rho}\sum_{i=1}^{n}|D^\alpha\Gamma_{x_i}(x-y;t-s)||u|\,dy\,ds\\ &= J_1+J_2+J_3.\end{aligned}$$

In the estimates to follow we denote by C and γ generic positive constants that can be different in different contexts. These may be quantitatively determined a priori only in terms of N and are independent of the multi-index α.

Lemma 12.1 *There exists a positive constant C such that*

$$|D^\alpha\Gamma(x-y;t-s)|\le C^{|\alpha|}\left[\left(\frac{\rho}{t-s}\right)^{|\alpha|}+\frac{|\alpha|!}{\rho^{|\alpha|}}\right]\Gamma(x-y;t-s)$$

for all $(x,t)\in Q_\rho$ and $(y,s)\in Q_{4\rho}$, and for every multi-index α.

Assuming the lemma for the moment, we proceed to estimate J_i. In estimating J_1 observe that within the domain of integration $t-s>\rho^2$. Therefore

$$|D^\alpha\Gamma(x-y;t-s)|\le\gamma\frac{C^{|\alpha|}|\alpha|!}{\rho^{|\alpha|+N}}$$

and

$$J_1\le\gamma\frac{C^{|\alpha|}|\alpha|!}{\rho^{|\alpha|}}\frac{1}{|Q_{4\rho}|}\int_{Q_{4\rho}}|u|\,dy\,ds.$$

The estimation of J_2 and J_3 hinges on the supremum of the function

$$g(\tau)=\frac{1}{\tau^m}e^{-A/\tau},\qquad\tau>0$$

where A and m are given positive constants. The supremum of g is achieved for $\tau=A/m$, and

$$|g(\tau)|\le\left(\frac{m}{A}\right)^m e^{-m}\qquad\text{for all }\ \tau\ge0.$$

Within the domain of integration of J_2 and J_3 one has $|x-y|>\rho$, provided $|x|<\rho$. Therefore

$$\begin{aligned}|D^\alpha\Gamma(x-y;t-s)|&\le\gamma C^{|\alpha|}\frac{\rho^{|\alpha|}}{[4(t-s)]^{|\alpha|+N/2}}e^{-\rho^2/4(t-s)}\\ &\quad+\gamma C^{|\alpha|}\frac{|\alpha|!}{\rho^{|\alpha|}}\frac{1}{[4(t-s)]^{N/2}}e^{-\rho^2/4(t-s)}\\ &\le\gamma\frac{C^{|\alpha|}}{\rho^{|\alpha|+N}}|\alpha|^{|\alpha|}e^{-|\alpha|}+\gamma\frac{C^{|\alpha|}|\alpha|!}{\rho^{|\alpha|+N}}.\end{aligned}$$

By Stirling's inequality $m^m e^{-m} \le \gamma m!$. Therefore by modifying the constants C and γ

$$|D^\alpha \Gamma(x-y;t-s)| \le \gamma \frac{C^{|\alpha|}|\alpha|!}{\rho^{|\alpha|}} \frac{1}{\rho^N}$$

for $(x,t) \in Q_\rho$ and $(y,s) \in Q_{4\rho} - Q_{2\rho}$. With this estimate in hand, we deduce that for $(x,t) \in Q_\rho$ and all multi-indices α

$$J_2 \le \gamma \frac{C^{|\alpha|}|\alpha|!}{\rho^{|\alpha|}} \frac{1}{|Q_{4\rho}|} \int_{Q_{4\rho}} |u|\,dy\,ds.$$

As for J_3, the previous calculations give

$$J_3 \le \gamma \frac{C^{|\alpha|+1}|\alpha|!(|\alpha|+1)}{\rho^{|\alpha|}} \frac{1}{|Q_{4\rho}|} \int_{Q_{4\rho}} |u|\,dy\,ds.$$

Now the constant C can be further modified so that

$$C^{|\alpha|+1}(|\alpha|+1) \le \bar{C}^{|\alpha|} \qquad \text{for all multi-indices } \alpha$$

and the theorem follows. ■

Proof (of Lemma 12.1). Fix a multi-index α of size $|\alpha| = n$ and let β be a multi-index of size $|\beta| = n+1$. Then

$$D^\beta \Gamma = D^\alpha \Gamma_{x_i} = D^\alpha \frac{(x-y)_i}{2(t-s)} \Gamma$$

for some $i = 1, \dots, N$. From this

$$2|D^\beta \Gamma| \le \left(\frac{\rho}{t-s}\right)|D^\alpha \Gamma| + \frac{n-1}{(t-s)}|D^{\bar{\alpha}}\Gamma|$$

where $\bar{\alpha}$ is a multi-index of size $|\bar{\alpha}| = n-1$. The lemma holds for $n = 1$. By induction, assuming that it does hold for multi-indices α of size $|\alpha| \le n$, we show that it continues to hold for multi-indices β of size $|\beta| = n+1$. Using the induction hypothesis

$$\frac{2}{\Gamma}|D^\beta \Gamma| \le C^n \bigg[\left(\frac{\rho}{t-s}\right)^{n+1} + \left(\frac{\rho}{t-s}\right)\frac{2n!}{\rho^n} + \left(\frac{\rho}{t-s}\right)^n \frac{n-1}{C\rho} \bigg].$$

By Young's inequality

$$\left(\frac{\rho}{t-s}\right)\frac{2n!}{\rho^n} \le \frac{1}{n+1}\left(\frac{\rho}{t-s}\right)^{n+1} + \frac{2^{\frac{n+1}{n}} n}{n+1} \frac{(n!)^{\frac{n+1}{n}}}{\rho^{n+1}}$$
$$\left(\frac{\rho}{t-s}\right)^n \frac{n-1}{C\rho} \le \frac{n}{n+1}\left(\frac{\rho}{t-s}\right)^{n+1} + \frac{(n-1)^{n+1}}{n+1} \frac{1}{C^{n+1}\rho^{n+1}}.$$

Using Stirling's inequality and choosing C sufficiently large

$$\frac{(n-1)^{n+1}}{n+1}\frac{1}{C^{n+1}} \le (n+1)!.$$

This in turn implies

$$\frac{|D^\beta \varGamma|}{\varGamma} \le \frac{C^n}{2}\left[2\left(\frac{\rho}{t-s}\right)^{n+1} + \left(1+\frac{4n}{(n+1)^2}(n!)^{1/n}\right)\frac{(n+1)!}{\rho^{n+1}}\right].$$

The number $(n!)^{1/n}$ is the geometric mean of the first n integers, which is majorized by its arithmetic mean. Therefore

$$\frac{4n}{(n+1)^2}(n!)^{1/n} \le \frac{4n}{(n+1)^2}\frac{\sum_{i=1}^n i}{n} = \frac{2n}{n+1}.$$

These remarks in the previous inequality yield

$$\frac{|D^\beta \varGamma|}{\varGamma} \le \frac{3}{2}C^n\left[\left(\frac{\rho}{t-s}\right)^{n+1} + \frac{(n+1)!}{\rho^{n+1}}\right].$$

It remains to choose C so that $\frac{3}{2}C^n \le C^{n+1}$. ■

12.1 Variable Cylinders

To simplify the symbolism, let us assume that (x_o, t_o) coincides with the origin. The estimates of Theorem 12.1 give information on $D^\alpha u$ on the cylinder Q_ρ in terms of the L^1-norm of u over the larger box $Q_{2\rho}$. The proof could be repeated with minor variations to derive a similar statement for any pair of boxes Q_ρ and $Q_{\sigma\rho}$ for $\sigma \in (0,1)$. Tracing the constant dependence on σ gives:

Proposition 12.2 *Let u be a solution of the heat equation in Q_ρ. There exist constants C and γ, depending only on N, such that for every multi-index α, for every non-negative integer k, and for all $\sigma \in (0,1)$*

$$\|D^\alpha u\|_{\infty,Q_{\sigma\rho}} \le \gamma\frac{C^{|\alpha|}|\alpha|!}{(1-\sigma)^{N+2+|\alpha|}\rho^{|\alpha|}}\fint_{Q_\rho}|u|\,dy\,ds \tag{12.3}$$

$$\|\frac{\partial^k}{\partial t^k}u\|_{\infty,Q_{\sigma\rho}} \le \gamma\frac{C^k(2k)!}{(1-\sigma)^{N+2+2k}\rho^{2k}}\fint_{Q_\rho}|u|\,dy\,ds. \tag{12.4}$$

Remark 12.1 Estimates (12.3)–(12.4) hold for any pair of boxes $(x_o,t_o)+Q_\rho$ and $(x_o,t_o)+Q_{\sigma\rho}$ contained in $\varOmega_T$.

12.2 The Case $|\alpha| = 0$

We state explicitly the estimate of Proposition 12.2 for the case $|\alpha| = 0$.

Corollary 12.1 *Let* $u \in \mathcal{H}(E_T)$ *be a local solution of the heat equation in* E_T*. There exists a constant* C *depending only on* N *such that for every box* $(x_o, t_o) + Q_\rho$ *contained in* E_T *and all* $\sigma \in (0,1)$

$$\sup_{(x_o,t_o)+Q_{\sigma\rho}} |u| \le \frac{C}{(1-\sigma)^{N+2}} \fint_{(x_o,t_o)+Q_\rho} |u| \, dy \, ds. \tag{12.5}$$

These estimates have a number of consequences for local or global non-negative solutions. In the next two sections we present some of them.

13 The Harnack Inequality

Non-negative local solutions of the heat equation in E_T satisfy an inequality similar to the Harnack estimate valid for non-negative harmonic functions(Section 5.1 of Chapter 2). This inequality can be stated as follows. For $\rho > 0$ consider the box $\mathcal{Q}_\rho = B_\rho \times (-\rho^2, \rho^2)$, with its "center" at the origin. If $(x_o, t_o) \in E_T$, let

$$(x_o, t_o) + \mathcal{Q}_\rho = [|x - x_o| < \rho] \times (t_o - \rho^2, t_o + \rho^2)$$

be the box congruent to $\mathcal{Q}_\rho$ and centered at (x_o, t_o).

Theorem 13.1. *Let* $u \in \mathcal{H}(E_T)$ *be a non-negative solution of the heat equation in* E_T*. There exists a constant* c *depending only upon* N *such that for every box* $(x_o, t_o) + \mathcal{Q}_{4\rho} \subset E_T$

$$\inf_{|x-x_o|<\rho} u(x, t_o + \rho^2) \ge c\, u(x_o, t_o). \tag{13.1}$$

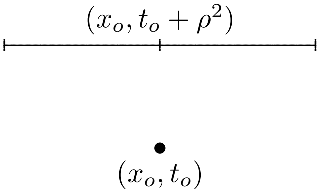

Fig. 13.1

Such an estimate can be given different equivalent forms. We illustrate one of them, assuming for simplicity of notation that $(x_o, t_o) = (0, 0)$. To distinguish between the upper part and the lower part of $\mathcal{Q}_\rho$, let us set

$$\mathcal{Q}_\rho^- = B_\rho \times (-\rho^2, 0), \qquad \mathcal{Q}_\rho^+ = B_\rho \times (0, \rho^2).$$

Fix $\sigma \in (0,1)$, and inside $\mathcal{Q}_\rho^+$ and $\mathcal{Q}_\rho^-$ construct the two sub-boxes

$$\mathcal{Q}_{\sigma\rho}^- = B_{\sigma\rho} \times (-\sigma\rho^2, 0), \qquad \mathcal{Q}_{\sigma\rho}^* = B_{\sigma\rho} \times \big((1-\sigma)\rho^2, \rho^2\big).$$

Theorem 13.2. *Let* $u \in \mathcal{H}(\mathcal{Q}_{4\rho})$ *be a non-negative solution of the heat equation in* $\mathcal{Q}_{4\rho}$. *For every* $\sigma \in (0,1)$ *there exists a constant* c *depending only upon* N *and* σ *such that*

$$\inf_{\mathcal{Q}^*_{\sigma\rho}} u \geq c \sup_{\mathcal{Q}^-_{\sigma\rho}} u. \tag{13.2}$$

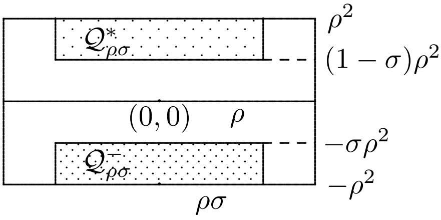

Fig. 13.2

In the case of harmonic functions, the main tool in the proof of the Harnack estimate was the explicit Poisson representation formula of the solution of the Dirichlet problem for the Laplacean over a ball (formula (3.9) of Chapter 2). The corresponding Dirichlet problem for the heat equation over cylinders whose cross section is a sphere does not have an explicit solution formula. However, local representations will play a major role via the regularity results of Proposition 12.1.

The form (13.1) of the Harnack estimate is due independently to Pini ([203]) and Hadamard ([112]). The form (13.2) was introduced by Moser in a more general context ([187]), which we will extensively deal with in Chapter 12. The proof we present here, based on an idea of Landis [153], is "nonlinear" in nature, and its main ideas can be applied to a large class of parabolic equations, including degenerate ones ([49], Chapters 6–7, and [55]). Alternative forms of the parabolic Harnack inequality that resemble the mean value property of harmonic functions are in [56]. We will consider them in Section 10.4 of Chapter 12.

The Harnack inequality is a property also shared by non-negative solutions of parabolic equations in nondivergence form, which are the subject of Chapter 13. In particular, Section 4 of such a chapter will be devoted to the Harnack estimate in this context.

13.1 Compactly Supported Sub-Solutions

For given positive numbers M, r, b, and $(x,t) \in S_\infty$, consider the function

$$\psi(x,t) = \frac{M\,r^{2b}}{(t+r^2)^b}(4-|z|^2)^2_+, \qquad \text{where } |z|^2 = \frac{|x|^2}{t+r^2}.$$

One verifies that $\psi \in \mathcal{H}(S_\infty) \cap C(\bar{S}_\infty)$, and it vanishes identically outside the paraboloid $|z| < 2$.

Lemma 13.1 *The number $b > 0$ can be chosen so that $H(\psi) \le 0$ in S_∞, for all $M > 0$.*

Proof. By direct calculation

$$H(\psi) = \frac{Mr^{2b}}{(t+r^2)^{b+1}}(4-|z|^2)_+ \left(-b(4-|z|^2)_+ + 4N - 2\frac{|z|^4}{4-|z|^2}\right).$$

For $\frac{4N}{N+1} \le |z|^2 < 4$, we have $H(\psi) \le 0$. For $|z|^2 \le \frac{4N}{N+1}$

$$H(\psi) \le \frac{Mr^{2b}}{(t+r^2)^{b+1}}(1-|z|^2)_+ \left(-\frac{4b}{N+1} + 4N\right).$$

To prove the lemma, choose $b = N(N+1)$. ∎

We will consider a version of ψ "centered" at points $(x_o, t_o) \in \mathbb{R}^{N+1}$. Precisely

$$\Psi_{(x_o,t_o)}(x,t) = \psi(x-x_o, t-t_o) = \frac{Mr^{2b}}{[t-t_o+r^2]^b}\left(4 - \frac{|x-x_o|^2}{t-t_o+r^2}\right)_+^2.$$

Corollary 13.1 *Let $b = N(N+1)$. Then*

$$H\big(\Psi_{(x_o,t_o)}\big) \le 0 \qquad \text{in } \mathbb{R}^N \times (t_o, \infty).$$

13.2 Proof of Theorem 13.1

We may assume that $(x_o, t_o) = (0,0)$, that $\rho = 1$, and that $u(x_o, t_o) = 1$. This is achieved by the change of variables

$$x \to \frac{x-x_o}{\rho}, \qquad t \to \frac{t-t_o}{\rho^2}, \qquad u \to \frac{u}{u(0,0)}.$$

Thus we have to show that if u is a solution of the heat equation in the box $\mathcal{Q}_4 = B_4 \times (-4,4)$ such that $u(0,0) = 1$, then $u(x,1) \ge c$, for all $x \in B_1$, for a positive constant c depending only on N. To prove this we proceed in three steps.

13.2.1 Locating the Supremum of u in Q_1

For $s \in [0,1)$, consider the family of nested and expanding boxes

$$Q_s = [|x| < s] \times (-s^2, 0]$$

and the nondecreasing family of numbers

$$M_s = \sup_{Q_s} u, \qquad N_s = (1-s)^{-\xi}$$

where ξ is a positive constant to be chosen later. One checks that $M_o = N_o = 1$, and as $s \to 1$

$$\lim_{s\to 1} M_s < \infty \quad \text{and} \quad \lim_{s\to 1} N_s = \infty.$$

Therefore the equation $M_s = N_s$ has roots. Denote by s_o the largest of such roots, so that

$$\sup_{Q_{s_o}} u = M_{s_o} = (1-s_o)^{-\xi} \quad \text{and } M_s < (1-s)^{-\xi} \quad \text{for } s > s_o.$$

Since $u \in C(\bar{Q}_4)$, the supremum M_{s_o} is achieved at some $(x_o,t_o) \in \bar{Q}_{s_o}$, i.e., $u(x_o,t_o) = (1-s_o)^{-\xi}$.

13.2.2 Positivity of u over a Ball

We show next that the "largeness" of u at (x_o,t_o) spreads over a small ball centered at x_o at the level t_o.

Lemma 13.2 *There exists $\varepsilon > 0$ depending only upon N and independent of s_o such that*

$$u(x,t_o) \ge \frac{1}{2}(1-s_o)^{-\xi} \quad \textit{for all } |x-x_o| < \varepsilon\frac{1-s_o}{2}.$$

Proof. Costruct the box with "vertex" at (x_o,t_o) and radius $\frac{1}{2}(1-s_o)$

$$(x_o,t_o) + Q_{\frac{1}{2}(1-s_o)} = \left[|x-x_o| < \frac{1-s_o}{2}\right] \times \left[t_o - \left(\frac{1-s_o}{2}\right)^2, t_o\right].$$

By construction, $(x_o,t_o) + Q_{\frac{1}{2}(1-s_o)} \subset Q_{\frac{1}{2}(1+s_o)}$, and by the definition of M_s and N_s

$$\sup_{(x_o,t_o)+Q_{\frac{1}{2}(1-s_o)}} u \le \sup_{Q_{\frac{1}{2}(1+s_o)}} u \le N_{\frac{1+s_o}{2}} = 2^{\xi}(1-s_o)^{-\xi}.$$

Apply Proposition 12.2 with $|\alpha| = 1$ over the pair of boxes $(x_o,t_o) + Q_{\frac{1}{8}(1-s_o)}$ and $(x_o,t_o) + Q_{\frac{1}{2}(1-s_o)}$, to obtain

$$\sup_{(x_o,t_o)+Q_{\frac{1}{8}(1-s_o)}} |Du| \le \frac{C}{1-s_o} \sup_{(x_o,t_o)+Q_{\frac{1}{2}(1-s_o)}} u$$

for a constant C dependent only upon N. Let $\varepsilon \in (0, \frac{1}{4})$ to be chosen later. Then forall $|x-x_o| < \varepsilon\frac{1}{2}(1-s_o)$

$$\begin{aligned} u(x,t_o) &\ge u(x_o,t_o) - \varepsilon\frac{1-s_o}{2} \sup_{(x_o,t_o)+Q_{\frac{1}{8}(1-s_o)}} |Du| \\ &\ge (1-s_o)^{-\xi}(1 - 2^{\xi}C\varepsilon). \end{aligned}$$

To prove the lemma we choose ε small enough that $1 - 2^{\xi}C\varepsilon = \frac{1}{2}$. ∎

13.2.3 Expansion of the Positivity Set

The point (x_o, t_o) being fixed, consider the comparison function $\Psi_{(x_o,t_o)}$ for the choice of parameters

$$M = \frac{1}{2}(1-s_o)^{-\xi}, \qquad r = \varepsilon\frac{1-s_o}{2}.$$

By Lemma 13.2

$$u(x,t_o) \ge \frac{1}{2}(1-s_o)^{-\xi} \ge \Psi_{(x_o,t_o)}(x,t_o), \qquad \text{for } |x-x_o| < r.$$

Therefore, by the maximum principle, $u \ge \Psi_{(x_o,t_o)}$ in the box $B_4 \times [t_o, 4)$. In particular, for $t = 1$ and $|x| < 1$

$$u(x,1) \ge 2^{1-4b}(1-s_o)^{-\xi+b}.$$

The knowledge of s_o is only qualitative. We render the estimate independent of s_o by choosing $\xi = b$. This gives $u(x,1) \ge 2^{1-4b} = c$. ■

Remark 13.1 A somewhat similar argument will be employed in Section 10.3 of Chapter 12.

14 Positive Solutions in S_T

We have shown that uniqueness for the Cauchy problem (1.4) holds within the class of functions satisfying the growth condition (5.1). However, the representation formula (2.7) is well defined for initial data $u_o \in L^1_{\text{loc}}(\mathbb{R}^N)$ for which the integral is convergent. This suggests that we consider the problem of uniqueness within the class of functions $u \in \mathcal{H}(S_T)$ such that

$$\int_{\mathbb{R}^N} |u(y,s)|\Gamma(x-y;t-s)dy < \infty \qquad \text{for } s \in (0,t). \tag{14.1}$$

It turns out that uniqueness for the Cauchy problem holds for functions in such a class. More important, every non-negative solution of the heat equation in S_T satisfies (14.1). Therefore, uniqueness for the Cauchy problem (1.4) holds within the class of non-negative solutions. This was observed by Widder in one space dimension ([276]). Here we give a different proof, valid in any space dimension.

Theorem 14.1. *Let $u \in \mathcal{H}(S_T)$ satisfy*

$$H(u) = 0 \ \text{ in } \ S_T, \qquad \text{and} \qquad u(\cdot,t) \to 0 \ \text{ in } \ L^1_{\text{loc}}(\mathbb{R}^N) \ \text{ as } \ t \to 0.$$

Then, if u satisfies (14.1), it vanishes identically in S_T.

Proof. Let $(x,t) \in S_T$ be arbitrary but fixed. For $\rho > 2|x|$ consider the balls $B_{2\rho}$ and let $y \to \zeta(y) \in C_o^2(B_{2\rho})$ be a non-negative cutoff function in $B_{2\rho}$ satisfying

$$\zeta = 1 \ \text{ in } \ B_\rho, \quad |D\zeta| \le \frac{1}{\rho}, \quad |\zeta_{y_i y_j}| \le \frac{2}{\rho}^2 .$$

For $\delta > 0$ let

$$h_\delta(u) = \begin{cases} 1 & \text{if } \ u > \delta \\ \dfrac{u}{\delta} & \text{if } \ |u| \le \delta \\ -1 & \text{if } \ u < -\delta. \end{cases} \tag{14.2}$$

Multiply the PDE by $h_\delta(u)\zeta(y)\Gamma(x-y;t-s)$, and integrate by parts in $dyds$ over the cylindrical domain $B_{2\rho} \times (\tau, t-\varepsilon)$ for $0 < \tau < t-\varepsilon$ and $0 < \varepsilon < t$. This gives

$$\begin{aligned} \int_{B_{2\rho}\times\{t-\varepsilon\}} &\left(\int_0^u h_\delta(\xi)d\xi \right) \Gamma(x-y;\varepsilon)\zeta(y)dy \\ &+ \frac{1}{\delta} \int_\tau^{t-\varepsilon} \int_{B_{2\rho}} |Du|^2 \Gamma(x-y;t-s)\chi_{[}|u| < \delta]\zeta(y)\, dy\, ds \\ &= \int_{B_{2\rho}\times\{\tau\}} \left(\int_0^u h_\delta(\xi)d\xi \right) \Gamma(x-y;t-\tau)\zeta(y)dy \\ &+ \int_\tau^{t-\varepsilon} \left(\int_0^u h_\delta(\xi)d\xi \right) (\Gamma \Delta\zeta + 2D\Gamma \cdot D\zeta)\, dy\, ds. \end{aligned}$$

As $\tau \to 0$, the first integral on the right-hand side tends to zero, since it is majorized by

$$\text{const} \int_{B_{2\rho}} |u(y,\tau)|dy \to 0 \ \text{ as } \ \tau \to 0.$$

Discard the second term on the left-hand side since it is non-negative and let first $\delta \to 0$ and then $\tau \to 0$ to obtain

$$\begin{aligned} \int_{B_{2\rho}} &|u(y,t-\varepsilon)|\Gamma(x-y;\varepsilon)\zeta(y)dy \\ &\le \frac{2}{\rho^2} \int_0^{t-\varepsilon} \int_{B_{2\rho}} |u(y,s)|\Gamma(x-y;t-s)\, dy\, ds \\ &\quad + \frac{2}{\rho} \int_0^{t-\varepsilon} \int_{B_{2\rho}-B_\rho} |u(y,s)||D\Gamma(x-y;t-s)|\, dy\, ds. \end{aligned}$$

The right-hand side of this inequality tends to zero as $\rho \to \infty$. This is obvious for the first term in view of (14.1). The second term is majorized by

$$\begin{aligned} \frac{1}{\rho} \int_0^{t-\varepsilon} &\int_{\rho<|y|<2\rho} |u(y,s)|\Gamma(x-y;t-s)\frac{|x-y|}{t-s}\, dy\, ds \\ &\le \frac{4}{\varepsilon} \int_0^{t-\varepsilon} \int_{\rho<|y|<2\rho} |u(y,s)|\Gamma(x-y;t-s)\, dy\, ds. \end{aligned}$$

Letting $\rho \to \infty$ gives

$$\int_{B_r} |u(y,t-\varepsilon)| \Gamma(x-y;\varepsilon)dy = 0$$

for all $r > 2|x|$ and all $\varepsilon \in (0,t)$. Finally, we let $\varepsilon \to 0$. Arguing as in Section 2, in the derivation of the representation formula (2.7), gives $u(x,t) = 0$. ■

14.1 Non-Negative Solutions

Theorem 14.2. *Let $u \in \mathcal{H}(S_T)$ be a non-negative solution of*

$$H(u) = 0 \;\; \text{in} \;\; S_T, \qquad \text{and} \qquad u(\cdot,t) \to 0 \;\; \text{in} \;\; L^1_{\text{loc}}(\mathbb{R}^N) \;\; \text{as} \;\; t \to 0.$$

Then u vanishes identically in S_T.

It will suffice to prove:

Proposition 14.1 *Let $u \in \mathcal{H}(S_T)$ be a non-negative solution of the heat equation in S_T. Then $\forall (x_o,t_o) \in S_T$*

$$\int_{\mathbb{R}^N} u(y,s)\Gamma(x_o-y;t_o-s)dy \le u(x_o,t_o) \quad \text{for all} \;\; 0 < s < t_o. \tag{14.3}$$

Proof. Fix $(x_o,t_o) \in S_T$ and $s \in (0,t_o)$ and introduce the change of variables

$$\tau = \frac{t-s}{t_o-s}, \qquad \eta = \frac{y-x_o}{\sqrt{t_o-s}}.$$

The function

$$U(\eta,\tau) = u\left(x_o + \sqrt{t_o-s}\,\eta, s + (t_o-s)\tau\right)$$

satisfies the heat equation in $\mathbb{R}^N \times [0,1]$. For such a function, (14.3) becomes

$$\int_{\mathbb{R}^N} U(\eta,0)\Gamma(\eta;1)d\eta \le U(0,1).$$

Thus it will be enough to prove that if $u \in \mathcal{H}(\bar{S}_1)$ is a non-negative solution of the heat equation in S_1 such that $u(\cdot,0) \in C^2(\mathbb{R}^N)$, then

$$\int_{\mathbb{R}^N} u(y,0)\Gamma(y;1)dy \le u(0,1). \tag{14.4}$$

To prove (14.4) fix $\rho > 0$ and consider the Cauchy problem

$$H(v) = 0 \;\; \text{in} \;\; S_1, \quad v(x,0) = \begin{cases} \zeta(x)u(x,0) & \text{if } |x| < 2\rho \\ 0 & \text{otherwise} \end{cases} \tag{14.5}$$

where $x \to \zeta(x) \in C_o^\infty(B_{2\rho})$ is non-negative and equals one on the ball B_ρ. Since the initial datum is compactly supported in $B_{2\rho}$, the unique bounded solution of (14.5) is given by

$$v(x,t) = \int_{|y|<2\rho} \zeta(y)u(y,0)\Gamma(x-y;t)dy.$$

Lemma 14.1 *$u \ge v$ in $\bar{S}_1$.*

Assuming this fact for the moment, it follows from the representation of v and the structure of the cutoff function ζ that

$$u(0,1) \ge \int_{B_\rho} u(y,0)\Gamma(y;1)dy.$$

This proves (14.4), since $\rho > 0$ is arbitrary. ■

Proof (of Lemma 14.1). The statement would follow from the maximum principle if u satisfed the growth condition (5.1). The positivity of u will replace such information. Let n_o be a positive integer larger than 2ρ, and for $n \ge n_o$, consider the sequence of homogeneous Dirichlet problems

$$\begin{aligned} H(v_n) &= 0 \ \text{ in } \ Q_n = B_n \times (0,1) \\ v_n \Big|_{|y|=n} &= 0 \\ v_n(x,0) &= \begin{cases} \zeta(x)u(x,0) & \text{if } |x| < 2\rho \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{14.6}$$

We regard the functions v_n as defined in the whole of S_1 by defining them to be zero outside Q_n. By the maximum principle applied over the bounded domains Q_n

$$0 \le v_n \le v_{n+1} \le \|u(\cdot,0)\|_{\infty,B_{2\rho}} \quad \text{and} \quad v_n \le u \tag{14.7}$$

for all $n \ge n_o$. By the second of these, the proof of the lemma reduces to showing that the increasing sequence $\{v_n\}$ converges to the unique solution of (14.5) uniformly over compact subsets of S_1. Consider compact subsets of the type $K = \bar{B}_R \times [\varepsilon, 1-\varepsilon]$ for $\varepsilon \in (0, \frac{1}{2})$ and $R \ge 2\rho$. By the estimates of Proposition 12.1 and the uniform upper bound of the first of (14.7), for every multi-index α and every positive integer k, there exists a constant C depending only on N, ε, R, $|\alpha|$, k and independent of n such that

$$\|D^\alpha v_n\|_{\infty,K} + \left\| \frac{\partial^k}{\partial t^k} v_n \right\|_{\infty,K} \le C \quad \text{for all } \ n \ge 2R.$$

It follows, by a diagonalization process, that $\{v_n\} \to w$, uniformly over compact subsets of S_1, where $w \in C^\infty(S_1)$ and satisfies the heat equation. It remains to prove that

$$w(\cdot,t) \to \zeta u(\cdot,0) \ \text{ in } \ L^1_{\text{loc}}(\mathbb{R}^N) \ \text{ as } \ t \to 0.$$

For this, rewrite (14.6) as

$$\begin{gathered} f_n = v_n - \zeta u(x,0), \quad f_{n,t} - \Delta f_n = \Delta\zeta u(x,0) \ \text{ in } \ S_1 \\ f_n\Big|_{|x|=n} = 0, \quad f_n(x,0) = 0. \end{gathered}$$

Let $h_\delta(\cdot)$ be the approximation to the Heaviside function introduced in (14.2). Multiply the PDE by $h_\delta(f_n)$ and integrate over $B_n \times (0,t)$ for $t \in (0,1)$ to obtain

$$\int_{B_n\times\{t\}} \left(\int_0^{f_n} h_\delta(\xi)d\xi\right)dy + \frac{1}{\delta}\int_0^t \int_{B_n} |Df_n|^2 \chi[|f_n| < \delta]\,dy\,ds$$
$$= \int_0^t \int_{B_{2\rho}} \Delta\zeta(y)u(y,0)h_\delta(f_n)\,dy\,ds.$$

Discard the second term on the left-hand side, which is non-negative, and let $\delta \to 0$ to get

$$\int_{B_R} |v_n(y,t) - \zeta(y)u(y,0)|dy \le t|B_{2\rho}|\|\Delta\zeta u(x,0)\|_{\infty,B_{2\rho}}.$$

Letting $n \to \infty$

$$\|w(y,t) - \zeta(y)u(y,0)\|_{1,B_R} \le t|B_{2\rho}|\|\Delta\zeta u(x,0)\|_{\infty,B_{2\rho}}.$$

By the first of (14.7), w is bounded; therefore by uniqueness of bounded solutions of the Cauchy problem, $w = v$. ■

Remark 14.1 The Tychonov function defined in (5.2) is of variable sign.

Problems and Complements

2c Similarity Methods

2.1c The Heat Kernel Has Unit Mass

To verify (2.2), disregard momentarily the factor $\pi^{-N/2}$, and introduce the change of variables $y - x = 2\sqrt{(t-s)}\eta$, whose Jacobian is $t[4(t-s)]^{N/2}$. This transforms the integral into

$$\begin{aligned}\int_{\mathbb{R}^N} e^{-|\eta|^2}d\eta &= \int_{\mathbb{R}^N} e^{\eta_1^2+\cdots+\eta_N^2}d\eta_1\cdots d\eta_N\\ &= \prod_{j=1}^N \int_{\mathbb{R}} e^{-\eta_j^2}d\eta_j = \left(\int_{\mathbb{R}} e^{-s^2}ds\right)^N\\ &= \left(\int_{\mathbb{R}} e^{-\eta_1^2}d\eta_1 \int_{\mathbb{R}} e^{-\eta_2^2}d\eta_2\right)^{N/2} = \left(\int_{\mathbb{R}^2} e^{-|\eta|^2}d\eta\right)^{N/2}\\ &= \left(2\pi\int_0^\infty re^{-r^2}dr\right)^{N/2} = \pi^{N/2}.\end{aligned}$$

2.2c The Porous Medium Equation

Find similarity solutions for the nonlinear evolution equation

$$u_t - \Delta u^m = 0 \quad u \geq 0, \quad m \geq 1.$$

This equation arises in the filtration of a fluid in a porous medium [226]. Similarity solutions were derived independently by Barenblatt [14], and Pattle [199]. Attempt solutions of the form $u(x,t) = h(t)f(\xi)$, where $\xi = \frac{|x|^2}{t^\sigma}$ and σ is a positive number to be found.

Solution: By straightforward computation, we have

$$\begin{aligned} u_t &= h'f - \xi\frac{\sigma}{t}hf' \\ Du^m &= mh^m\, f^{m-1}f'\frac{2x}{t^\sigma} \\ \Delta u^m &= mh^m\Big\{\Big[(m-1)f^{m-2}f'^2 + f^{m-1}f''\Big]\frac{4\xi}{t^\sigma} + f^{m-1}f'\frac{2N}{t^\sigma}\Big\}. \end{aligned}$$

Enforcing the equation gives

$$h'f - mh^m f^{m-1}f'\frac{2N}{t^\sigma} = \frac{\xi}{t}h\Big\{\sigma f' + \frac{4m}{t^{\sigma-1}}h^{m-1}\Big[(m-1)f^{m-2}f'^2 + f^{m-1}f''\Big]\Big\}.$$

Stipulate to take

$$h(t) = t^\theta \quad \text{where} \quad \theta(m-1) = \sigma - 1.$$

This choice makes both sides homogeneous with respect to t. Next, to compute f, set to zero the term $\{\cdots\}$, i.e.,

$$\sigma f' + 4m\big[(m-1)f^{m-2}f'^2 + f^{m-1}f''\big] = 0,$$

which is equivalent to

$$\sigma f' + 4m\big(f^{m-1}f'\big)' = 0.$$

Choose the integration constant to be zero to obtain

$$\sigma + 4mf^{m-2}f' = 0 \quad \Longrightarrow \quad f(\xi) = \Big(C - \frac{\sigma(m-1)}{4m}\xi\Big)_+^{\frac{1}{m-1}}$$

for a constant C to be chosen. Setting to zero the left-hand side of the equation gives

$$h'f - \frac{2N}{t^\sigma}h^m mf^{m-1}f' = 0 \quad \Longrightarrow \quad h'f - \frac{2N}{t^\sigma}h^m(f^m)' = 0.$$

Using the previously obtained form of f, compute

$$(f^m)' = -\frac{\sigma}{4} f.$$

Therefore,

$$h'f + \frac{\sigma}{4}\frac{2N}{t^\sigma} h^m f = 0 \quad \Longrightarrow \quad h' + \frac{N\sigma}{2t^\sigma} h^m = 0.$$

This is integrated as

$$h^{-m}h' = -\frac{N\sigma}{2} t^{-\sigma} \quad \Longrightarrow \quad h^{1-m} = \frac{N(m-1)\sigma}{1-\sigma} t^{1-\sigma},$$

and it is compatible with the choice $h(t) = t^\theta$ if

$$1 = \frac{N(m-1)\sigma}{1-\sigma} = \frac{-N[1+\theta(m-1)]}{2\theta}.$$

This implies

$$\theta = -\frac{N}{\kappa} \quad \text{and} \quad \sigma = \frac{2}{\kappa}, \quad \text{where} \quad \kappa = N(m-1)+2.$$

Hence, we arrive at

$$\Gamma_m(x,t) = \frac{1}{t^{N/\kappa}} \left[1 - c\gamma_m \left(\frac{|x|^2}{t^{2/\kappa}}\right)\right]_+^{\frac{1}{m-1}}, \quad t > 0$$

$$\gamma_m = \frac{m-1}{2\kappa}, \quad \kappa = N(m-1)+2,$$

where $c > 0$ is an arbitrary constant.

2.1. Show that as $m \to 1$, $\Gamma_m(x,t)$ tends to the fundamental solution of the heat equation.

2.2. Find the constant c such that the total mass of Γ_m is 1, i.e.,

$$\int_{\mathbb{R}^N} \Gamma_m(x-y; t-\tau)dy = 1 \qquad\qquad (c = 4\pi).$$

2.3. Show that if $m > 1$, possible solutions of the Cauchy problem

$$u_t - \Delta u^m = 0 \text{ in } S_T, \quad u \geq 0$$
$$u(\cdot,0) = u_o \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$$

cannot be represented as the convolution of Γ_m with the initial datum u_o.

2.4. Attempt to find similarity solutions when $0 < m < 1$.

2.3c The p-Laplacean Equation

Carry on the same analysis for the nonlinear evolution equation

$$u_t - \operatorname{div} |Du|^{p-2} Du = 0, \quad p > 2.$$

A version of this equation arises in modelling certain non-Newtonian fluids ([148]). Then for $p = 2$ this reduces to the heat equation. The similarity solutions are

$$\begin{gathered} \Gamma_p(x,t) = \frac{1}{t^{N/\lambda}} \left[1 - c\gamma_p \left(\frac{|x|}{t^{1/\lambda}} \right)^{\frac{p}{p-1}} \right]_+^{\frac{p-1}{p-2}}, \quad t > 0 \\ \gamma_p = \left(\frac{1}{\lambda} \right)^{\frac{1}{p-1}} \frac{p-2}{p}, \quad \lambda = N(p-2) + p. \end{gathered}$$

Prove that $\Gamma_p \to \Gamma$ as $p \to 2$. Find the constant c so that Γ_p has mass 1. Attempt to find similarity solutions when $1 < p < 2$.

2.4c The Error Function

Prove that the unique solution of the Cauchy problem

$$u_t - u_{xx} = 0 \ \text{ in } \ \mathbb{R} \times \mathbb{R}^+, \quad u(x,0) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

is given by

$$u(x,t) = \frac{1}{2} \left[1 + E\left(\frac{x}{\sqrt{4t}} \right) \right], \quad \text{where} \quad E(s) = \frac{2}{\sqrt{\pi}} \int_0^s e^{-r^2} dr.$$

The function $s \to E(s)$ is the *error function*.

2.5c The Appell Transformation ([10])

Let u be a solution of the heat equation in $\mathbb{R} \times \mathbb{R}^+$. Then

$$w(x,t) = \Gamma(x,t) u \left(\frac{x}{t}, -\frac{1}{t} \right)$$

is also a solution of the heat equation in $\mathbb{R} \times \mathbb{R}^+$.

2.6c The Heat Kernel by Fourier Transform

For $f \in L^1(\mathbb{R}^N)$, let $\hat{f}$ denote its Fourier transform

$$\hat{f}(x) \overset{\text{def}}{=} \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} f(y) e^{-i\langle x,y\rangle} dy.$$

Here i is the imaginary unit and $\langle x, y\rangle = x_j y_j$. In general, assuming that $f \in L^1(\mathbb{R}^N)$ or even that f is compactly supported in $\mathbb{R}^N$, does not guarantee that $\hat{f} \in L^1(\mathbb{R}^N)$, as shown by the following examples.

2.5 Compute the Fourier transform of the characteristic function of the unit interval in $\mathbb{R}^1$. Show that $x \to (\chi_{[0,1]})^\wedge(x) \notin L^1(\mathbb{R})$.

2.6. Let $N = 1$, and let m be a positive integer larger than 2. Compute the Fourier transform of

$$f(x) = \begin{cases} 0 & \text{for } x < 1 \\ x^{-m} & \text{for } x \geq 1 \end{cases}$$

and show that $\hat{f} \notin L^1(\mathbb{R})$.

2.7c Rapidly Decreasing Functions

These examples show that $L^1(\mathbb{R}^N)$ is not closed under the operation of Fourier transform, and raise the question of finding a class of functions that is closed under such an operation. The class of smooth and rapidly decreasing functions in $\mathbb{R}^N$, or the Schwartz class, is defined by ([228])

$$\mathcal{S}_N \overset{\text{def}}{=} \left\{ \begin{array}{c} f \in C^\infty(\mathbb{R}^N) \,\Big|\, \sup\limits_{x\in\mathbb{R}^N} |x|^m |D^\alpha f(x)| < \infty \\ \text{for all } m \in \mathbb{N} \text{ and all multi-indices } \alpha \text{ of size } |\alpha| \geq 0 \end{array} \right\}.$$

Proposition 2.1c $f \in \mathcal{S}_N \Longrightarrow \hat{f} \in \mathcal{S}_N$.

Proof. For $f \in \mathcal{S}_N$ and multi-indices α and β, compute

$$\begin{aligned} x^\beta D^\alpha \hat{f}(x) &= \frac{x^\beta}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} f(y) D_x^\alpha e^{-i\langle x,y\rangle} dy \\ &= \frac{(-i)^{|\alpha|}}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} x^\beta y^\alpha f(y) e^{-i\langle x,y\rangle} dy \\ &= \frac{(-i)^{|\alpha|-|\beta|}}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} y^\alpha f(y) D_y^\beta e^{-i\langle x,y\rangle} dy \\ &= \frac{(-i)^{|\alpha+\beta|}}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} D^\beta [y^\beta D^\alpha f(y)] e^{-i\langle x,y\rangle} dy. \end{aligned}$$ ■

2.8c The Fourier Transform of the Heat Kernel

Proposition 2.2c *Let* $\varphi(x) = e^{-\frac{1}{2}|x|^2}$. *Then* $\hat{\varphi} = \varphi$.

Proof. Assume first that $N = 1$. One verifies that φ and $\hat{\varphi}$ satisfy the same ODE

$$\varphi' + x\varphi = 0, \qquad \hat{\varphi}' + x\hat{\varphi} = 0, \qquad x \in \mathbb{R}.$$

Therefore $\hat{\varphi} = C\varphi$ for a constant C. From (2.2) with $t - s = \frac{1}{2}$ and $N = 1$

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2}y^2} dy = \hat{\varphi}(0) = 1.$$

Since also $\varphi(0) = 1$, we conclude that $C = 1$, and the proposition follows in the case of one dimension. If $N \geq 2$, by Fubini's theorem

$$\begin{aligned} \hat{\varphi}(x) &= \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{-\frac{1}{2}|y|^2} e^{-i\langle x,y\rangle} dy \\ &= \prod_{j=1}^{N} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2}y_j^2} e^{-ix_j y_j} dy_j \\ &= \prod_{j=1}^{N} \hat{\varphi}(x_j) = \prod_{j=1}^{N} \varphi(x_j) = \varphi(x). \end{aligned}$$ ■

2.7. Prove the rescaling formula $\hat{\psi}(\varepsilon x) = \varepsilon^{-N}\hat{\psi}(x/\varepsilon)$, valid for all $\psi \in \mathcal{S}_N$ and all $\varepsilon > 0$.

2.8. Verify the formula

$$\left(e^{-|x-y|^2(t-\tau)}\right)^{\wedge} = \frac{1}{[2(t-\tau)]^{N/2}} e^{-|x-y|^2/4(t-\tau)}$$

for all $t - \tau > 0$ fixed.

2.9c The Inversion Formula

Theorem 2.1c. *Let* $f \in \mathcal{S}_N$. *Then*

$$f(x) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} \hat{f}(y) e^{i\langle x,y\rangle} dy.$$

Proof. The formula follows by computing the limit

$$\frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} \hat{f}(y) e^{i\langle x,y\rangle} dy = \lim_{\tau \to t} \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} \hat{f}(y) e^{-|y|^2(t-\tau)} e^{i\langle x,y\rangle} dy.$$

The integral on the right-hand side is computed by repeated application of Fubini's theorem:

$$\begin{aligned}
&\frac{1}{(2\pi)^{N/2}}\int_{\mathbb{R}^N}\hat{f}(y)e^{-|y|^2(t-\tau)}e^{i\langle x,y\rangle}dy\\
&\qquad=\frac{1}{(2\pi)^N}\int_{\mathbb{R}^N}f(\eta)e^{-i\langle y,\eta\rangle}e^{-|y|^2(t-\tau)}e^{i\langle x,y\rangle}dyd\eta\\
&\qquad=\frac{1}{(2\pi)^{N/2}}\int_{\mathbb{R}^N}f(\eta)\left(\frac{1}{(2\pi)^{N/2}}\int_{\mathbb{R}^N}e^{-|y|^2(t-\tau)}e^{-i\langle\eta-x,y\rangle}dy\right)d\eta\\
&\qquad=\frac{1}{(2\pi)^{N/2}}\int_{\mathbb{R}^N}f(\eta)\left(e^{-|y|^2(t-\tau)}\right)^{\wedge}(\eta-x)d\eta\\
&\qquad=\frac{1}{[4\pi(t-\tau)]^{N/2}}\int_{\mathbb{R}^N}f(\eta)e^{-|x-\eta|^2/4(t-\tau)}d\eta.
\end{aligned}$$

Therefore

$$\frac{1}{(2\pi)^{N/2}}\int_{\mathbb{R}^N}\hat{f}(y)e^{i\langle x,y\rangle}dy=\lim_{\tau\to t}\int_{\mathbb{R}^N}\Gamma(x-\eta;t-\tau)f(\eta)d\eta=f(x)$$

where the last limit is computed by the same technique leading to the representation formula (2.7). ■

3c The Maximum Principle in Bounded Domains

Let E be a bounded domain in $\mathbb{R}^N$ with smooth boundary ∂E.

3.1. Let u be a solution of the Dirichlet problem (1.2) with $g=0$. Prove that

$$\|u(\cdot,t)\|_{\infty,E}\le\frac{1}{(4\pi t)^{N/2}}\|u_o\|_{1,E}.$$

3.2. State and prove a maximum principle for $u\in\mathcal{H}(E_T)\cap C(\bar{E}_T)$ satisfying

$$H(u)=\mathbf{v}\cdot Du+c\ \text{ in }\ E_T.$$

where $\mathbf{v}\in\mathbb{R}^N$ and $c\in\mathbb{R}$ are given.

3.3. Discuss a possible maximum principle for $H(u)=\lambda u$ for $\lambda\in\mathbb{R}$.

3.4. Let $f\in C(\mathbb{R}^+)$ and consider the boundary value problem

$$\begin{aligned}
&u\in\mathcal{H}(E_\infty)\cap C(\bar{E}_\infty)\\
&u_t-\Delta u=f(t)\left(u-\frac{|x|^2}{2N}\right)-1\ \text{ in }\ B_1\times\mathbb{R}^+\\
&u(\cdot,t)\big|_{\partial_* B_1}=\frac{1}{2N}
\end{aligned}$$

Prove that this problem has at most one solution, the solution is non-negative and satisfies

$$0\le u(x,t)\le\frac{1}{2N}\exp\left(\int_0^t f(s)ds\right)+\frac{|x|^2}{2N}.$$

In particular, if $f\le 0$ then $u(x,t)\le 1/N$.

3.5. In the previous problem assume that

$$f(t) \le -\frac{C}{1+t} \quad \text{for all } \ t \ge t_*$$

for some $C > 0$ and some $t_* \ge 0$. Prove that

$$\lim_{t\to\infty} u(x,t) = \frac{|x|^2}{2N}.$$

Moreover, if $u(\cdot,0) = |x|^2/2N$, then $u(\cdot,t) = u(\cdot,0)$, for all f.

3.6. Let $f \in C(\bar{E}_T)$ and $\alpha \in (0,1)$. Prove that a non-negative solution of $H(u) = u^\alpha$ in E_T satisfies

$$\|u\|_{\infty,E_T} \le \frac{1}{1-\alpha}\|u\|_{\infty,\partial_* E_T} + \Big[(e^{\operatorname{diam}(E)} - 1)\|f\|_{\infty,E_T}\Big]^{\frac{1}{1-\alpha}}.$$

3.1c The Blow-Up Phenomenon for Super-Linear Equations

Consider non-negative classical solutions of

$$u_t - \Delta u = u^\alpha \quad \text{in } \ E \times \mathbb{R}^+, \qquad \text{for some } \ \alpha \ge 1$$

that are bounded on the parabolic boundary of E_T, say

$$\sup_{\partial_* E_\infty} u \le M \qquad \text{for some } \ M > 0.$$

Prove that if $\alpha = 1$, then $u \le Me^t$. Therefore if $\alpha \in [0,1)$ the solution remains bounded for all $t \ge 0$, and if $\alpha = 1$, it remains bounded for all $t \ge 0$ with bound increasing with t. If $\alpha > 1$, an upper bound is possible only for finite times.

Lemma 3.1c *Let $\alpha > 1$. Then*

$$u(x,t) \le \frac{M}{[1-(\alpha-1)M^{\alpha-1}t]^{1/(\alpha-1)}}.$$

Proof (Hint). Divide the PDE by u^α and introduce the function

$$w = u^{1-\alpha} + (\alpha-1)t.$$

Using that $\alpha > 1$, prove that $H(w) \ge 0$ in E_∞. Therefore, by the maximum principle

$$\frac{1}{u^{\alpha-1}} + (\alpha-1)t \ge \frac{1}{M^{\alpha-1}}. \qquad \blacksquare$$

Remark 3.1c This estimate is *stable* as $\alpha \to 1$ in the sense that as $\alpha \to 1$, the right-hand side converges to the corresponding exponential upper bound valid for $\alpha = 1$.

3.1.1c An Example for $\alpha = 2$

Even though the boundary data are uniformly bounded, the solution might indeed *blow up* at interior points of E in finite time, as shown by the following example ([81]).

$$\begin{aligned} u_t - u_{xx} = u^2 \ \text{ in } \ (0,1)\times(0,\infty), \quad u(\cdot,0)=u_o \\ u(0,t)=h_o(t), \quad u(1,t)=h_1(t), \quad \text{ for all } \ t\ge 0. \end{aligned} \tag{3.1c}$$

Assume that

$$u_o,\, h_o,\, h_1 \ge c = \frac{c_1}{c_2}$$

for positive constants c_1 and c_2 to be chosen. These constants can be chosen such that (3.1c) has no solution that remains bounded for finite times. Introduce the comparison function

$$v = \frac{c_1}{c_2 - x(1-x)t}.$$

By direct calculation

$$\begin{aligned} v_t - v_{xx} &= \frac{c_1 x(1-x)}{[c_2 - x(1-x)t]^2} + \frac{2c_1 t}{[c_2 - x(1-x)t]^2} - \frac{2c_1 t^2(1-2x)^2}{[c_2 - x(1-x)t]^3} \\ &\le \frac{v^2}{c_1}\left(\frac{1}{4} + 2t\right). \end{aligned}$$

Taking $t \in (0, 4c_2)$ and choosing c_1 sufficiently large, this last term is majorized by v^2. Therefore

$$v_t - v_{xx} \le v^2 \quad \text{ in } \ (0,1)\times(0,4c_2).$$

Fix any time $T \in (0,4c_2)$ and consider the domain $E_T = (0,1)\times(0,T)$. If u is a solution of (3.1c), the function $w = (v-u)e^{-\lambda t}$ for $\lambda > 0$ satisfies

$$w_t - w_{xx} \le -(\lambda - (v+u))w \ \text{ in } \ E_T, \quad w\big|_{\partial_* E_T} \le 0.$$

Therefore, by choosing λ sufficiently large, the maximum principle implies that $w \le 0$ in E_T.

3.2c The Maximum Principle for General Parabolic Equations

Let $\mathcal{L}_o(\cdot)$ be the differential operator introduced in (4.1c) of the Complements of Chapter 2. By using a technique similar to that of Theorem 4.1c, prove

Theorem 3.1c. *Let $u \in \mathcal{H}(E_T) \cap C(\bar{E}_T)$ and let $c \le 0$. then*

$$u_t - \mathcal{L}_o(u) \le 0 \ \text{ in } \ E_T \quad \Longrightarrow \quad u(x,t) \le \sup_{\partial_* E} u \ \text{ in } \ E_T.$$

3.6. The maximum principle gives one-sided estimates for merely sub(super)-solutions of the heat equation. An important class of sub(super)-solutions is determined as follows. Let $u \in \mathcal{H}(E_T)$ be a solution of the heat equation in E_T. Prove that for every convex(concave) function $\varphi(\cdot) \in C^2(\mathbb{R})$, the composition $\varphi(u)$ is a sub(super)-solution of the heat equation in E_T.

4c The Maximum Principle in $\mathbb{R}^N$

4.1. Show that $u = 0$ is the only solution of the Cauchy problem

$$u \in \mathcal{H}(S_T) \cap C(\bar{S}_T) \cap L^2(S_T), \quad u_t - \Delta u = 0 \text{ in } S_T, \quad u(\cdot, 0) = 0. \tag{4.1c}$$

Hint: Let $x \to \zeta(x) \in C_o^2(B_{2\rho})$ be a non-negative cutoff function in $B_{2\rho}$ satisfying

$$\zeta = 1 \text{ in } B_\rho, \quad |D\zeta| \le \begin{cases} 0 & \text{if } |x| < \rho \\ \dfrac{2}{\rho} & \text{if } \rho < |x| < 2\rho, \end{cases}$$

$$|\zeta_{x_i x_j}| \le \begin{cases} 0 & \text{if } |x| < \rho \\ \dfrac{4}{\rho^2} & \text{if } \rho < |x| < 2\rho. \end{cases}$$

Multiply the PDE by $u\zeta^2$ and integrate over $B_{2\rho} \times (0, t)$ to derive

$$\int_{B_{2\rho}} u^2(t)\zeta^2 dx + 2\int_0^t \int_{B_{2\rho}} |Du|^2 \zeta^2 dx\, ds = 4\int_0^t \int_{B_{2\rho}} \zeta u D u D\zeta dx\, ds.$$

By the Cauchy-Schwarz inequality, the last integral is majorized by

$$2\int_0^t \int_{B_{2\rho}} |Du|^2 \zeta^2 dx\, ds + 2\int_o^t \int_{B_{2\rho}} u^2 |D\zeta|^2 dx\, ds$$

and

$$2\int_0^t \int_{B_{2\rho}} u^2 |D\zeta|^2 dx\, ds \le \frac{8}{\rho^2} \int_0^t \int_{\rho<|x|<2\rho} u^2 |D\zeta|^2 dx\, ds.$$

Combine these estimates and let $\rho \to \infty$.

4.2. Prove that the same conclusion holds if in (4.1c) one replaces $L^2(S_T)$ with $L^1(S_T)$.
Hint: Let $h_\delta(\cdot)$ be the approximation to the Heaviside function introduced in (14.2). Multiply the PDE by $h_\delta(u)\zeta$ and integrate over $B_\rho \times (0, t)$ to obtain

$$\int_{B_{2\rho}\times\{t\}} \left(\int_0^u h_\delta(\xi) d\xi \right) \zeta dx + \frac{1}{\delta} \int_0^t \int_{B_{2\rho}} |Du|^2 \chi(|u| < \delta) \zeta \, dx\, ds$$
$$= -\int_0^t \int_{B_{2\rho}} D\left(\int_0^u h_\delta(\xi) d\xi \right) D\zeta \, dx\, ds.$$

The last term is transformed and majorized by

$$\int_0^t \int_{B_{2\rho}} \left(\int_0^u h_\delta(\xi) d\xi \right) \Delta\zeta \, dx\, ds \le \frac{\text{const}}{\rho^2} \int_0^t \int_{\rho<|x|<2\rho} |u| \, dx\, ds.$$

Combining these estimates and letting $\delta \to 0$ we arrive at

$$\int_{B_\rho \times \{t\}} |u| dxds \le \frac{\text{const}}{\rho^2} \int_0^t \int_{\rho<|x|<2\rho} |u| \, dx \, ds.$$

To conclude, let $\rho \to \infty$.

4.3. Prove that the same conclusion holds if u satisfies either one of the weaker conditions

$$\frac{u}{(1+|x|)} \in L^2(S_T), \qquad \frac{u}{(1+|x|^2)} \in L^1(S_T).$$

4.1c Counterexamples of the Tychonov Type

4.1. Prove that the function

$$\begin{aligned} u(x,t) = \int_0^\infty & [e^{xy} \cos(xy + 2ty^2) \\ & + e^{-xy} \cos(xy - 2ty^2)] y e^{-y^{4/3}} \cos(\sqrt{3}\, y^{4/3}) dy \end{aligned}$$

is another nontrivial solution of the Cauchy problem in $\mathbb{R} \times \mathbb{R}^+$ with vanishing initial data ([220]). The interesting feature is that u is of class C^∞ over the whole plane.

4.2. Prove that the function

$$u(x,t) = \int_{a-i\infty}^{a+i\infty} e^{st + x\sqrt{s} - s^{2/3}} \, ds, \qquad a > 0,$$

where the integration is along the line $\operatorname{Re} s = a$ of the complex s-plane, is a third nontrivial solution of the Cauchy problem in $\mathbb{R} \times \mathbb{R}^+$ with vanishing initial data. Here u has the additional feature that it vanishes identically for negative t, which does not occur for the previous case. This example was proposed in [220] as an application of the theory developed in [126].

7c Remarks on the Cauchy Problem

7.1. Write down the explicit solution of

$$\begin{aligned} u_t - \Delta u &= u + \mathbf{b} \cdot \nabla u + e^t \sin(x_1 - b_1 t) \ \text{ in } \ S_T \\ u(x,0) &= |x| \end{aligned}$$

for a given $\mathbf{b} \in \mathbb{R}^N$.

Hint: The function $v(x,t) = u(x - \mathbf{b}t, t)$ satisfies the PDE with $\mathbf{b} = 0$.

7.2. Using the reflection technique, solve the homogeneous mixed boundary value problems

$$\begin{array}{ll} u_t - u_{xx} = 0 \ \text{ in } \ \mathbb{R}^+ \times \mathbb{R}^+ & \qquad u_t - u_{xx} = 0 \ \text{ in } \ \mathbb{R}^+ \times \mathbb{R}^+ \\ u_x(0,t) = 0 \ \text{ for } \ t > 0 & \qquad u(0,t) = 0 \ \text{ for } \ t > 0 \\ u(x,0) = u_o \in C^1(\bar{\mathbb{R}}^+) & \qquad u(x,0) = u_o \in C(\bar{\mathbb{R}}^+) \\ u_{o,x}(0) = 0 & \qquad u_o(0) = 0. \end{array}$$

where u_o is bounded in $\mathbb{R}^+$.

7.3. Solve the inhomogeneous mixed boundary value problems

$$\begin{array}{ll} u_t - u_{xx} = \ \ \text{ in } \ \mathbb{R}^+ \times \mathbb{R}^+ & \qquad u_t - u_{xx} = 0 \ \text{ in } \ \mathbb{R}^+ \times \mathbb{R}^+ \\ u_x(0,t) = h(t) \in C^1(\mathbb{R}^+) & \qquad u(0,t) = h(t) \in C^1(\mathbb{R}^+) \\ u(x,0) = u_o \in C^1(\bar{\mathbb{R}}^+) & \qquad u(x,0) = u_o \in C(\bar{\mathbb{R}}^+) \\ u_{o,x}(0) = h(0) & \qquad u_o(0) = h(0). \end{array}$$

12c On the Local Behavior of Solutions

Proposition 12.1c *Let $u \in \mathcal{H}(E_T)$ be a local solution of the heat equation in E_T. For every $p > 0$ there exists a constant C, depending only on N and p such that, for all $(x_o, t_o) + Q_\rho \subset E_T$*

$$\sup_{(x_o,t_o)+Q_\rho} |u| \le C \left(\fint_{(x_o,t_o)+Q_{2\rho}} |u|^p dy\, ds \right)^{1/p}.$$

Proof. The case $p = 1$ is the content of Corollary 12.1. The case $p > 1$ follows from this and Hölder's inequality. To prove the estimate for $0 < p < 1$, one may assume that $(x_o, t_o) = (0, 0)$. Consider the increasing sequence of radii $\{\rho_n\}$, the family of nested expanding boxes $\{Q_n\}$, and the nondecreasing sequence of numbers $\{M_n\}$, defined by

$$\rho_n = \rho \sum_{i=0}^{n} 2^{-i}, \quad Q_n = B_{\rho_n} \times (-\rho_n^2, 0), \quad M_n = \sup_{Q_n} |u|, \quad n = 0, 1, \dots .$$

Apply Corollary 12.1 to the pair of cylinders Q_n and Q_{n+1} to obtain

$$M_n \le C 2^{(n+1)(N+2)} \fint_{Q_{n+1}} |u|\, dy\, ds.$$

Fix $p \in (0, 1)$. Then by Young's inequality, for all $\delta > 0$

$$\begin{aligned} M_n &\le C 2^{(n+1)(N+2)} M_{n+1}^{1-p} \fint_{Q_{n+1}} |u|^p dy\, ds \\ &\le \delta M_{n+1} + p \delta^{1-\frac{1}{p}} \left(C 2^{(n+2)(N+2)} \right)^{1/p} \left(\fint_{Q_{2\rho}} |u|^p dy\, ds \right)^{1/p}. \end{aligned}$$

Setting

$$K = p\delta^{1-\frac{1}{p}}\left(C2^{(N+2)}\right)^{1/p}, \qquad b = 2^{\frac{N+2}{p}}$$

we arrive at the recursive inequalities

$$M_n \le \delta M_{n+1} + b^n K \left(\fint_{Q_{2\rho}} |u|^p dy\, ds\right)^{1/p}.$$

By iteration

$$M_o \le \delta^n M_{n+1} + bK \sum_{i=0}^{n} (\delta b)^i \left(\fint_{Q_{2\rho}} |u|^p dy\, ds\right)^{1/p}.$$

Choose δ small enough that $\delta b = \frac{1}{2}$, so that the series $\sum_{i=0}^{\infty}(\delta b)^i$ is convergent. Then let $n \to \infty$. ■

6

THE WAVE EQUATION

1 The One-Dimensional Wave Equation

Consider the hyperbolic equation in two variables

$$u_{tt} - c^2 u_{xx} = 0. \tag{1.1}$$

The variable t stands for time, and *one-dimensional* refers to the number of space variables. A general solution of (1.1) in a *convex* domain $E \subset \mathbb{R}^2$, is given by

$$u(x,t) = F(x - ct) + G(x + ct) \tag{1.2}$$

where $s \to F(s), G(s)$ are of class C^2 within their domain of definition. Indeed, the change of variables

$$\xi = x - ct, \qquad \eta = x + ct \tag{1.3}$$

transforms E into a convex domain $\tilde{E}$ of the (ξ,η)-plane, and in terms of ξ and η, equation (1.1) becomes

$$U_{\xi\eta} = 0 \quad \text{where} \quad U(\xi,\eta) = u\Big(\frac{\xi+\eta}{2}, \frac{\eta-\xi}{2c}\Big).$$

Therefore $U_\xi = F'(\xi)$ and

$$U(\xi,\eta) = \int F'(\xi)d\xi + G(\eta).$$

Rotating the axes back of an angle $\theta = \arctan(c^{-1})$, maps $\tilde{E}$ into E back in the (x,t)-plane and

$$u(x,t) = F(x - ct) + G(x + ct).$$

The graphs of $\xi \to F(\xi)$ and $\eta \to G(\eta)$ are called *undistorted waves* propagating to the right and left respectively (right and left here refer to the positive

© Springer Nature Switzerland AG 2023

E. DiBenedetto, U. Gianazza, *Partial Differential Equations*,
Cornerstones, https://doi.org/10.1007/978-3-031-46618-2_7

orientation of the x- and t-axes). The two lines obtained from (1.3) by making ξ and η constants are called characteristic lines. Write them in the parametric form

$$x_1(t) = ct + \xi, \quad x_2(t) = -ct + \eta, \qquad \text{for } t \in \mathbb{R}$$

and regard the abscissas $t \to x_i(t)$ for $i = 1, 2$ as points traveling on the x-axis, with velocities ±c respectively.

1.1 A Property of Solutions

Consider any parallelogram of vertices A, B, C, D with sides parallel to the characteristics $x = \pm ct + \xi$ and contained in some *convex* domain $E \subset \mathbb{R}^2$.

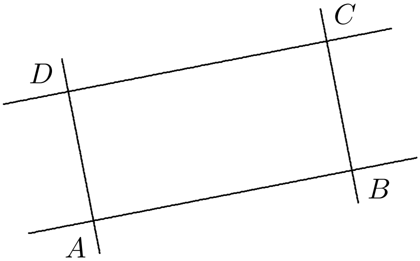

Fig. 1.1

We call it a characteristic parallelogram. Let

$$A = (x, t), \quad B = (x + cs, t + s)$$
$$C = (x + cs - c\tau, t + s + \tau), \quad D = (x - c\tau, t + \tau)$$

be the coordinates of the vertices of a characteristic parallelogram, where s and τ are positive parameters. If a function $u \in C(E)$ is of the form (1.2), for two continuous functions $F(\cdot)$ and $G(\cdot)$, then

$$\begin{aligned} u(A) &= F(x - ct) + G(x + ct) \\ u(C) &= F(x - 2c\tau - ct) + G(x + 2cs + ct) \\ u(B) &= F(x - ct) + G(x + 2cs + ct) \\ u(D) &= F(x - 2c\tau - ct) + G(x + ct). \end{aligned}$$

Therefore

$$u(A) + u(C) = u(B) + u(D). \tag{1.4}$$

Therefore any solution of (1.1) satisfies (1.4). Vice versa if $u \in C^2(E)$ is of the form (1.2) for F and G of class C^2 and satisfies (1.4) for any characteristic parallelogram, rewrite (1.4) as

$$[u(x,t) - u(x+cs,t+s)] = [u(x-c\tau,t+\tau) - u(x+cs-c\tau,t+s+\tau)].$$

Using the Taylor formula one verifies that u satisfies the PDE (1.1). Since (1.4) only requires that u be continuous, it might be regarded as some sort of *weak* formulation of (1.1).

2 The Cauchy Problem

On the noncharacteristic line $t = 0$, prescribe the shape and speed of the undistorted waves, and seek to determine the shape and speed of the solution of (1.1), for all the later and previous times. Formally, seek to solve the Cauchy problem

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0 && \text{in } \mathbb{R}^2 \\ u(\cdot,0) &= \varphi && \text{in } \mathbb{R} \\ u_t(\cdot,0) &= \psi && \text{in } \mathbb{R} \end{aligned} \tag{2.1}$$

for given $\varphi \in C^2(\mathbb{R})$ and $\psi \in C^1(\mathbb{R})$. According to (1.2) one has to determine the form of F and G from the initial data, i.e.,

$$F + G = \varphi, \quad F' + G' = \varphi', \quad -F' + G' = \frac{1}{c}\psi.$$

From this

$$F' = \frac{1}{2}\varphi' - \frac{1}{2c}\psi, \qquad G' = \frac{1}{2}\varphi' + \frac{1}{2c}\psi.$$

This, in turn, implies

$$\begin{aligned} F(\xi) &= \frac{1}{2}\varphi(\xi) - \frac{1}{2c}\int_0^\xi \psi(s)ds + c_1 \\ G(\eta) &= \frac{1}{2}\varphi(\eta) + \frac{1}{2c}\int_0^\eta \psi(s)ds + c_2 \end{aligned}$$

for two constants c_1 and c_2. Therefore

$$u(x,t) = \frac{1}{2}[\varphi(x-ct) + \varphi(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds \tag{2.2}$$

since, in view of the second of (2.1), $c_1 + c_2 = 0$. Formula (2.2) is the explicit d'Alembert representation of the *unique* solution of the Cauchy problem (2.1). The right-hand side of (2.2) is well defined whenever $\varphi \in C_{\text{loc}}(\mathbb{R})$ and $\psi \in L^1_{\text{loc}}(\mathbb{R})$. However, in such a case, the corresponding function $(x,t) \to u(x,t)$ need not satisfy the PDE in the classical sense. For this reason, (2.2) might be regarded as some sort of *weak* solution of the Cauchy problem (2.1) whenever the data satisfy merely the indicated reduced regularity.

Remark 2.1 (Domain of Dependence) The value of u at (x, t) is determined by the restriction of the initial φ and ψ, data to the interval $[x - ct, x + ct]$. If the initial speed ψ vanishes on such an interval, then $u(x, t)$ depends only n the datum φ at the points $x \pm ct$ of the x-axis.

Remark 2.2 (Propagation of Disturbances) The value of the initial data $\varphi(\xi)$, $\psi(\xi)$ at a point ξ of the x-axis is felt by the solution only at points (x, t) within the sector

$$[x - ct \leq \xi] \cap [x + ct \geq \xi].$$

If $\psi \equiv 0$, it is felt only at points of the characteristic curves $x = \pm ct + \xi$.

Remark 2.3 (Well Posedness) The Cauchy problem (2.1) is well posed in the sense of Hadamard, i.e., (a) there exists a solution; (b) the solution is unique; (c) the solution is stable. Statement (c) asserts that small perturbations of the data φ and ψ yield small changes in the solution u. This is also referred to as *continuous dependence* on the data. Such a statement becomes precise only when a topology is introduced to specify the meaning of "small" and "continuous".

Since the problem is linear, to prove (c) it will suffice to show that "small data" yield "small solutions". As a smallness condition on φ and ψ, take

$$\|\varphi\|_{\infty,\mathbb{R}}, \quad \|\psi\|_{\infty,\mathbb{R}} < \varepsilon \ \text{ for some } \ \varepsilon > 0.$$

Then formula (2.2) gives that the solution u corresponding to such data satisfies

$$\|u(\cdot, t)\|_{\infty,\mathbb{R}} \leq (1 + t)\varepsilon.$$

This proves the continuous dependence on the data in the topology of $L^\infty(\mathbb{R})$. If in addition, the initial velocity ψ is compactly supported in $\mathbb{R}$, say in the interval $(-L, L)$, then

$$\|u\|_{\infty,\mathbb{R}^2} < \Big(1 + \frac{L}{c}\Big)\varepsilon.$$

3 Inhomogeneous Problems

Let $f \in C^1(\mathbb{R}^2)$ and consider the inhomogeneous Cauchy problem

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= f \qquad \text{in } \mathbb{R}^2 \\ u(\cdot, 0) &= \varphi \qquad \text{in } \mathbb{R} \\ u_t(\cdot, 0) &= \psi \qquad \text{in } \mathbb{R}. \end{aligned} \tag{3.1}$$

The solution of (3.1) can be constructed by superposing the unique solution of (2.1) with a solution of

$$\begin{aligned} v_{tt} - c^2 v_{xx} &= f \qquad && \text{in } \mathbb{R}^2 \\ v(\cdot, 0) = v_t(\cdot, 0) &= 0 && \text{in } \mathbb{R}. \end{aligned} \tag{3.2}$$

To solve the latter, introduce the change of variables (1.3), which transforms (3.2) into

$$U_{\xi\eta}(\xi,\eta) = -\frac{1}{4c^2}F(\xi,\eta), \qquad \text{where} \qquad F(\xi,\eta) = f\Big(\frac{\xi+\eta}{2}, -\frac{\xi-\eta}{2c}\Big).$$

The initial conditions translate into

$$U(s,s) = U_\xi(s,s) = U_\eta(s,s) = 0 \qquad \forall s \in \mathbb{R}.$$

Integrate the transformed PDE in the first variable, over the interval (η,ξ). Taking into account the initial conditions

$$U_\eta(\xi,\eta) = -\frac{1}{4c^2}\int_\eta^\xi F(s,\eta)ds.$$

Next integrate in the second variable, over (ξ,η). This gives

$$U(\xi,\eta) = \frac{1}{4c^2}\int_\xi^\eta \int_\xi^z F(s,z)ds\,dz. \tag{3.3}$$

In (3.3) perform the change of variables

$$-\frac{s-z}{2c} = \tau, \qquad \frac{s+z}{2} = \sigma$$

whose Jacobian is $2c$. The domain of integration is transformed into

$$x - ct = \xi < \sigma - c\tau < \sigma + c\tau < \eta = x + ct.$$

Therefore, in terms of x and t, (3.3) gives the unique solution of (3.2) in the form

$$v(x,t) = \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} f(\sigma,\tau)d\sigma\,d\tau. \tag{3.4}$$

Remark 3.1 (Duhamel's Principle ([61])) Consider the one-parameter family of initial value problems

$$\begin{aligned} v_{tt} - c^2 v_{xx} &= 0 && \text{in } \mathbb{R}\times(\tau,\infty)\\ v(\cdot,\tau) &= 0 && \text{in } \mathbb{R}\\ v_t(\cdot,\tau) &= f(\cdot,\tau) && \text{in } \mathbb{R}. \end{aligned}$$

By the d'Alembert formula (2.2)

$$v(x,t;\tau) = \frac{1}{2c}\int_{x-c(t-\tau)}^{x+c(t-\tau)} f(\sigma,\tau)d\sigma.$$

Therefore, it follows from (3.4), that the solution of (3.2) is given by "superposing" $\tau \to v(x,t;\tau)$ for $\tau \in (0,t)$. This is a particular case of Duhamel's principle (see Section 3.1c of the Complements).

Remark 3.2 It follows from the solution formula (3.4) that if $x \to f(x,t)$ is odd about some x_o, then $x \to v(x,t)$ is also odd about x_o for all $t \in \mathbb{R}$. In particular, $u(x_o,t) = 0$ for all $t \in \mathbb{R}$.

4 A Boundary Value Problem (Vibrating String)

A string of length L vibrates with its end-points kept fixed. Let $(x,t) \to u(x,t)$ denote the vertical displacement at time t of the point $x \in (0,L)$. Assume that at time $t = 0$ the shape of the string and its speed are known, say $\varphi, \psi \in C^2[0,L]$. At all times $t \in \mathbb{R}$ the phenomenon is described by the boundary value problem

$$\begin{aligned} & u_{tt} = c^2 u_{xx} && \text{in } (0,L)\times\mathbb{R} \\ & u(0,\cdot) = u(L,\cdot) = 0 && \text{in } \mathbb{R} \\ & u(\cdot,0) = \varphi,\ u_t(\cdot,0) = \psi && \text{in } (0,L). \end{aligned} \tag{4.1}$$

The data φ and ψ are required to satisfy the compatibility conditions

$$\varphi(0) = \varphi(L) = \psi(0) = \psi(L) = 0.$$

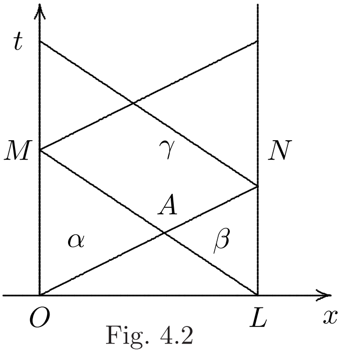

Fig. 4.2

At each point of $[0,L]\times\mathbb{R}$, the solution $u(x,t)$ of (4.1), can be determined by making use of the solution formula (2.2) for the Cauchy problem, and formula (1.4). First draw the characteristic $x = ct$ originating at $(0,0)$, and the characteristic $x = -ct + L$ originating at $(L,0)$, and let A be their intersection. As they intersect the vertical axes $x = 0$ and $x = L$, reflect them by following the characteristic of opposite slope, as in Figure 4.2. The solution $u(x,t)$ is determined for all (x,t) in the closed triangle OAL by means of (2.2). Every point P of the triangle OAM is a vertex of a parallelogram with sides parallel to the characteristics, and such that of the three remaining vertices, two lie on the characteristic $x = ct$, where u is known, and the other is on the vertical line $x = 0$, where $u = 0$. Thus $u(P)$ can be calculated from (1.4). Analogously u can be computed at every point of the closure of LAN. We may now proceed in this fashion to determine u progressively at every point of the closure of the regions α, β, etc.

4.1 Separation of Variables

Seek a solution of (4.1) in the form $u(x,t) = X(x)T(t)$. The equation yields

$$\begin{array}{ll} T'' = c^2 \lambda T & \text{in } \mathbb{R} \\ X'' = \lambda X & \text{in } (0,L) \end{array} \qquad \lambda \in \mathbb{R}. \tag{4.2}$$

The first of these implies that only negative values of λ yield bounded solutions. Setting $\lambda = -\gamma^2$, the second gives the one-parameter family of solutions

$$X(x) = C_1 \sin \gamma x + C_2 \cos \gamma x.$$

These will satisfy the boundary conditions at $x = 0$ and $x = L$ if $C_2 = 0$ and $\gamma = n\pi/L$ for $n \in \mathbb{N}$. Therefore, the functions

$$X_n(x) = \sin \frac{n\pi}{L} x, \qquad n \in \mathbb{N}$$

represent a family of solutions for the second of (4.2). With the indicated choice of γ, the first of (4.2) gives

$$T_n(t) = A_n \sin\left(\frac{n\pi c}{L} t\right) + B_n \cos\left(\frac{n\pi c}{L} t\right).$$

The solutions $u_n = X_n T_n$ can be superposed to give the general solution in the form

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \sin\left(\frac{n\pi c}{L} t\right) + B_n \cos\left(\frac{n\pi c}{L} t\right)\right] \sin\left(\frac{n\pi}{L} x\right). \tag{4.3}$$

The numbers A_n and B_n are called the *Fourier coefficients* of the series in (4.3), and are computed from the initial conditions, i.e.,

$$\sum_{n=1}^{\infty} B_n \sin \frac{\pi n}{L} x = \varphi(x), \qquad \sum_{n=1}^{\infty} A_n \frac{n\pi c}{L} \sin \frac{n\pi}{L} x = \psi(x).$$

Since the system $\left\{\sin \frac{n\pi x}{L}\right\}$ is orthogonal and complete in $L^2(0,L)$ (**10.2** of the Complements of Chapter 4), one computes

$$A_n = \frac{2}{n\pi c} \int_0^L \sin \frac{n\pi x}{L} \psi(x) dx, \quad B_n = \frac{2}{L} \int_0^L \sin \frac{n\pi x}{L} \varphi(x) dx. \tag{4.4}$$

Remark 4.1 We have assumed $\varphi, \psi \in C^2[0,L]$. Actually, the method leading to (4.3) requires only that φ and ψ be in $L^2(0,L)$. Therefore, one might define the solutions obtained by (4.3) as weak solutions of (4.1), whenever merely $\varphi, \psi \in L^2(0,L)$. The PDE, however, need not be satisfied in the classical sense.

Remark 4.2 The nth term in (4.3) is called the nth mode of vibration or the nth *harmonic*. We rewrite the n^{th} harmonic as

$$G_n \sin \frac{n\pi}{L} x \cos \frac{n\pi c}{L}(t - \tau_n)$$

where G_n and τ_n are two new constants called *amplitude* and *phase angle* respectively. The solution u can be thought of as the superposition of independent harmonics, each vibrating with amplitude G_n, phase angle τ_n, and frequency $\nu_n = n\pi c/L$.

The method of separation of variables and the principle of superposition were introduced by D. Bernoulli ([16, 18]), even though not in the context of a formal PDE. In the context of the wave equation, the method was suggested, on a more formal basis by d'Alembert; it was employed by Poisson and developed by Fourier [78].

4.2 Odd Reflection

We describe another method to solve (4.1) by referring to the Cauchy problem (2.1). If the initial data φ and ψ are odd with respect to $x = 0$, then u is odd with respect to $x = 0$. Analogously, if φ and ψ are odd about $x = L$, the same holds for u. It follows that the solution of the Cauchy problem (2.1) with φ and ψ odd about both points $x = 0$ and $x = L$ must be zero at $x = 0$ and $x = L$, for all $t \in \mathbb{R}$, i.e., it satisfies the boundary conditions at $x = 0$ and $x = L$ prescribed by (4.1). This suggests constructing a solution of (4.1) by converting it into an initial value problem (a Cauchy problem) with initial data given by the odd extension of φ and ψ about both $x = 0$ and $x = L$. For φ, such an extension is given by

$$\tilde{\varphi}(x) = \begin{cases} \varphi(x - nL) & \text{for } x \in \big(nL, (n+1)L\big) \;\; n \in \mathbb{Z} \;\text{ even} \\ -\varphi\big((n+1)L - x\big) & \text{for } x \in \big(nL, (n+1)L\big) \;\; n \in \mathbb{Z} \;\text{ odd.} \end{cases}$$

An analogous formula holds for $\tilde{\psi}$. Then the solution of (4.1) is given by the restriction to $(0, L) \times \mathbb{R}$ of

$$\tilde{u}(x,t) = \frac{1}{2}[\tilde{\varphi}(x - ct) + \tilde{\varphi}(x + ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \tilde{\psi}(s)ds$$

constructed by the d'Alembert formula.

Remark 4.3 Even if φ and ψ are in $C^2[0, L]$, their odd extensions might fail to be of class C^2 across $x = nL$. However, for $(x, t) \in (0, L) \times \mathbb{R}$, the points $x \pm ct$ are in the interior of some interval $\big(nL, (n+1)L\big)$ for some $n \in \mathbb{N}$, so that u is actually a classical solution of (4.1).

4.3 Energy and Uniqueness

Let $u \in C^2([0,L]\times\mathbb{R})$ be a solution of (4.1). The quantity

$$\mathcal{E}(t) = \int_0^L (u_t^2 + c^2 u_x^2)(x,t)dx \tag{4.5}$$

is called the energy of the system at the instant t. Multiplying the first of (4.1) by u_t, integrating by parts over $(0,L)$, and using the boundary conditions at $x=0$ and $x=L$ gives

$$\frac{d}{dt}\int_0^L (u_t^2 + c^2 u_x^2)(x,t)dx = \mathcal{E}'(t) = 0.$$

Thus $\mathcal{E}(t) = \mathcal{E}(0)$ for all $t \in \mathbb{R}$, and the energy is conserved. Also, if $\varphi = \psi = 0$, then $u = 0$ in $(0,L)\times\mathbb{R}$. In view of the linearity of the PDE one concludes that C^2 solutions of (4.1) are unique.

4.4 Inhomogeneous Problems

Let $f \in C^1\big((0,L)\times\mathbb{R}\big)$, and consider the inhomogeneous boundary value problem

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= f && \text{in } (0,L)\times\mathbb{R} \\ u(0,\cdot) = u(L,\cdot) &= 0 && \text{in } \mathbb{R} \\ u(\cdot,0) = \varphi,\ u_t(\cdot,0) &= \psi && \text{in } (0,L). \end{aligned} \tag{4.6}$$

The solution $u(x,t)$ represents the position, at point x and at time t, of a string vibrating under the action of a load f applied at time t at its points $x \in (0,L)$. The solution of (4.6) can be constructed by superposing the unique solution of (4.1) with the unique solution of

$$\begin{aligned} v_{tt} - c^2 v_{xx} &= f && \text{in } (0,L)\times\mathbb{R} \\ v(0,\cdot) = v(L,\cdot) &= 0 && \text{in } \mathbb{R} \\ v(\cdot,0) = v_t(\cdot,0) &= 0 && \text{in } (0,L). \end{aligned}$$

This, in turn, can be solved by reducing it to an initial value problem, through an odd reflection of $x \to f(x,t)$, for all $t \in \mathbb{R}$, about $x = 0$ and $x = L$, as suggested by Remark 3.1.

5 The Initial Value Problem in N Dimensions

Introduce formally the d'Alembertian

$$\Box \overset{\text{def}}{=} \frac{\partial^2}{\partial t^2} - c^2 \Delta.$$

and, given $\varphi \in C^3(\mathbb{R}^N)$ and $\psi \in C^2(\mathbb{R}^N)$, consider the Cauchy problem

$$\begin{aligned} \Box u &= 0 \qquad \text{in } \mathbb{R}^N \times \mathbb{R} \\ u(\cdot, 0) &= \varphi \\ u_t(\cdot, 0) &= \psi \qquad \text{in } \mathbb{R}^N. \end{aligned} \tag{5.1}$$

If $N \geq 3$, the problem (5.1) can be solved by the Poisson method of spherical means, and if $N = 2$ by the Hadamard method of descent.

5.1 Spherical Means

Let ω_N denote the measure of the unit sphere in $\mathbb{R}^N$ and let $d\omega$ denote the surface measure on the unit sphere of $\mathbb{R}^N$, that is, the infinitesimal solid angle in $\mathbb{R}^N$. If $v \in C(\mathbb{R}^N)$, the spherical mean of v at x of radius ρ is

$$\begin{aligned} M(v; x, \rho) &= \frac{1}{\operatorname{meas}[\partial B_\rho(x)]} \int_{\partial B_\rho(x)} v d\sigma \\ &= \frac{1}{\omega_N \rho^{N-1}} \int_{|x-y|=\rho} v(y) d\sigma(y) = \frac{1}{\omega_N} \int_{|\nu|=1} v(x + \rho\nu) d\omega \end{aligned}$$

where ν ranges over the unit sphere of $\mathbb{R}^N$.

Remark 5.1 The function $\rho \to M(v; x, \rho)$ can be defined in all of $\mathbb{R}$ by an even reflection about the origin since

$$\int_{|\nu|=1} v(x + \rho\nu) d\omega = \int_{|\nu|=1} v(x - \rho(-\nu)) d\omega = \int_{|\nu|=1} v(x - \rho\nu) d\omega.$$

Remark 5.2 If $v \in C^s(\mathbb{R}^N)$ for some $s \in \mathbb{N}$, then $x \to M(v; x, \rho) \in C^s(\mathbb{R}^N)$.

Remark 5.3 Knowing $(x, \rho) \to M(v; x, \rho)$ permits one to recover $x \to v(x)$, since

$$\lim_{\rho \to 0} M(v; x, \rho) = v(x) \quad \text{for all } x \in \mathbb{R}^N.$$

5.2 The Darboux Formula

Assume that $v \in C^2(\mathbb{R}^N)$. By the divergence theorem

$$\begin{aligned} \int_{|x-y|<\rho} \Delta v(y) dy &= \int_{|x-y|=\rho} \nabla v(y) \cdot \nu d\sigma(y) \\ &= \rho^{N-1} \int_{|\nu|=1} \nabla v(x + \rho\nu) \cdot \nu d\omega \\ &= \rho^{N-1} \frac{d}{d\rho} \int_{|\nu|=1} v(x + \rho\nu) d\omega. \end{aligned}$$

Therefore

$$\begin{aligned}\frac{\partial}{\partial\rho}M(v;x,\rho) &= \frac{1}{\omega_N\rho^{N-1}}\int_{|x-y|<\rho}\Delta v(y)dy\\ &= \frac{1}{\omega_N\rho^{N-1}}\int_0^\rho r^{N-1}\Delta_x\int_{|\nu|=1}v(x+r\nu)d\omega dr.\end{aligned}$$

Multiplying by ρ^{N-1} and taking the derivative with respect to ρ yields

$$\frac{\partial}{\partial\rho}\left(\rho^{N-1}\frac{\partial}{\partial\rho}M(v;x,\rho)\right) = \Delta_x(\rho^{N-1}M(v;x,\rho)).$$

This, in turn, gives Darboux's formula

$$\left(\frac{\partial^2}{\partial\rho^2}+\frac{N-1}{\rho}\frac{\partial}{\partial\rho}\right)M(v;x,\rho) = \Delta_x M(v;x,\rho) \tag{5.2}$$

valid for all $v \in C^2(\mathbb{R}^N)$.

5.3 An Equivalent Formulation of the Cauchy Problem

Let $u \in C^2(\mathbb{R}^N\times\mathbb{R})$ be a solution of (5.1). Then for all $x \in \mathbb{R}^N$ and for all $\rho > 0$

$$\begin{aligned}\Delta_x M(u;x,\rho) &= \frac{1}{\omega_N}\int_{|\nu|=1}\Delta_x u(x+\rho\nu,t)d\omega\\ &= \frac{1}{c^2\omega_N}\int_{|\nu|=1}\frac{\partial^2}{\partial t^2}u(x+\rho\nu,t)\,d\omega = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}M(u;x,\rho).\end{aligned}$$

Therefore, setting

$$M(\rho,t) = M(u(x,t);x,\rho)$$

and recalling Remarks 5.1 and 5.3, one concludes that $u \in C^2(\mathbb{R}^N\times\mathbb{R})$ is a solution of (5.1) if and only if

$$\begin{aligned}\frac{\partial^2}{\partial t^2}M(\rho,t) &= c^2\left(\frac{\partial^2}{\partial\rho^2}+\frac{N-1}{\rho}\frac{\partial}{\partial\rho}\right)M(\rho,t)\\ M(\rho,0) &= M(\varphi;x,\rho) = M_\varphi(x,\rho)\\ M_t(\rho,0) &= M(\psi;x,\rho) = M_\psi(x,\rho).\end{aligned} \tag{5.3}$$

6 The Cauchy Problem in $\mathbb{R}^3$

If $N = 3$, the initial value problem (5.3) becomes, on multiplication by ρ

$$\begin{aligned} \frac{\partial^2}{\partial t^2}(\rho M(\rho,t)) &= c^2\frac{\partial^2}{\partial \rho^2}(\rho M(\rho,t)) \text{ in } \mathbb{R}\times\mathbb{R} \\ \rho M(\rho,0) &= \rho M_\varphi(x,\rho) \\ \rho M_t(\rho,0) &= \rho M_\psi(x,\rho). \end{aligned} \tag{6.1}$$

By the d'Alembert formula (2.2)

$$\begin{aligned} \rho M(\rho,t) = {}& \frac{1}{2}[(\rho-ct)M_\varphi(x,\rho-ct)+(\rho+ct)M_\varphi(x,\rho+ct)] \\ &+ \frac{1}{2c}\int_{\rho-ct}^{\rho+ct} sM_\psi(x,s)ds. \end{aligned}$$

Differentiating with respect to ρ

$$\begin{aligned} M(\rho,t)+\rho M_\rho(\rho,t) = {}& \frac{1}{2}[M_\varphi(x,\rho-ct)+M_\varphi(x,\rho+ct)] \\ &+ \frac{1}{2}\left[(\rho-ct)\frac{\partial}{\partial\rho}M_\varphi(x,\rho-ct)+(\rho+ct)\frac{\partial}{\partial\rho}M_\varphi(x,\rho+ct)\right] \\ &+ \frac{1}{2c}\left[(\rho+ct)M_\psi(x,\rho+ct)-(\rho-ct)M_\psi(x,\rho-ct)\right]. \end{aligned}$$

Letting $\rho\to 0$ gives the solution formula for (5.1)

$$\begin{aligned} u(x,t) = {}& \frac{1}{8\pi}\left(\int_{|\nu|=1}\varphi(x+c\nu t)d\omega+\int_{|\nu|=1}\varphi(x-c\nu t)d\omega\right) \\ &+ \frac{1}{8\pi}\left(ct\int_{|\nu|=1}\nabla\varphi(x+\nu ct)\cdot\nu\, d\omega - ct\int_{|\nu|=1}\nabla\varphi(x-c\nu t)\cdot\nu\, d\omega\right) \\ &+ \frac{1}{8\pi}\left(t\int_{|\nu|=1}\psi(x+\nu ct)d\omega+t\int_{|\nu|=1}\psi(x-c\nu t)d\omega\right). \end{aligned} \tag{6.2}$$

From this and Remark 5.1

$$u(x,t)=\frac{1}{4\pi}\frac{\partial}{\partial t}\left(t\int_{|\nu|=1}\varphi(x+\nu ct)d\omega\right)+\frac{1}{4\pi}t\int_{|\nu|=1}\psi(x+\nu ct)d\omega. \tag{6.3}$$

This can be written in the equivalent form

$$4\pi c^2u(x,t)=\frac{\partial}{\partial t}\left(\frac{1}{t}\int_{|x-y|=ct}\varphi(y)d\sigma\right)+\frac{1}{t}\int_{|x-y|=ct}\psi(y)d\sigma. \tag{6.4}$$

By carrying out the differentiation under the integral in (6.3)

$$4\pi c^2u(x,t)=\frac{1}{t^2}\int_{|x-y|=ct}[t\psi(y)+\varphi(y)+\nabla\varphi\cdot(x-y)]d\sigma. \tag{6.5}$$

Theorem 6.1. *Let $N=3$ and assume that $\varphi\in C^3(\mathbb{R}^3)$ and $\psi\in C^2(\mathbb{R}^3)$. Then there exists a unique solution to the Cauchy problem (5.1), and it is given by (6.2)–(6.5).*

Proof. We have only to prove the uniqueness. If $u, v \in C^2(\mathbb{R}^3 \times \mathbb{R})$ are two solutions, the spherical mean of their difference

$$\tilde{M} = \frac{3}{4\pi} \int_{|\nu|=1} (u - v)(x + \rho\nu) d\omega$$

satisfies (6.1) with homogeneous data. By the uniqueness of solutions to the one dimensional Cauchy problem, $\tilde{M} = 0$ for all $\rho > 0$. Thus $u = v$. ■

Formulas (6.2)–(6.5) are the *Kirchoff formulas*; they permit one to read the relevant properties of the solution u.

Remark 6.1 (Domain of Dependence) The solution at a point $(x,t) \in \mathbb{R}^{N+1}$ for $N = 3$ depends on the data φ and ψ and the derivatives φ_{x_i} on the sphere $|x - y| = ct$. Unlike the 1-dimensional case, the data in the interior of $B_{ct}(x)$ are not relevant to the value of u at (x,t).

Remark 6.2 (Regularity) In the case $N = 1$ the solution is as regular as the data. If $N = 3$, because of the t-derivative intervening in the representation (6.4), solutions of (5.1) are less regular than the data φ and ψ. In general, if $\varphi \in C^{m+1}(\mathbb{R}^3)$ and $\psi \in C^m(\mathbb{R}^3)$ for some $m \in \mathbb{N}$, then $u \in C^m(\mathbb{R}^3 \times \mathbb{R})$. Thus if φ and ψ are merely of class C^2 in $\mathbb{R}^3$, then $u_{x_i x_i}$ might blow up at some point $(x,t) \in \mathbb{R}^3 \times \mathbb{R}$ even though $\varphi_{x_i x_j}$, and $\psi_{x_i x_j}$ are bounded. This is known as the *focussing effect*. In view of Remark 6.1, the set of singularities might become *compressed* for $t > 0$ into a smaller set called the *caustic*.

Remark 6.3 (Compactly Supported Data) In the remainder of this section we assume that the initial data φ and ψ are compactly supported, say in the ball $B_r(0)$, and discuss the stability in $L^\infty(\mathbb{R}^3)$ for all $t \in \mathbb{R}$. From the solution formula (6.3), it follows that $x \to u(x,t)$ is supported in the spherical annulus $(ct - r)_+ \le |x| \le r + ct$. A disturbance concentrated in $B_r(0)$ affects the solution only within such a spherical annulus.

Remark 6.4 (Decay for Large Times) We continue to assume that the data φ and ψ are supported in the ball $B_r(0)$. By Remark 6.3, the solution $x \to u(x,t)$ is also compactly supported in $\mathbb{R}^3$. The solution is also compactly supported in the t variable, in the following sense:

$$t \to u(x,t) = 0 \;\text{ if for fixed }\; |x|,\; |t| \;\text{ is sufficiently large.}$$

A stronger statement holds, i.e., $\|u\|_{\infty,\mathbb{R}}(t) \to 0$ as $t \to \infty$. Indeed, from (6.5), for large times

$$\|u(t)\|_{\infty,\mathbb{R}^3} \le \frac{(1+c)r^2}{c^2 t} (\|\varphi\|_{\infty,\mathbb{R}^3} + \|\nabla\varphi\|_{\infty,\mathbb{R}^3} + \|\psi\|_{\infty,\mathbb{R}^3}) \qquad (6.6)$$

since the sphere $|x - y| = ct$ intersects the support of the data, at most in a disc of radius r.

Remark 6.5 (Energy) Let $\mathcal{E}(t)$ denote the energy of the system at time t

$$\mathcal{E}(t) = \int_{\mathbb{R}^3} \left(u_t^2 + c^2|Du|^2\right) dx$$

where D denotes the gradient with respect to the space variables only. Multiplying the PDE $\Box u = 0$ by u_t and integrating by parts in $\mathbb{R}^3$ yields

$$\frac{d}{dt}\mathcal{E}(t) = 0.$$

The compactly supported nature of $x \to u(x,t)$ is employed here in justifying the integration by parts. The same result would hold for a solution $u \in C^2(\mathbb{R}^3 \times \mathbb{R})$ satisfying

$$|Du|(\cdot,t) \in L^2(\mathbb{R}^3) \quad \text{for all } \ t \in \mathbb{R}. \tag{6.7}$$

A consequence is

Lemma 6.1 *There exists at most one solution to the Cauchy problem (5.1) within the class (6.7).*

Also, taking into account (6.6) and Theorem 6.1,

Theorem 6.2. *Let $N = 3$ and assume that φ and ψ are supported in the ball B_r for some $r > 0$. Assume further that $\varphi \in C^3(\mathbb{R}^3)$ and $\psi \in C^2(\mathbb{R}^3)$. Then there exists a unique solution to the Cauchy problem (5.1), and it is given by (6.2)–(6.4). Moreover, such a solution is stable in $L^\infty(\mathbb{R}^3)$.*

Therefore, for smooth and compactly supported initial data, (5.1) is well posed in the sense of Hadamard, in the topology of $L^\infty(\mathbb{R}^3)$.

7 The Cauchy Problem in $\mathbb{R}^2$

Consider the Cauchy problem for the wave equation in two space dimensions

$$\begin{aligned} u_{tt} - c^2(u_{x_1x_1} + u_{x_2x_2}) &= 0 \qquad && \text{in } \mathbb{R}^2 \times \mathbb{R} \\ u(\cdot,0) &= \varphi && \text{in } \mathbb{R}^2 \\ u_t(\cdot,0) &= \psi && \text{in } \mathbb{R}^2. \end{aligned} \tag{7.1}$$

Theorem 7.1. *Assume that $\varphi \in C^3(\mathbb{R}^2)$ and $\psi \in C^2(\mathbb{R}^2)$. Then the Cauchy problem (7.1) has the unique solution*

$$\begin{aligned} u(x_1,x_2,t) = {}& \frac{\partial}{\partial t}\left(\frac{1}{2\pi c}\int_{D_{ct}(x_1,x_2)} \frac{\varphi(y_1,y_2)dy_1dy_2}{\sqrt{c^2t^2 - (y_1-x_1)^2 - (y_2-x_2)^2}}\right) \\ & + \frac{1}{2\pi c}\int_{D_{ct}(x_1,x_2)} \frac{\psi(y_1,y_2)dy_1y_2}{\sqrt{c^2t^2 - (y_1-x_1)^2 - (y_2-x_2)^2}} \end{aligned} \tag{7.2}$$

where $D_{ct}(x_1,x_2)$ is the disc of center (x_1,x_2) and radius ct.

The Hadamard method of descent ([111]), consists in viewing the solution of (7.1) as an x_3-independent solution of (5.1) for $N = 3$, for which one has the explicit representations (6.2)–(6.5). Let S be the sphere in $\mathbb{R}^3$, of center $(x_1, x_2, 0)$ and radius ct

$$S = \left\{(y_1, y_2, y_3) \in \mathbb{R}^3 \;\middle|\; (x_1 - y_1)^2 + (x_2 - y_2)^2 + y_3^2 = c^2t^2\right\}.$$

From (6.5)

$$\begin{aligned} u(x_1, x_2, t) &= u(x_1, x_2, 0, t) \\ &= \frac{\partial}{\partial t}\left(\frac{1}{4\pi c^2 t}\int_S \varphi(y_1, y_2)d\sigma\right) + \frac{1}{4\pi c^2 t}\int_S \psi(y_1, y_2)\, d\sigma. \end{aligned}$$

If $P = (y_1, y_2, y_3) \in S$ and if $\nu(P)$ is the outward unit normal to S at P, then for $|y_3| > 0$

$$\nu \cdot \frac{y_3}{|y_3|} = \frac{y_3}{ct} \qquad \text{and} \qquad d\sigma = \frac{ct}{|y_3|}dy_1 dy_2$$

where $dy = dy_1 dy_2$ is the Lebesgue measure in $\mathbb{R}^2$ and (y_1, y_2) ranges over the disc $(y_1 - x_1)^2 + (y_2 - x_2)^2 < (ct)^2$. Also

$$|y_3| = \sqrt{c^2t^2 - [(y_1 - x_1)^2 + (y_2 - x_2)^2]}.$$

Carry these remarks in the previous formula and denote by $x = (x_1, x_2)$ and $y = (y_1, y_2)$ points in $\mathbb{R}^2$ to obtain

$$\begin{aligned} u(x, t) = \frac{\partial}{\partial t}&\left(\frac{1}{2\pi c}\int_{|y-x|<ct} \frac{\varphi(y)}{\sqrt{c^2t^2 - |y - x|^2}}dy\right) \\ &+ \frac{1}{2\pi c}\int_{|y-x|<ct} \frac{\psi(y)}{\sqrt{c^2t^2 - |y - x|^2}}dy \end{aligned} \tag{7.3}$$

where we have used that as $P = (y_1, y_2, y_3) = (y, y_3)$ runs over S, y runs twice over the disc $|y-x| < ct$. Formula (7.3) is the Poisson formula for the solution of (7.1).

Remark 7.1 (Domain of Dependence) The solution u at a point $(x, t) \in \mathbb{R}^2 \times \mathbb{R}$ depends on the values of the initial data φ, $\nabla\varphi$, and ψ on the whole disc $|y - x| < ct$. This is in contrast to the three-dimensional case in which only the values on the sphere of center x and radius ct were relevant.

Remark 7.2 (Disturbances and the Huygens Principle) The values of the data φ, $\nabla\varphi$, and ψ at some $x_o \in \mathbb{R}^2$ (initial disturbances at x_o) will not affect a point x until time $ct(x) = |x - x_o|$, and will affect $u(x, t)$ at all further times $t > t(x)$. Therefore a signal starting at x_o at time $t = 0$ is received by x at $t = t(x)$ and keeps being "received" thereafter. This explains the propagation of circular waves in still water originating from a "nearly-a-point" disturbance. In the three-dimensional case, an initial disturbance $\varphi(x_o)$, $\nabla\varphi(x_o)$, and $\psi(x_o)$ at $x_o \in \mathbb{R}^3$ reaches x at time $ct = |x - x_o|$ and will not affect $u(x,t)$ for all later times. This is a special case of the Huygens principle, which states that if $N \ge 3$ and N is odd, signals originating at some $x_o \in \mathbb{R}^N$ are received by an observer at $x \in \mathbb{R}^N$ only at a single instant.

8 The Inhomogeneous Cauchy Problem

Consider the inhomogeneous initial value problem

$$\begin{aligned} \Box u &= f \in C^2(\mathbb{R}^N \times \mathbb{R}) && \text{in } \mathbb{R}^N \times \mathbb{R},\ N = 2,3 \\ u(\cdot,0) &= \varphi \in C^3(\mathbb{R}^N) && \text{in } \mathbb{R}^N \\ u_t(\cdot,0) &= \psi \in C^2(\mathbb{R}^N) && \text{in } \mathbb{R}^N. \end{aligned} \tag{8.1}$$

The solution is the sum of the unique solution of (5.1) ($f = 0$), and

$$\begin{aligned} \Box v &= f && \text{in } \mathbb{R}^N \times \mathbb{R} \\ v(x,0) &= v_t(x,0) = 0 && \text{in } \mathbb{R}^N. \end{aligned} \tag{8.2}$$

The Duhamel principle permits one to reduce the solution of (8.2) to the solution of the family of homogeneous problems ($f = 0$)

$$\begin{aligned} \Box w(x,t;\tau) &= 0 && \text{in } \mathbb{R}^N \times (t > \tau) \\ w(\cdot,\tau;\tau) &= 0 && \text{in } \mathbb{R}^N \\ w_t(\cdot,\tau;\tau) &= f(\cdot,\tau) \end{aligned}$$

By Duhamel's principle, the solution of (8.2) is given by

$$v(x,t) = \int_0^t w(x,t;\tau)d\tau.$$

Indeed, by direct calculation

$$v_t(x,t) = \int_0^t w_t(x,t;\tau)d\tau$$

since $w(x,t;t) = 0$. Therefore $v(x,0) = v_t(x,0) = 0$. Next

$$\begin{aligned} v_{tt} &= w_t(x,t;t) + \int_0^t w_{tt}(x,t;\tau)d\tau \\ &= f(x,t) + c^2 \int_0^t \Delta w(x,t;\tau)d\tau = f + c^2 \Delta v \end{aligned}$$

so that (8.2) holds. If $N = 3$ and $t \ge 0$

$$v(x,t) = \frac{1}{4\pi c^2} \int_0^t \frac{1}{(t-\tau)} \int_{|x-y|=c(t-\tau)} f(y,\tau)d\sigma d\tau. \tag{8.3}$$

If $N = 2$ and $t \ge 0$

$$v(x,t) = \frac{1}{2\pi c}\int_0^t \int_{|x-y|\le c(t-\tau)} \frac{f(y,\tau)}{\sqrt{c^2(t-\tau)^2 - |x-y|^2}} dy\, d\tau. \tag{8.4}$$

Remark 8.1 (Domain of Dependence) If $N = 3$, the value of v at a point (x,t), for $t > 0$, depends only on the values of the forcing term f on the surface of the truncated backward characteristic cone

$$[|x-y| = c(t-\tau)] \cap [0 \le \tau \le t].$$

If $N = 2$, the domain of dependence is the full truncated backward characteristic cone

$$[|x-y| < c(t-\tau)] \cap [0 \le \tau \le t].$$

Remark 8.2 (Disturbances) The effect of a source disturbance at a point (x_o, t_o) is not felt at x until the time

$$t(x) = t_o + \frac{1}{c}|x - x_o|.$$

Notice that $\frac{1}{c}|x - x_o|$ is the time it takes for an *initial* disturbance at x_o to affect x. Thus $f(x_o, t_o)$ can be viewed as an *initial datum* delayed to a time t_o. For this reason, the solution formulas (8.3), (8.4) are referred to as *retarded potentials.*

9 The Cauchy Problem for Inhomogeneous Surfaces

The methods introduced for the inhomogeneous initial value problem permit one to solve the following noncharacteristic Cauchy problem

$$\begin{aligned} \Box u &= f && \text{in } \mathbb{R}^3 \times (t > \Phi) \\ u(\cdot, \Phi) &= \varphi && \text{in } \mathbb{R}^3 \\ u_t(\cdot, \Phi) &= \psi && \text{in } \mathbb{R}^3. \end{aligned} \tag{9.1}$$

The data φ, and ψ are now given on the surface $\Sigma = [t = \Phi]$. Such a surface must be noncharacteristic in the sense that $c|\nabla\Phi| \ne 1$ in $\mathbb{R}^3$. We require that Σ is nearly flat, in the sense

$$c\|\nabla\Phi\|_{\infty,\mathbb{R}^3} < 1. \tag{9.2}$$

To convey the main ideas of the technique, we will assume that φ, ψ, and Φ are as smooth as needed to carry out the calculations below. Finally, without loss of generality, we may assume that $\Phi \ge 0$.

9.1 Reduction to Homogeneous Data on $t = \Phi$

First consider the problem of finding $v \in C^3(\mathbb{R}^3 \times \mathbb{R})$, a solution of

$$\begin{gathered}(\Box v - f)\big|_{t=\Phi} = (\Box v - f)_t\big|_{t=\Phi} = (\Box v - f)_{tt}\big|_{t=\Phi} = 0 \\ v(\cdot, \Phi) = \varphi, \quad v_t(\cdot, \Phi) = \psi.\end{gathered} \tag{9.3}$$

Lemma 9.1 *Let (9.2) hold. Then there exists a solution to problem (9.3).*

Proof. Seek v of the form

$$v(x,t) = \sum_{i=0}^{4} a_i(x)(t - \Phi(x))^i$$

where $x \to a_i(x)$, for $i = 1, \dots, 4$, are smooth functions to be calculated. The last two of (9.3) give $a_o = \varphi$ and $a_1 = \psi$. Next, by direct calculation

$$\begin{aligned}\Box v = &\sum_{i=2}^{4} i(i-1)a_i(x)(t - \Phi(x))^{i-2} \\ &- c^2 \sum_{i=0}^{4} \Delta a_i(x)(t - \Phi(x))^i \\ &- 2c^2 \sum_{i=1}^{4} i\nabla a_i(x)\nabla(t - \Phi(x))(t - \Phi(x))^{i-1} \\ &- c^2 \sum_{i=1}^{4} a_i(t - \Phi(x))^{i-1}\Delta(t - \Phi(x)) \\ &- c^2 \sum_{i=2}^{4} i(i-1)a_i(x)(t - \Phi(x))^{i-2}|\nabla(t - \Phi(x))|^2.\end{aligned}$$

From this and (9.2)–(9.3)

$$\begin{aligned}2(1 - c^2|\nabla\Phi|^2)a_2 &= c^2[\Delta(\varphi - a_1\Phi) + \Phi\Delta a_1] + f \\ 6(1 - c^2|\nabla\Phi|^2)a_3 &= c^2[\Delta(\psi - 2a_2\Phi) + 2\Phi\Delta a_2] + f_t \\ 24(1 - c^2|\nabla\Phi|^2)a_4 &= 2c^2[\Delta(a_2 - 3a_3\Phi) + 3\Phi\Delta a_3] + f_{tt}.\end{aligned}$$ ■

9.2 The Problem with Homogeneous Data

Look for a solution of (9.1) of the form $w = u - v$, and set $F = f - \Box v$. Then w satisfies

$$\begin{aligned}\Box w &= F &&\text{in } \mathbb{R}^3 \times (t > \Phi) \\ w(\cdot, \Phi) &= w_t(\cdot, \Phi) = 0 &&\text{in } \mathbb{R}^3.\end{aligned} \tag{9.4}$$

By the construction process of the solution of (9.3) $F = F_t = F_{tt} = 0$ on $t = \Phi$, so that the function

$$F_o(x,t) = \begin{cases} F(x,t) & \text{for } t \ge \Phi(x) \\ 0 & \text{for } t \le \Phi(x) \end{cases}$$

is of class C^2 in $\mathbb{R}^3 \times \mathbb{R}$. Then solve

$$\begin{aligned} \Box \bar{w} &= F_o && \text{in } \mathbb{R}^3 \times (t > 0) \\ \bar{w}(x,0) &= \bar{w}_t(x,0) = 0 && \text{in } \mathbb{R}^3 \end{aligned}$$

whose solution is given by the representation formula (8.3). The restriction of $\bar{w}$ to $[t > \Phi]$ is the solution of (9.4). This will follow from (8.3) and the next lemma.

Lemma 9.2 *Let (9.2) hold. Then* $\bar{w}(x,t) = 0$ *for* $t \le \Phi(x)$.

Proof. In (8.3), written for $\bar{w}$ and F_o, fix x and $t \le \Phi(x)$. For all y on the lateral surface of the backward truncated characteristic cone

$$[|x-y| = c(t-\tau)] \cap [0 \le \tau < t \le \Phi(x)]$$

we must have $\tau < \Phi(y)$. Indeed, if not

$$|x-y| \le c(\Phi(x) - \Phi(y)) \le c|\nabla\Phi(\xi)||x-y|$$

for some ξ on the line segment $\tau x + (1-\tau)y$ for $\tau \in (0,1)$. In view of (9.2) this yields a contradiction. Since F_o vanishes for (y,τ) such that $\tau \le \Phi(y)$, the lemma follows. ■

The solution obtained this way is unique. This is shown as in Theorem 6.1. Unlike the Cauchy–Kowalewski theorem, the data are not required to be analytic and the solution is global. Analytic data would yield analytic solutions only near Σ.

10 Solutions in Half Space. The Reflection Technique

Consider the initial boundary value problem

$$\begin{aligned} \Box u &= f && \text{in } (\mathbb{R}^2 \times \mathbb{R}^+) \times \mathbb{R} \\ u(\cdot,0) &= \varphi && \text{for } x_3 \ge 0 \\ u_t(\cdot,0) &= \psi && \text{for } x_3 \ge 0 \\ u(x_1,x_2,0,t) &= h(x_1,x_2,t) && \text{for } x_3 = 0,\ t \ge 0. \end{aligned} \tag{10.1}$$

If the data are sufficiently smooth, and there is a solution of class C^3 in the closed half-space $\mathbb{R}^2 \times [x_3 \ge 0] \times [t \ge 0]$, the following compatibility conditions must be satisfied

$$\begin{aligned} h(x_1,x_2,0) &= \varphi(x_1,x_2)\\ h_t(x_1,x_2,0) &= \psi(x_1,x_2)\\ c^2\Delta\varphi + f(x_1,x_2,0,t) &= h_{tt}(x_1,x_2,0)\\ c^2\Delta\psi + f_t(x_1,x_2,0,t) &= h_{ttt}(x_1,x_2,0) \end{aligned} \tag{10.2}$$

Assume henceforth that (10.2) are satisfied and reduce the problem to one with homogeneous data on the hyperplane $x_3 = 0$.

10.1 An Auxiliary Problem

First find a solution $v \in C^3(\mathbb{R}^3 \times \mathbb{R})$ of the problem

$$\begin{gathered} v\big|_{x_3=0} = h,\ v_{x_3}\big|x_3 = 0 = v_{x_3x_3}\big|_{x_3=0} = 0\\ (\Box v - f)\big|_{x_3=0} = (\Box v - f)x_3\big|_{x_3=0} = (\Box v - f)_{x_3x_3}\big|_{x_3=0} = 0. \end{gathered} \tag{10.3}$$

Lemma 10.1 *There exists a smooth solution to (10.3).*

Proof. Look for solutions of the form

$$v(x,t) = h(x_1,x_2,t) + \sum_{i=2}^{4} a_{i-1}(x_1,x_2,t)x_3^i$$

and calculate

$$\begin{aligned} \Box v - f = (h_{tt} - c^2\Delta h)(x_1,x_2,t) &+ \sum_{i=2}^{3} [\Box a_{i-1}(x_1,x_2,t)]x_3^i\\ - c^2 \sum_{i=2}^{4} i(i-1)a_{i-1}x_3^{i-2} &- f(x,t). \end{aligned}$$

Therefore the conditions (10.3) yield

$$2c^2a_1 = \Box h - f\big|_{x_3=0},\ 6c^2a_2 = -f_{x_3}\big|_{x_3=0},\ 24c^2a_3 = -f_{x_3x_3}\big|_{x_3=0}. \qquad \blacksquare$$

10.2 Homogeneous Data on the Hyperplane $x_3 = 0$

Set $w = u - v$ and $F = f - \Box v$. Then

$$\begin{aligned} \Box w &= F && \text{in } (\mathbb{R}^2 \times [x_3 > 0]) \times [t > 0]\\ w(\cdot,0) &= \varphi_o \overset{\text{def}}{=} \varphi - v(\cdot,0) && \text{in } \mathbb{R}^2 \times [x_3 \ge 0]\\ w_t(\cdot,0) &= \psi_o \overset{\text{def}}{=} \psi - v_t(\cdot,0) && \text{in } \mathbb{R}^2 \times [x_3 \ge 0]\\ w\big|_{x_3=0} &= 0 && \text{for } x_3 = 0,\ t \ge 0. \end{aligned}$$

Let $\tilde{F}$, $\tilde{\varphi}_o$, and $\tilde{\psi}_o$ be the odd extensions of F, φ_o, and ψ_o about $x_3 = 0$, and consider the problem

$$\begin{aligned} \square \tilde{w} &= \tilde{F} && \text{in } \mathbb{R}^3 \times \mathbb{R} \\ \tilde{w}(\cdot, 0) &= \tilde{\varphi}_o(x) && \text{in } \mathbb{R}^3 \\ \tilde{w}_t(\cdot, 0) &= \tilde{\psi}_o(x) && \text{in } \mathbb{R}^3. \end{aligned}$$

If this problem has a smooth solution $\tilde{w}$, it must be odd about $x_3 = 0$, that is, $\tilde{w}(x_1, x_2, 0, t) = 0$, so that the restriction of $\tilde{w}$ to $x_3 \ge 0$ is the unique solution of the indicated problem with homogeneous data on $x_3 = 0$. To establish the existence of $\tilde{w}$ we have only to check that $\tilde{\varphi}_o \in C^3(\mathbb{R}^3)$, $\tilde{\psi}_o \in C^2(\mathbb{R}^3)$ and $\tilde{F} \in C^2(\mathbb{R}^3 \times \mathbb{R})$. For this it will suffice to check that

$$\begin{aligned} &F = F_{x_3} = F_{x_3x_3} = 0 \\ &\tilde{\varphi}_o = \tilde{\varphi}_{0,x_3} = \tilde{\varphi}_{0,x_3x_3} = 0 \qquad \text{for } x_3 = 0. \\ &\tilde{\psi}_o = \tilde{\psi}_{0,x_3} = \tilde{\psi}_{0,x_3x_3} = 0 \end{aligned}$$

These conditions follow from the definition of odd reflection about $x_3 = 0$, the compatibility conditions (10.2), and the construction (10.3) of the auxiliary function v.

11 A Boundary Value Problem

Let E be a bounded open set in $\mathbb{R}^N$ with smooth boundary ∂E and consider the initial boundary value problem

$$\begin{aligned} \square u &= 0 && \text{in } E \times \mathbb{R}^+ \\ u(\cdot, t)\big|_{\partial E} &= 0 && \text{in } \mathbb{R}^+ \\ u(\cdot, 0) &= \varphi && \text{in } E \\ u_t(\cdot, 0) &= \psi && \text{in } E. \end{aligned} \tag{11.1}$$

Here $u(x, t)$ represents the displacement, at the point x at time t, of a vibrating ideal body, kept at rest at the boundary at ∂E. By the energy method, (11.1) has at most one solution. To find such a solution we use an N-dimensional version of the method of separation of variables of Section 4.1. Solutions of the type $T(t)X(x)$ yield

$$\begin{aligned} -\Delta X_n &= \lambda_n X && \text{in } E \\ X_n &= 0 && \text{on } \partial E \end{aligned} \quad n \in \mathbb{N} \tag{11.2}$$

and

$$T_n''(t) = -c^2 \lambda_n T_n(t) \quad \text{for } t > 0, \quad n \in \mathbb{N}. \tag{11.3}$$

The next proposition is a consequence of Theorem 11.1 of Section 11 of Chapter 4.

Proposition 11.1 *There exists an increasing sequence $\{\lambda_n\}$ of positive numbers and a sequence of corresponding functions $\{v_n\} \subset C^2(E)$ satisfying (11.2). Moreover $\{v_n\}$ form a complete orthonormal system in $L^2(E)$.*

Using this fact, write the solution u as

$$u(x,t) = \sum T_n(t) v_n(x) \tag{11.4}$$

and deduce that the initial conditions to be associated to (11.3) are derived from (11.4) and the initial data in (11.1), i.e.,

$$T_n(0) = \int_E v_n \varphi \, dx, \qquad T_n'(0) = \int_E v_n \psi \, dx.$$

Thus

$$T_n(t) = \int_E \big[\psi \frac{\sin(c\sqrt{\lambda_n} t)}{c\sqrt{\lambda_n}} + \varphi \cos(c\sqrt{\lambda_n} t)\big] v_n \, dx.$$

Even though the method is elegant and simple, the eigenvalues and eigenfunctions for the Laplace operator in E can be calculated explicitly only for domains with a simple geometry (see Section 8c of the Complements of Chapter 3). The approximate solutions

$$u_n(x,t) = \sum_{i=1}^{n} T_i(t) v_i(x)$$

satisfy, for all $i \in \mathbb{N}$, the approximating problems

$$\begin{aligned} \Box u_n &= 0 & &\text{in } E \times \mathbb{R} \\ u_n(\cdot,t)\big|_{\partial E} &= 0 & &\text{in } \mathbb{R} \\ u_n(x,0) &= \varphi_n(x) \overset{\text{def}}{=} \sum_{i=1}^{n} \langle \varphi, v_i \rangle v_i(x) & &\text{in } E \\ u_{n,t}(x,0) &= \psi_n(x) \overset{\text{def}}{=} \sum_{i=1}^{n} \langle \psi, v_i \rangle v_i(x) & &\text{in } E \end{aligned} \tag{11.5}$$

The function $u(\cdot,t)$ defined by (11.4) is meant as the limit of $u_n(\cdot,t)$ in $L^2(E)$, uniformly in $t \in \mathbb{R}$. The PDE in (11.1) and the initial data are verified in the following weak sense. Let f be any function in $C^2(\bar{E} \times \mathbb{R})$, and vanishing on ∂E. Multiply the PDE in (11.5) by any such f and integrate by parts over $E \times (0,t)$, where $t \in \mathbb{R}$ is arbitrary but fixed. This gives

$$\begin{aligned} \int_E u_n(x,t) f(x,t)\, dx &+ \int_0^t \int_E u_n(x,t)(x,\tau) \Box f \, dx \, d\tau \\ &= \int_E \psi_n f(x,0)\, dx - \int_E \varphi_n f_t(x,0)\, dx. \end{aligned}$$

Letting $n \to \infty$ gives the weak form of (11.1)

$$\begin{aligned} \int_E u(x,t) f(x,t)\, dx &+ \int_0^t \int_E u(x,t)(x,\tau) \Box f \, dx \, dt \\ &= \int_E \psi f(x,0)\, dx - \int_E \varphi f_t(x,0)\, dx \end{aligned}$$

for all $f \in C^2(\bar{E} \times \mathbb{R})$ vanishing on ∂E.

12 Hyperbolic Equations in Two Variables

The most general linear hyperbolic equation in two variables $x = (x_1, x_2)$ takes the form

$$\mathcal{L}(u) = \frac{\partial^2 u}{\partial x_1 \partial x_2} + \mathbf{b} \cdot \nabla u + cu = f \tag{12.1}$$

where $\mathbf{b} = (b_1, b_2)$ and c, f are given continuous functions in $\mathbb{R}^2$. For this, the characteristics are the lines $x_i = (\text{const})_i$ for $i = 1, 2$. If $\mathbf{b} = c = f = 0$, then, up to a change of variables, (12.1) can be rewritten in the form of the wave equation

$$v_{tt} - v_{xx} = 0 \quad \text{in } \mathbb{R}^2 \tag{12.2}$$

where

$$\begin{aligned} x_1 &= x - t \\ x_2 &= x + t \end{aligned} \qquad \text{and} \qquad v(x,t) = u(x - t, x + t).$$

Therefore if v is prescribed on the characteristics $x \pm t = \text{const}$, the method of the characteristic parallelograms of Section 1.1 permits one to solve (12.2) in the whole of $\mathbb{R}^2$.

13 The Characteristic Goursat Problem

The characteristic Goursat problem consists in finding $u \in C^2(\mathbb{R}^2)$ satisfying[1]

$$\mathcal{L}(u) = f \ \text{ in } \mathbb{R}^2, \qquad u \,\big|_{x_i=0} = \varphi_i \in C^2(\mathbb{R}), \quad i = 1, 2. \tag{13.1}$$

Theorem 13.1. *There exists a unique solution to the characteristic Goursat problem (13.1).*

13.1 Proof of Theorem 13.1: Existence

Setting $\nabla u = (w_1, w_2) = \mathbf{w}$, by virtue of (12.1)

$$\frac{\partial}{\partial x_2} w_1 = \frac{\partial}{\partial x_1} w_2 = f - \mathbf{b} \cdot \mathbf{w} - cu.$$

Integrate the first of these equations over $(0, x_2)$ and the second over $(0, x_1)$. Taking into account the data φ_i on the characteristics $x_i = 0$, $i = 1, 2$, recast (13.1) into the equivalent form

---

[1] The problem is also referred to as the Darboux–Goursat problem. For $\mathcal{L}(\cdot)$ linear, the problem was posed and solved by Darboux, [42](Tome II, pages 91-94). The nonlinear case of $u_{x_1x_2} = F(x_1, x_2, u, x_{x_1}, u_{x_2})$ was solved by E. Goursat, [108, Vol. 3 part I]. See also J. Hadamard, [110](pages 107-108).

$$
\begin{aligned}
w_1(x) &= \varphi_2'(x_1) + \int_0^{x_2} (f - \mathbf{b} \cdot \mathbf{w} - cu)(x_1, s)ds \\
w_2(x) &= \varphi_1'(x_2) + \int_0^{x_1} (f - \mathbf{b} \cdot \mathbf{w} - cu)(s, x_2)ds \qquad (13.2) \\
u(x) &= \varphi_2(x_1) + \int_0^{x_2} w_2(x_1, s)ds.
\end{aligned}
$$

The last equation could be equivalently replaced by

$$u(x) = \varphi_1(x_2) + \int_0^{x_1} w_1(s, x_2)ds.$$

To solve (13.2), define

$$u_o = \varphi_2, \quad w_{1,o} = \varphi_2', \quad w_{2,o} = \varphi_1'$$

and recursively, for $n = 0, 1, \dots$

$$
\begin{aligned}
w_{1,n+1}(x) &= \varphi_2'(x_1) + \int_0^{x_2} [f - \mathbf{b} \cdot (w_{1,n}, w_{2,n}) - cu_n](x_1, s)ds \\
w_{2,n+1}(x) &= \varphi_1'(x_2) + \int_0^{x_1} [f - \mathbf{b} \cdot (w_{1,n}, w_{2,n}) - cu_n](s, x_2)ds \qquad (13.2)_n \\
u_{n+1} &= \varphi_2(x_1) + \int_0^{x_2} w_{2,n}(x_1, s)ds.
\end{aligned}
$$

A solution of (13.2) can be found by letting $n \to \infty$ in $(13.2)_n$, provided the sequences $\{u_n\}$ and $\{w_{i,n}\}$ for $i = 1, 2$ are uniformly convergent over compact subsets of $\mathbb{R}^2$. For this it suffices to prove that the telescopic series

$$u_o + \sum(u_n - u_{n-1}) \quad \text{and} \quad w_{i,o} + \sum(w_{i,n} - w_{i,n-1}) \qquad (13.3)$$

are absolutely and uniformly convergent on compact subsets $K \subset \mathbb{R}^2$. Having fixed one such K, one may assume that it is a square about the origin with sides parallel to the coordinate axes, and such that $\text{meas}(K) \le 1$. Set

$$
\begin{gathered}
V_n = (u_n, w_{1,n}, w_{2,n}), \quad |x| = |x_1| + |x_2| \\
\|V_n - V_{n-1}\| = |u_n - u_{n-1}| + |w_{1,n} - w_{1,n-1}| + |w_{2,n} - w_{2,n-1}| \\
C_K = 1 + \|\mathbf{b}\|_{\infty,K} + \|c\|_{\infty,K} + \|f\|_{\infty,K}, \quad A_K = 1 + \|V_o\|_{\infty,K}.
\end{gathered}
$$

Lemma 13.1 *For all $x \in K$ and all $n \in \mathbb{N}$*

$$\|V_n - V_{n-1}\|(x) \le A_K(2C_K)^n \frac{|x|^n}{n!}. \qquad (13.4)$$

Proof. From $(13.2)_{n=0}$

$$\begin{aligned} w_{1,1} - w_{1,o} &= \int_0^{x_2} [f - \mathbf{b} \cdot (w_{1,o}, w_{2,o}) - cu_o](x_1, s)ds \\ w_{2,1} - w_{2,o} &= \int_0^{x_1} [f - \mathbf{b} \cdot (w_{1,o}, w_{2,o}) - cu_o](s, x_2)ds \\ u_1 - u_o &= \int_0^{x_2} w_{2,o}(x_1, s)ds. \end{aligned}$$

From this

$$\begin{aligned} \|V_1 - V_o\|(x) \le \|f\|_{\infty,K} &+ (\|\mathbf{b}\|_{\infty,K} + \|c\|_{\infty,K})\|V_o\|_{\infty,K} \int_0^{x_2} ds \\ &+ \|f\|_{\infty,K} + (\|\mathbf{b}\|_{\infty,K} + \|c\|_{\infty,K})\|V_o\|_{\infty,K} \int_0^{x_1} ds \\ &+ \|V_o\|_{\infty,K} \int_0^{x_2} ds \\ &\le A_K C_K |x|. \end{aligned}$$

Therefore (13.4) holds for $n = 1$. We show by induction that if it does hold for n it continues to hold for $n + 1$. From (13.2), for all $x \in K$

$$\begin{aligned} &\|V_{n+1} - V_n\|(x) \\ &\quad \le C_K \left(\int_0^{x_2} \|V_n - V_{n-1}\|(x_1, s)ds + \int_0^{x_1} \|V_n - V_{n-1}\|(s, x_2)ds \right) \\ &\quad \le A_K \frac{2^n C_K^{n+1}}{(n-1)!} \left(\int_0^{x_2} |(x_1, s)|^n ds + \int_0^{x_1} |(s, x_2)|^n ds \right) \\ &\quad \le A_K (2C_K)^{n+1} \frac{|x|^{n+1}}{(n+1)!}. \end{aligned}$$ ■

Returning to the absolute convergence of the series in (13.3), it follows from the lemma that for all $x \in K$

$$\|V_o\|(x) + \sum \|V_n - V_{n-1}\|(x) \le A_K \left(1 + \sum (2C_K)^n \frac{|x|^n}{n!} \right) = A_K e^{2C_K |x|}.$$ ■

13.2 Proof of Theorem 13.1: Uniqueness

Let us assume that there exist two locally bounded solutions of the system (13.2), say $(u^{(i)}, w_1^{(i)}, w_2^{(i)}) = V^{(i)}$ for $i = 1, 2$, and set

$$\|V^{(1)} - V^{(2)}\| = |u^{(1)} - u^{(2)}| + |w_1^{(1)} - w_1^{(2)}| + |w_2^{(1)} - w_2^{(2)}|.$$

Write the system (13.2) for $V^{(1)}$ and $V^{(2)}$, and subtract the resulting equations, to obtain for all $x \in K$

$$\|V^{(1)} - V^{(2)}\|(x) \le \|V^{(1)} - V^{(2)}\|_{\infty,K} B_K |x|$$

where $B_K = \|\mathbf{b}\|_{\infty,K} + \|c\|_{\infty,K}$. Since K is an arbitrary compact subset of $\mathbb{R}^2$, this implies $V^{(1)} = V^{(2)}$ identically. ■

13.3 Goursat Problems in Rectangles

Let $\alpha_1 < \beta_1$ and $\alpha_2 < \beta_2$, and let R be the rectangle $[\alpha_1, \beta_1] \times [\alpha_2, \beta_2]$. Prescribe data $\varphi_1 \in C^2[\alpha_1, \beta_1]$ and $\varphi_2 \in C^2[\alpha_2, \beta_2]$ on the segments $[\alpha_1, \beta_1]$ and $[\alpha_2, \beta_2]$, and consider the problem of finding $u \in C^2(R)$ satisfying

$$\begin{aligned} \mathcal{L}(u) &= f && \text{in } R \\ u(x_1, \alpha_2) &= \varphi_2(x_1) && \text{for } x_1 \in [\alpha_1, \beta_1] \\ u(\alpha_1, x_2) &= \varphi_1(x_2) && \text{for } x_2 \in [\alpha_2, \beta_2]. \end{aligned} \tag{13.5}$$

The same proof applies, and one may conclude that (13.5) has a unique solution. Analogously, there exists a unique solution to the characteristic problem

$$\begin{aligned} \mathcal{L}(u) &= f && \text{in } R \\ u(x_1, \beta_2) &= \varphi_2(x_1) && \text{for } x_1 \in [\alpha_1, \beta_1] \\ u(\beta_1, x_2) &= \varphi_1(x_2) && \text{for } x_2 \in [\alpha_2, \beta_2]. \end{aligned} \tag{13.6}$$

14 The Noncharacteristic Cauchy Problem and the Riemann Function

Let Γ be a regular curve in $\mathbb{R}^2$ whose tangent is nowhere parallel to either of the coordinate axes. For example

$$\Gamma = \begin{cases} x_1 = s & s \in \mathbb{R} \\ x_2 = h(s) \in C^1(\mathbb{R}) & \\ h'(s) < 0 & \text{for all } s \in \mathbb{R}. \end{cases}$$

Consider the problem of finding $u \in C^2(\mathbb{R}^2)$ satisfying

$$\mathcal{L}(u) = f \text{ in } \mathbb{R}^2, \quad u\big|_\Gamma = u_{x_2}\big|_\Gamma = 0 \tag{14.1}$$

where $\mathcal{L}(\cdot)$ is defined in (12.1). As an example, take the case $\mathbf{b} = c = 0$, and Γ is the line $x_2 = -x_1$. Then (14.1) reduces to the Cauchy problem for the wave equation

$$\begin{aligned} v_{tt} - v_{xx} &= \tilde{f} && \text{in } \mathbb{R}^2 \\ v(\cdot, 0) &= 0 && t = x_1 + x_2 \\ v_t(\cdot, 0) &= 0 && x = -x_1 + x_2, \end{aligned}$$

where

$$v(x,t) = u\Big(\frac{t-x}{2}, \frac{t+x}{2}\Big), \qquad \tilde{f}(x,t) = f\Big(\frac{t-x}{2}, \frac{t+x}{2}\Big).$$

This problem has a unique solution is given by the representation formula (3.4). We will prove that (14.1) has a unique solution and will exhibit a representation formula for it.

Through a point $x \in \mathbb{R}^2 - \Gamma$, draw two lines parallel to the coordinate axes and let E_x be the region enclosed by these lines and Γ, as in Figure 14.3

$$E_x = \{(\sigma, s) \bigm| h(\sigma) < s < x_2;\ \alpha < \sigma < x_1\}.$$

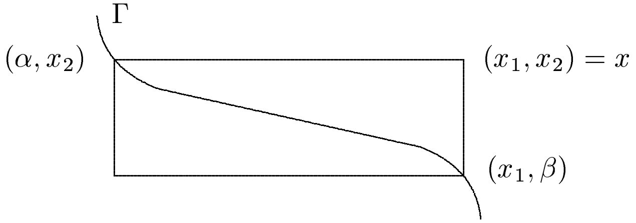

Fig. 14.3

Let $\mathcal{L}^*(\cdot)$ denote the adjoint operator to $\mathcal{L}(\cdot)$

$$\mathcal{L}^*(v) = \frac{\partial^2 v}{\partial x_1 \partial x_2} - \operatorname{div}(\mathbf{b}v) + cv.$$

This is well defined if $b_i \in C^1(\mathbb{R}^2)$, which we assume henceforth. Let u, v be a pair of functions in $C^2(\mathbb{R}^2)$, and compute the quantity

$$\iint_{E_x} [v\mathcal{L}(u) - u\mathcal{L}^*(v)]dy.$$

The outward unit normal to E_x on Γ is $\mathbf{n} = (-h', 1)/\sqrt{1 + h'^2}$. Therefore by Green's theorem

$$\begin{aligned}
&\iint_{E_x} [v\mathcal{L}(u) - u\mathcal{L}^*(v)]dy = (uv)(x) - (uv)(\alpha, x_2) \\
&- \int_\beta^{x_2} u[v_{x_2} - vb_1](x_1, s)ds - \int_\alpha^{x_1} u[v_{x_1} - vb_2](s, x_2)ds \qquad (14.2)\\
&- \int_\alpha^{x_1} vh'(s)[u_{x_2} + ub_1](s, h(s))ds - \int_\alpha^{x_1} u[vb_2 - v_{x_1}](s, h(s))ds.
\end{aligned}$$

If u is a solution to (14.1), then (14.2) reduces to

$$\begin{aligned}
\iint_{E_x} [vf - u\mathcal{L}^*(v)]dy = (uv)(x) &- \int_\beta^{x_2} u[v_{x_2} - b_1 v](x_1, s)ds \\
&- \int_\alpha^{x_1} u[v_{x_1} - b_2 v](s, x_2)ds.
\end{aligned} \qquad (14.3)$$

Next, in (14.3), we make a particular choice of the function v. For each fixed $x \in \mathbb{R}^2$, let $y \to \mathcal{R}(y; x) \in C^2(\mathbb{R}^2)$ satisfy

$$\begin{aligned} \mathcal{L}_y^*[\mathcal{R}(y;x)] &= 0 \quad \text{in } \mathbb{R}^2 \\ \mathcal{R}(x_1,y_2;x) &= \exp\left[\int_{x_2}^{y_2} b_1(x_1,s)ds\right] \\ \mathcal{R}(y_1,x_2;x) &= \exp\left[\int_{x_1}^{y_1} b_2(s,x_2)ds\right]. \end{aligned} \tag{14.4}$$

Such a function exists, and it can be constructed by the method of successive approximations of the previous section. The last two of (14.4) imply that $\mathcal{R}(x;x) = 1$. Therefore, writing (14.3) for $y \to v(y) = \mathcal{R}(y;x)$ yields the representation formula

$$u(x) = \iint_{E_x} \mathcal{R}(y;x)f(y)dy. \tag{14.5}$$

This formula, derived under the assumption that a solution of (14.1) exists, indeed does give the unique solution of such a noncharacteristic problem, as can be verified by direct calculation. The function $y \to \mathcal{R}(y;x)$ is called the Riemann function ([218]), with pole at x, for the operator $\mathcal{L}(\cdot)$ in $\mathbb{R}^2$.[2]

Remark 14.1 The integral formula (14.3) and the Riemann function $\mathcal{R}(\cdot;\cdot)$ permit us to give a representation formula for the unique solution of the noncharacteristic problem (12.1) with inhomogeneous data on Γ

$$\mathcal{L}(u) = f \ \text{ in } \ \mathbb{R}^2, \quad u\big|_\Gamma = \varphi, \ u_{x_2}\big|_\Gamma = \psi \tag*{(14.1)$'$}$$

for given smooth functions in $\mathbb{R}$.

15 Symmetry of the Riemann Function

The Riemann function $y \to \mathcal{R}^*(y;x)$, with pole at x, for $\mathcal{L}^*(\cdot)$ satisfies

$$\begin{aligned} \mathcal{L}[\mathcal{R}^*(y;x)] &= 0 \ \text{ in } \ \mathbb{R}^2 \\ \mathcal{R}^*(y_1,x_2;x) &= \exp\left[-\int_{x_1}^{y_1} b_2(s,x_2)ds\right] \\ \mathcal{R}^*(x_1,y_2;x) &= \exp\left[-\int_{x_2}^{y_2} b_1(x_1,s)ds\right]. \end{aligned} \tag{15.1}$$

It follows from this that $\mathcal{R}^*(x;x) = 1$.

Lemma 15.1 *$\mathcal{R}(y;x) = \mathcal{R}^*(x;y)$.*

Proof. Let $x = (x_1,x_2)$ and $y = (y_1,y_2)$ be fixed in $\mathbb{R}^2$ and be such that the line through them is not parallel to either coordinate axis. Without loss of generality may assume that $y_1 < x_1$ and $y_2 < x_2$, and construct the rectangle

---

[2] For an N-dimensional version of the Riemann function, see Hadamard [110].

$$Q_{x,y} = [y_1 < s < x_1] \times [y_2 < \tau < x_2].$$

By Green's theorem, for every pair of functions $u, v \in C^2(\mathbb{R}^2)$

$$\begin{aligned}\iint_{Q_{x,y}} [v\mathcal{L}(u) - u\mathcal{L}^*(v)]ds d\tau &= (uv)(x) - (uv)(y) \\ &- \int_{y_2}^{x_2} v[u_{x_2} + b_1 u](y_1, \tau)d\tau - \int_{y_1}^{x_1} v[u_{x_1} + b_2 u](s, y_2)ds \qquad (15.2) \\ &- \int_{y_2}^{x_2} u[v_{x_2} - b_1 v](x_1, \tau)d\tau - \int_{y_1}^{x_1} u[v_{x_1} - b_2 v](s, x_2)d\tau.\end{aligned}$$

Write this identity for $v = \mathcal{R}(\cdot; x)$ and $u = \mathcal{R}^*(\cdot; y)$. ■

Remark 15.1 (The Characteristic Goursat Problem) The integral formula (15.2) and the Riemann function permit one to give a representation formula in terms of $\mathcal{R}(\cdot;\cdot)$ of the characteristic Goursat problems (13.5) and (13.6).

Problems and Complements

2c The d'Alembert Formula

2.1. Solve the Cauchy problems

$$\begin{aligned} u_{tt} - u_{xx} = f \ \text{ in } \ \mathbb{R}\times\mathbb{R} \\ u(\cdot, 0) = u_t(\cdot, 0) = 0 \end{aligned} \qquad \text{for} \qquad \begin{aligned} f(x,t) = e^{x-t} \\ f(x,t) = x^2. \end{aligned}$$

3c Inhomogeneous Problems

3.1c The Duhamel Principle ([61])

A linear differential operator with constant coefficients and of order $n \in \mathbb{N}$ in the space variables $x = (x_1, \dots, x_N)$ is defined by

$$\mathcal{L}(w) = \sum_{|\alpha|\le n} A_\alpha D^\alpha w, \quad A_\alpha \in \mathbb{R}, \quad w \in C^n(\mathbb{R}^N).$$

Let $f \in C(\mathbb{R}^{N+1})$, and for a positive integer $m \ge 2$ let

$$(x, t; \tau) \to v(x, t; \tau), \quad x \in \mathbb{R}^N, \ t \in (\tau, \infty), \ \tau \in \mathbb{R}$$

be a family of solutions to the homogeneous Cauchy problems

$$\begin{aligned}
&\frac{\partial^m}{\partial t^m} v = \mathcal{L}(u) && \text{in } \ \mathbb{R}^N \times (\tau, \infty), \ m \ge 2 \\
&\frac{\partial^j}{\partial t^j} v(\cdot, \tau; \tau) = 0 && \text{for } \ j = 0, 1, \dots, m-2 \\
&\frac{\partial^{m-1}}{\partial t^{m-1}} v(\cdot, \tau; \tau) = f(\cdot, \tau)
\end{aligned} \tag{3.1c}$$

parametrized with $\tau \in \mathbb{R}$. Then, the inhomogeneous Cauchy problem

$$\begin{aligned}
&\frac{\partial^m}{\partial t^m} u = \mathcal{L}(u) + f(x, t) && \text{in } \ \mathbb{R}^N \times \mathbb{R} \\
&\frac{\partial^j}{\partial t^j} u(\cdot, 0) = 0 && \text{for } j = 0, 1, \dots, m-1
\end{aligned} \tag{3.2c}$$

has a solution given by

$$u(x, t) = \int_0^t v(x, t; \tau) d\tau \qquad (x, t) \in \mathbb{R}^N \times \mathbb{R}. \tag{3.3c}$$

Formulate a general Duhamel's principle ([61]) if $m = 1$.

4c Solutions for the Vibrating String

4.1. Solve the boundary value problems

$$\begin{aligned}
u_{tt} - u_{xx} &= f \ \text{ in } \ (0, L) \times \mathbb{R} & & f(x, t) = e^x \\
u(0, \cdot) &= u(L, \cdot) = 0 & \text{for} \quad & f(x, t) = \sin \pi x \\
u(\cdot, 0) &= u_t(\cdot, 0) = 0 & & f(x, t) = x^2.
\end{aligned}$$

4.2. Solve the boundary value problem

$$\begin{aligned}
u_{tt} - u_{xx} = x \ \text{ in } \ (0, 1) \times \mathbb{R} \\
u(\cdot, 0) = x^2(1 - x), \ u_t(\cdot, 0) = 0 \\
u_x(0, \cdot) = 0, \ u(1, \cdot) = 0.
\end{aligned}$$

4.3. Let $\beta \in \mathbb{R}$ be a given constant. Solve

$$\begin{aligned}
u_{tt} - u_{xx} = \beta(2u_t - \beta u) \ \text{ in } \ (0, 1) \times \mathbb{R} \\
u(\cdot, 0) = \varphi \in C^2(0, 1), \ u_t(\cdot, 0) = 0 \\
u(0, \cdot) = u(1, \cdot) = 0.
\end{aligned}$$

4.4. Solve the previous problem for $\varphi \in C(0,1)$ but not necessarily of class $C^2(0,1)$. Take, for example

$$\varphi(x) = \begin{cases} 2hx & \text{for } x \in (0, \frac{1}{2}) \\ 2h(1-x) & \text{for } x \in (\frac{1}{2}, 1) \end{cases}$$

where h is a given positive constant.

4.5. Let $a \in \mathbb{R}$, and consider the boundary value problem

$$\begin{aligned} u_{tt} + au_t - u_{xx} &= 0 && \text{in } (0,1) \times (t > 0) \\ u(0,t) = u(1,t) &= 0 && \text{for } t > 0 \\ u(\cdot, 0) = \varphi,\ u_t(\cdot, 0) &= \psi && \text{in } (0,1). \end{aligned}$$

Find an expression for the energy $\mathcal{E}(t)$, introduced in (4.5) in terms of u_t, and estimate $\mathcal{E}(t)$ in terms of the initial data only.

Hint: Setting

$$f(t) = \int_0^t \int_0^1 u_t^2(x,s)\, dx\, ds$$

derive a differential inequality $f' \le A - Bf$, for suitable constants A, B.

4.6. In the previous problem take

$$a = 1, \quad \varphi(x) = \sin \pi x + 2 \sin 5\pi x, \quad \psi = 0.$$

Write down the explicit solution. Find constants c_1 and c_2 such that $|u| + |u_t| \le c_1 e^{c_2 t}$.

4.7. Solve by the separation of variables

$$\begin{aligned} u_{tt} - u_{xx} &= \cos 2t && \text{in } (0,1) \times \mathbb{R}^+ \\ u(0,t) = u(1,t) &= 0 && \text{for } t > 0 \\ u(x,0) &= 0 && \text{in } (0,1) \\ u_t(x,0) &= \sum_{n=1}^{\infty} \sin 2n\pi x && \text{in } (0,1). \end{aligned}$$

4.8. Solve by the separation of variables

$$\begin{aligned} u_{tt} - u_{xx} &= 0 && \text{in } (0,1) \times \mathbb{R}^+ \\ u(0,t) = u(1,t) &= 0 && \text{for } t > 0 \\ u(x,0) &= x^3(1-x)^3 && \text{in } (0,1) \\ u_t(x,0) &= 0 && \text{in } (0,1). \end{aligned}$$

Discuss the regularity of u.

4.9. Solve by the separation of variables

$$\begin{aligned} u_{tt} - u_{xx} &= 0 && \text{in } (0,\pi) \times \mathbb{R}^+ \\ u(0,t) = u(\pi,t) &= 0 && \text{for } t > 0 \\ u(x,0) &= 3\sin^2 x && \text{in } (0,\pi) \\ u_t(x,0) &= 0 && \text{in } (0,\pi). \end{aligned}$$

Discuss the regularity of u.

4.10. Solve by the separation of variables

$$\begin{aligned} u_{tt} - u_{xx} &= 0 && \text{in } (0,\pi)\times\mathbb{R}^+ \\ u_x(0,t) = u_x(\pi,t) &= 0 && \text{for } t>0 \\ u(x,0) &= \frac{\pi}{2} - \left|x - \frac{\pi}{2}\right| && \text{in } (0,\pi) \\ u_t(x,0) &= 0 && \text{in } (0,\pi). \end{aligned}$$

Hint: It might be useful to draw a graph of $f(x) = \frac{\pi}{2} - |x - \frac{\pi}{2}|$.

4.11. Solve by the separation of variables

$$\begin{aligned} u_{tt} - u_{xx} + u &= 0 && \text{in } (0,\pi)\times\mathbb{R}^+ \\ u(0,t) = u(\pi,t) &= 0 && \text{for } t>0 \\ u(x,0) &= x && \text{in } (0,\pi) \\ u_t(x,0) &= \chi_{[0,\frac{\pi}{2}]}(x) && \text{in } (0,\pi). \end{aligned}$$

4.12. Relying on (1.4) solve the problem

$$\begin{aligned} u_{tt} - u_{xx} &= 0 && \text{in } (0,1)\times\mathbb{R}^+ \\ u(0,t) = u(1,t) &= 0 && \text{for } t>0 \\ u(x,0) &= 0 && \text{in } (0,1) \\ u_t(x,0) &= 1 && \text{in } (0,1). \end{aligned}$$

Hint: It might be useful to solve first in $[0,1]\times[0,1]$ and then in $[0,1]\times\mathbb{R}^+$.

4.13. Relying on (1.4) in $D = [0,1]\times[0,1]$ solve the problem

$$\begin{aligned} u_{tt} - u_{xx} &= 0 && \text{in } (0,1)\times\mathbb{R}^+ \\ u(0,t) &= 1 && \text{for } t>0 \\ u(1,t) &= 0 && \text{for } t>0 \\ u(x,0) &= 1 - x^2 && \text{in } (0,1) \\ u_t(x,0) &= 0 && \text{in } (0,1). \end{aligned}$$

4.14. Let u be the solution of (4.1) defined in (4.3)–(4.4). Discuss questions of convergence of the formal approximating solutions

$$u_n = \sum_{j=1}^{n} (A_j \sin j\pi t + B_j \cos j\pi t)\sin j\pi x.$$

Take $L = c = 1$ and verify that for all $p, q, j \in \mathbb{N}$

$$\left\| \frac{\partial^p}{\partial x^p}\frac{\partial^q}{\partial t^q} u_n \right\|_{\infty,(0,1)\times\mathbb{R}} \le \sum_{j=1}^{n} (j\pi)^{p+q}(|A_j| + |B_j|).$$

4.15. Let m be a positive even integer, and let $C^m_{\text{odd}}(0,1)$ be defined as in **10.3** of the Complements of Chapter 4. Assume that φ and ψ are in $C^m_{\text{odd}}(0,1)$, and prove that $u_n \to u$ in $C^m[(0,1)\times\mathbb{R}]$.

6c Cauchy Problems in $\mathbb{R}^3$

6.1. Solve the Cauchy problem

$$\Box u = 0 \text{ in } \mathbb{R}^3 \times \mathbb{R}, \quad u(x,0) = |x|^2, \quad u_t(x,0) = x_3 \text{ in } \mathbb{R}^3.$$

6.2. Find a space-independent solution of $\Box u = e^{-t}$ in $\mathbb{R}^3 \times \mathbb{R}$, and use it to find the solution of

$$\Box u = e^{-t} \text{ in } \mathbb{R}^3 \times \mathbb{R}, \quad u(\cdot,0) = x_1, \quad u_t(\cdot,0) = x_2 x_3.$$

6.1c Asymptotic Behavior

6.3. Let u be the solution of

$$\Box u = 0 \text{ in } \mathbb{R}^3 \times \mathbb{R}, \quad u(\cdot,0) = 0, \quad u_t(\cdot,0) = |x|^k$$

for some $k > 0$. Compute the limit of $u(0,t)$ as $t \to \infty$. Prove that as $|x| \to \infty$ the solution has the form

$$u(x,t) = a(x,t)(1 + |x|^k)$$

for a smooth function $a(x,t)$ uniformly bounded on compact subsets of the t-axis.

6.4. Let u_ε be the unique solution of $\Box u_\varepsilon = 0$ in $\mathbb{R}^3 \times \mathbb{R}$, with initial data

$$u_\varepsilon(\cdot,0) = 0, \qquad \frac{\partial}{\partial t} u_\varepsilon(x,0) = \begin{cases} e^{\frac{-\varepsilon^2}{\varepsilon^2 - |x|^2}} & \text{for } |x| < \varepsilon \\ 0 & \text{for } |x| \geq \varepsilon. \end{cases}$$

Study the limit of u_ε, as $\varepsilon \to 0$, in some appropriate topology.

6.2c Radial Solutions

6.5. Let B be the unit ball about the origin in $\mathbb{R}^3$ and consider the problem (internal vibrations of a contracted sphere)

$$\begin{aligned} u_{tt} - \Delta u &= 0 && \text{in } B \times \mathbb{R} \\ u(\cdot,t)\big|_{\partial B} &= 0 && \text{for } t \in \mathbb{R} \\ u(x,0) &= 0 && \text{in } B \\ u_t(x,0) &= \cos\frac{\pi}{2}|x| && \text{in } B. \end{aligned} \tag{6.1c}$$

Find a radial solution of (6.1c) by the following steps:

(i) Set $|x| = \rho$ and recast the problem as

$$\begin{aligned} &u_{tt} - u_{\rho\rho} - \frac{2}{\rho} u_\rho = 0 && \text{in } (0,1) \times \mathbb{R} \\ &u(1,t) = u_\rho(0,t) = 0 && \text{for } t \in \mathbb{R} \\ &u(\rho,0) = 0, \; u_t(\rho,0) = \cos\frac{\pi}{2}\rho && \text{for } \rho \in (0,1). \end{aligned} \tag{6.2c}$$

(ii) Let v be the symmetric extension of $u(\cdot,t)$ about the origin

$$v(\rho,t)=\begin{cases} u(\rho,t) & \text{for} \quad 0<\rho<1 \\ u(-\rho,t) & \text{for} \quad -1<\rho<0. \end{cases}$$

Verify that $v_\rho(0,t)=0$, and that v solves

$$\begin{aligned} & v_{tt}-v_{\rho\rho}-\frac{2}{\rho}v_\rho=0 && \text{in } (-1,1)\times\mathbb{R} \\ & v(-1,t)=v(1,t)=0 && \text{for } t\in\mathbb{R} \\ & v(\rho,0)=0,\ v_t(\rho,0)=\cos\frac{\pi}{2}\rho && \text{for } \rho\in(-1,1). \end{aligned} \tag{6.3c}$$

(iii) Solve (6.3c) and verify that

$$v(0,t)=t\cos\frac{\pi}{2}t \quad \text{for } |t|<1.$$

6.6. Now consider (6.2c) in the whole of $\mathbb{R}^3$, i.e.,

$$\begin{aligned} u_{tt}-\Delta u=0 & \quad \text{in } \mathbb{R}^3\times\mathbb{R} \\ u(x,0)=0 & \quad \text{in } \mathbb{R}^3 \\ u_t(x,0)=\cos\frac{\pi}{2}|x| & \quad \text{in } \mathbb{R}^3. \end{aligned}$$

Write down the explicit solution and check that $u=v$.

6.7. Prove that all radial solutions of the wave equation in $\mathbb{R}^3\times\mathbb{R}$ are of the form

$$u(x,t)=\frac{F(|x|-ct)+G(|x|+ct)}{|x|}$$

for functions $F(\cdot)$ and $G(\cdot)$ of class $C^2(\mathbb{R})$.

6.8. Write down the explicit solution of

$$\Box u=0 \text{ in } \mathbb{R}^3\times\mathbb{R}, \quad u(\cdot.0)=0,\ u_t(\cdot,0)=\psi, \tag{6.4c}$$

where ψ is radial.

6.9. In the case $c=1$ and

$$\psi(|x|)=\begin{cases} 1 & \text{for } |x|<1 \\ 0 & \text{for } |x|\ge 1 \end{cases}$$

prove that the unique solution of (6.4c) is

$$u(x,t)=\begin{cases} t & \text{if } 0<|x|<1-t, & t\in(0,1] \\ \dfrac{1-(|x|-t)^2}{4|x|} & \text{if } |1-t|<|x|<t+1, & t\ge 0 \\ 0 & \text{if } 0\le|x|\le t+1. & \end{cases}$$

In particular, the solution is discontinuous at $(0,1)$.

6.3c Solving the Cauchy Problem by Fourier Transform

We will use here notions and techniques introduced in Sections 2.6c–2.9c of the Complements of Chapter 5. Consider the Cauchy problem

$$v_{tt} - \Delta v = 0 \text{ in } \mathbb{R}^N \times \mathbb{R}^+, \quad v(\cdot, 0) = 0, \quad v_t(\cdot, 0) = \psi. \tag{6.5c}$$

We assume that ψ is in the class of the rapidly decreasing functions or the Schwartz class $\mathcal{S}_N$, and seek a solution $v(\cdot, t)$ in the same class with respect to the space variables. Taking the Fourier transform of the PDE in (6.5c) with respect to the space variables gives

$$\hat{v}_{tt} + |y|^2 \hat{v} = 0, \quad \hat{v}(y, 0) = 0, \quad \hat{v}_t(y, 0) = \hat{\psi}.$$

This can be solved explicitly to give

$$\hat{v}(y, t) = \frac{\sin |y|t}{|y|} \hat{\psi}(y).$$

Prove that

$$\sup_{y \in \mathbb{R}^N} \left| D^\alpha \frac{\sin |y|t}{|y|} \right| < \infty, \qquad \text{for every multi-index } \alpha.$$

Deduce that $\hat{v}(\cdot, t) \in \mathcal{S}_N$. By the inversion formula obtain the solution of (6.5c) in the form

$$v(x, t) = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} \frac{\sin |y|t}{|y|} \hat{\psi}(y) e^{i\langle x, y \rangle} dy. \tag{6.6c}$$

Now consider the general Cauchy problem

$$u_{tt} - \Delta u = 0 \text{ in } \mathbb{R}^N \times \mathbb{R}^+, \quad u(\cdot, 0) = \varphi, \quad u_t(\cdot, 0) = \psi \tag{6.7c}$$

with both φ and ψ in $\mathcal{S}_N$. Verify that if w solves (6.5c) with the initial condition $w_t(\cdot, 0) = \varphi$, then the solution of (6.7c) is given by

$$u(x, t) = v(x, t) + w_t(x, t).$$

It follows from (6.6c) that the solution of (6.7c) can be represented by the formula

$$u(x, t) = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} \left(\frac{\sin |y|t}{|y|} \hat{\psi}(y) + \cos |y|t \, \hat{\varphi}(y) \right) e^{i\langle x, y \rangle} dy.$$

6.3.1c The 1-Dimensional Case

If $N = 1$, by the inversion formula

$$\begin{aligned}u(x,t) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{e^{iy(x+t)} + e^{iy(x-t)}}{2} \hat{\varphi}(y)dy \\ &\quad + \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{e^{iy(x+t)} - e^{iy(x-t)}}{2iy} \hat{\psi}(y)dy \\ &= \frac{1}{2}[\varphi(x+t) + \varphi(x-t)] + \frac{1}{2} \int_{x-t}^{x+t} \left(\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\eta y} \hat{\psi}(y)dy \right) d\eta \\ &= \frac{1}{2}[\varphi(x+t) + \varphi(x-t)] + \frac{1}{2} \int_{x-t}^{x+t} \psi(y)dy.\end{aligned}$$

6.3.2c The Case $N = 3$

We refer to the representation formula (6.6c). Let $N = 3$ and prove the formula

$$\frac{\sin |y|t}{|y|} = \frac{1}{4\pi t} \int_{|\eta|=t} e^{i\langle \eta, y\rangle} d\sigma = \frac{t}{4\pi} \int_{|\eta|=1} e^{it\langle \eta, y\rangle} d\sigma.$$

Hint: If T is a rotation matrix in $\mathbb{R}^3$, then

$$\int_{|\eta|=t} e^{i\langle \eta, y\rangle} d\sigma(\eta) = \int_{|\eta|=t} e^{i\langle \eta, Ty\rangle} d\sigma(y).$$

Next, choose T such that $Ty = |y|(0,0,1)$ and compute the integral by introducing polar coordinates. By the Fubini theorem and the inversion formula, one computes from (6.6c)

$$\begin{aligned}v(x,t) &= \frac{1}{4\pi t} \int_{\mathbb{R}^3} \left[\frac{1}{(2\pi)^{3/2}} \int_{|\eta|=t} e^{i\langle y, \eta\rangle} d\sigma \right] \hat{\psi}(y) e^{i\langle x, y\rangle} dy \\ &= \frac{1}{4\pi t} \int_{|\eta|=t} \psi(x+\eta) d\sigma = \frac{1}{4\pi t} \int\limits_{|x-y|=t} \psi(y) d\sigma(y).\end{aligned}$$

The solution of (6.7c) is given by

$$u(x,t) = \frac{\partial}{\partial t} \left(\frac{1}{4\pi t} \int_{|x-y|=t} \varphi(y) d\sigma(y) \right) + \frac{1}{4\pi t} \int_{|x-y|=t} \psi(y) d\sigma(y).$$

7c Cauchy Problems in $\mathbb{R}^2$ and the Method of Descent

7.1. Solve the Cauchy problem

$$u_{tt} - \Delta u = 0 \text{ in } \mathbb{R}^2 \times \mathbb{R}, \quad u(x,0) = |x|^2, \quad u_t(x,0) = 1 \text{ in } \mathbb{R}^2.$$

7.2. Write down the explicit solution of

$$u_{tt} - \Delta u = 0 \text{ in } \mathbb{R}^2 \times \mathbb{R}, \quad u(\cdot,0) = p(\cdot), \quad u_t(\cdot,0) = 0$$

where p is a homogeneous polynomial of degree 10 in x_1 and x_2. Write down $u(0,0,t)$ in the case $p(x_1,x_2) = (x_1^2 + x_2^2)^5$.

7.3. Solve the problem

$$u_{tt} - \Delta u = 0 \text{ in } \mathbb{R}^2 \times \mathbb{R}, \quad u(x,0) = 0, \quad u_t(x,0) = \sin(x_1 + x_2).$$

Find the solution in the form $u(x,t) = a(t)\sin(x_1 + x_2)$.

7.4. Recover the d'Alembert formula (2.2) from the Poisson formula (7.3) and the method of descent.

7.1c The Cauchy Problem for $N = 4, 5$

In the Darboux formula (5.2) take $N = 5$ and let

$$w(x,\rho,t) = \rho^2 \frac{\partial}{\partial\rho} M(u;x,\rho,t) + 3\rho M(u;x,\rho,t).$$

Verify that $w_{tt} = c^2 w_{\rho\rho}$ and solve (5.1) for $N = 5$. Prove that the solution of (5.1) is given by

$$\begin{aligned} u(x,t) =& \left(\frac{1}{3}t^2 \frac{\partial}{\partial t} + t\right)\left(\frac{1}{\omega_5 (ct)^4} \int_{|x-y|=ct} \psi(y) d\sigma\right) \\ &+ \frac{\partial}{\partial t}\left(\frac{1}{3}t^2 \frac{\partial}{\partial t} + t\right)\left(\frac{1}{\omega_5 (ct)^4} \int_{|x-y|=ct} \varphi(y) d\sigma\right). \end{aligned}$$

Use the previous result and the method of descent to solve (5.1) for $N = 4$.

8c Inhomogeneous Cauchy Problems

8.1c The Wave Equation for the N and $(N+1)$-Laplacean

Denote by Δ_N the Laplacean with respect to the N variables $x = (x_1, \dots, x_N)$ and by Δ_{N+1} the Laplacean with respect to the $(N+1)$ variables (x, x_{N+1}). Let $k \in \mathbb{R}$ be a given constant and let $u \in C^2(\mathbb{R}^N \times \mathbb{R})$ be a solution of

$$u_{tt} = c^2 \Delta_N u - k^2 u \quad \text{in } \mathbb{R}^N \times \mathbb{R}.$$

Then, for any two given constants A and B, the function

$$v(x, x_{N+1}, t) = \left[A\cos\left(\frac{k}{c}x_{N+1}\right) + B\sin\left(\frac{k}{c}x_{N+1}\right)\right]u(x,t)$$

solves

$$v_{tt} = c^2 \Delta_{N+1} v \quad \text{in } \mathbb{R}^{N+1} \times \mathbb{R}. \tag{8.1c}$$

Similarly, if u solves

$$u_{tt} = c^2 \Delta_N u + k^2 u \quad \text{in } \mathbb{R}^N \times \mathbb{R}$$

then

$$v(x, x_{N+1}, t) = \left[A \cosh\left(\frac{k}{c} x_{N+1}\right) + B \sinh\left(\frac{k}{c} x_{N+1}\right)\right] u(x,t)$$

solves (8.1c). Use these remarks and the method of descent to solve the Cauchy problems

$$\begin{aligned} u_{tt} &= c^2 \Delta_2 u \pm \lambda^2 u && \text{in } \mathbb{R}^2 \times \mathbb{R} \\ u(\cdot, 0) &= \varphi \in C^3(\mathbb{R}^2) && \text{in } \mathbb{R}^2 \\ u_t(\cdot, 0) &= \psi \in C^2(\mathbb{R}^2) && \text{in } \mathbb{R}^2, \end{aligned}$$

where $\lambda \in \mathbb{R}$ is a given constant.

8.1.1c The Telegraph Equation

Solve the Cauchy problems

$$\begin{aligned} u_{tt} &= u_{xx} \pm \lambda^2 u && \text{in } \mathbb{R} \times \mathbb{R} \\ u(\cdot, 0) &= \varphi \in C^3(\mathbb{R}) && \text{in } \mathbb{R} \\ u_t(\cdot, 0) &= \psi \in C^2(\mathbb{R}) && \text{in } \mathbb{R}. \end{aligned} \tag{T$_\pm$}$$

The equation $(\mathrm{T})_-$ is called the telegraph equation. Set

$$B^+(s) = \frac{2}{\pi} \int_0^{\pi/2} \cos(s \sin\theta) d\theta, \quad B^-(s) = \frac{2}{\pi} \int_0^{\pi/2} \cosh(s \sin\theta) d\theta$$

where s is a real parameter. Prove that the solutions $u_\pm$ of $(\mathrm{T})_\pm$ are

$$\begin{aligned} u_\pm(x,t) &= \frac{1}{2} \int_{x-t}^{x+t} \psi(s) B^\pm\left(\lambda\sqrt{t^2 - (x-s)^2}\right) ds \\ &\quad + \frac{\partial}{\partial t}\left[\frac{1}{2} \int_{x-t}^{x+t} \varphi(s) B^\pm\left(\lambda\sqrt{t^2 - (x-s)^2}\right) ds\right]. \end{aligned}$$

The functions $B^\pm(\cdot)$ are the Bessel functions of order zero [19].

8.2c Miscellaneous Problems

8.1. Let a, b, c be given constants. Solve

$$\begin{gathered} u_{tt} = u_{xx} + a u_x + b u_t + c u \quad \text{in } \mathbb{R} \times \mathbb{R} \\ u(\cdot, 0) = \varphi \in C^3(\mathbb{R}), \quad u_t(\cdot, 0) = \psi \in C^2(\mathbb{R}). \end{gathered}$$

Hint: Reduce the problem to the previous one by exponential shifts in x and t. The solution is

$$2u(x,t) = e^{\frac{bt-ax}{2}} \int_{x-t}^{x+t} e^{\frac{as}{2}} \left(\psi(s) - \frac{b}{2}\varphi(s) \right) A\left(\lambda \sqrt{t^2 - (x-s)^2} \right) ds$$
$$+ e^{\frac{bt-ax}{2}} \frac{\partial}{\partial t} \left(\int_{x-t}^{x+t} e^{\frac{as}{2}} \psi(s) A\left(\lambda\sqrt{t^2-(x-s)^2} \right) ds \right),$$

where

$$\text{if } a^2 \geq b^2 + 4c, \quad \lambda = \sqrt{\frac{a^2-b^2}{4} - c}, \quad A(s) = B^-(s)$$

$$\text{if } a^2 \leq b^2 + 4c, \quad \lambda = \sqrt{c - \frac{a^2-b^2}{4}}, \quad A(s) = B^+(s).$$

8.2. Solve

$$u_{tt} = u_{xx} + au_x + bu_t + cu + f \text{ in } \mathbb{R} \times \mathbb{R}$$

$$u(\cdot, 0) = \varphi \in C^3(\mathbb{R}), \quad u_t(\cdot, 0) = \psi \in C^2(\mathbb{R}), \quad f \in C^2(\mathbb{R}^2).$$

8.3. Let β be a given constant. Solve the boundary value problem

$$u_{tt} - u_{xx} = -\beta u_t \text{ in } (0,1) \times \mathbb{R}^+, \quad u(0,\cdot) = u(1,\cdot) = 0$$

$$u(\cdot,0) = \varphi \in C^2(0,1) \cap C[0,1], \quad u_t(\cdot,0) = 0.$$

8.4. Solve the problem

$$\begin{aligned} u_{tt} - u_{xx} - \frac{2}{x}u_x &= 0 && \text{in } (-1,1) \times \mathbb{R} \\ u(-1,\cdot) = u(1,\cdot) &= 0 && \text{in } \mathbb{R} \\ u(x,0) &= 0 && \text{for } x \in (-1,1) \\ u_t(x,0) &= \cos\frac{\pi}{2}x && \text{for } x \in (-1,1). \end{aligned}$$

Verify that the solution is symmetric about the origin, that $u_x(0,t) = 0$ for all t, and moreover

$$u(0,t) = t\cos\frac{\pi}{2}t \quad \text{for } |t| < 1.$$

Hint: $\Box(xu) = 0$.

8.5. Using the Fourier Transform, solve the problem

$$\begin{aligned} u_{tt} - u_{xx} &= -a^2\, u && \text{in } \mathbb{R} \times \mathbb{R}^+ \\ \lim_{|x| \to +\infty} u(x,t) &= 0 && \text{for } t > 0 \\ u(x,0) &= e^{-b|x|}, \quad 0 < b < a && \text{for } x \in \mathbb{R} \\ u_t(x,0) &= 0 && \text{for } x \in \mathbb{R}. \end{aligned}$$

Hint: After determining the expression of $\hat{u}(y,t)$, use the Inversion Theorem for the Fourier Transform and leave $u(x,t)$ written as an integral.

8.6. Solve explicitly

$$\begin{aligned}
u_{tt}-(u_{xx}+u_{yy})&=0 && \text{in } \left(-\frac{\pi}{2},\frac{\pi}{2}\right)\times\mathbb{R}\times\mathbb{R}^+\\
u\left(\pm\frac{\pi}{2},y,t\right)&=0 && \text{for } t>0 \text{ and } y\in\mathbb{R}\\
u(x,y,0)&=\cos x\cos y && \text{in } \left(-\frac{\pi}{2},\frac{\pi}{2}\right)\times\mathbb{R}\\
u_t(x,y,0)&=0 && \text{in } \left(-\frac{\pi}{2},\frac{\pi}{2}\right)\times\mathbb{R}
\end{aligned}$$

Solution: Direct inspection from the data suggests that the solution might be

$$u(x,y,t)=\cos x\cos y\cos\sqrt{2}\,t.$$

The point is to arrive at such a solution in a constructive way. Periodic extension of $\cos x$ reduces the problem to a Cauchy problem set in $\mathbb{R}^2\times\mathbb{R}^+$ with the same initial data. The solution of such a Cauchy problem is

$$\begin{aligned}
2\pi u(x,y,t)&=\frac{\partial}{\partial t}\int_{D_t(x,y)}\frac{\cos\xi\cos\eta}{\sqrt{t^2-(x-\xi)^2-(y-\eta)^2}}d\xi d\eta\\
&=\frac{\partial}{\partial t}\int_{D_t(0,0)}\frac{\cos(x-u)\cos(y-v)}{\sqrt{t^2-u^2-v^2}}dudv.
\end{aligned}$$

Therefore, the expression of $u(x,y,t)$ hinges upon computing the integral

$$I=\int_{D_t(0,0)}\frac{\cos(x-u)\cos(y-v)}{\sqrt{t^2-u^2-v^2}}dudv.$$

The numerator of the integrand separates into

$$\begin{aligned}
\cos(x-u)\cos(y-v)&=(\cos x\cos u+\sin x\sin u)(\cos y\cos v+\sin y\sin v)\\
&=\cos x\cos y\ \cos u\cos v+\cos x\sin y\ \cos u\sin v\\
&\quad+\sin x\cos y\ \sin u\cos v+\sin x\sin y\ \sin u\sin v.
\end{aligned}$$

Therefore, I is the sum of the four integrals

$$\begin{aligned}
I_1&=\cos x\cos y\int_{D_t(0,0)}\frac{\cos u\cos v}{\sqrt{t^2-(u^2+v^2)}}dudv\\
I_2&=\cos x\sin y\int_{D_t(0,0)}\frac{\cos u\sin v}{\sqrt{t^2-(u^2+v^2)}}dudv\\
I_3&=\sin x\cos y\int_{D_t(0,0)}\frac{\sin u\cos v}{\sqrt{t^2-(u^2+v^2)}}dudv\\
I_4&=\sin x\sin y\int_{D_t(0,0)}\frac{\sin u\sin v}{\sqrt{t^2-(u^2+v^2)}}dudv.
\end{aligned}$$

We claim that $I_2=I_3=I_4=0$. To see this, disregard the (x,y)-dependent coefficients and call by I_j' the resulting integrals for $j=2,3,4$. In all of these effect a polar change of variables $u=r\cos\theta$ and $v=r\sin\theta$, to get

$$
\begin{aligned}
I_2' &= \int_0^t \frac{r}{\sqrt{t^2 - r^2}} \Big(\int_0^{2\pi} \cos(r\cos\theta)\sin(r\sin\theta)d\theta \Big) dr \\
I_3' &= \int_0^t \frac{r}{\sqrt{t^2 - r^2}} \Big(\int_0^{2\pi} \sin(r\cos\theta)\cos(r\sin\theta)d\theta \Big) dr \\
I_4' &= \int_0^t \frac{r}{\sqrt{t^2 - r^2}} \Big(\int_0^{2\pi} \sin(r\cos\theta)\sin(r\sin\theta)d\theta \Big) dr.
\end{aligned}
$$

Now, $I_2' = I_4' = 0$ since the integrand is odd. The integrand of I_3' is even and hence

$$
\begin{aligned}
\frac{1}{2}\Big(\int_0^{2\pi} \sin(r\cos\theta)\cos(r\sin\theta)d\theta \Big) &= \int_0^{\frac{\pi}{2}} \sin(r\cos\theta)\cos(r\sin\theta)d\theta \\
&\quad + \int_{\frac{\pi}{2}}^{\pi} \sin(r\cos\theta)\cos(r\sin\theta)d\theta \\
&= I_{3,1}' + I_{3,2}'.
\end{aligned}
$$

In $I_{3,2}$ effect the change of variables $\theta \to \frac{1}{2}\pi + \alpha$ to find

$$
I_{3,2}' = -I_{3,1}' \quad \Longrightarrow \quad I_3' = 0.
$$

Thus, the only nonzero integral is I_1'. To compute it recall that $I_4' = 0$ and write

$$
\begin{aligned}
I_1' &= I_1' + I_4' \\
&= \int_{D_t(0,0)} \frac{\cos u \cos v + \sin u \sin v}{\sqrt{t^2 - (u^2 + v^2)}} dudv \\
&= \int_{D_t(0,0)} \frac{\cos(u - v)}{\sqrt{t^2 - (u^2 + v^2)}} dudv.
\end{aligned}
$$

In this last integral effect the change of variables

$$
u + v = \sqrt{2}\xi, \qquad u - v = \sqrt{2}\eta.
$$

This is a rotation of (u, v) of $\frac{1}{4}\pi$ about the origin. Therefore,

$$
\{u^2 + v^2 < t^2\} \longrightarrow \{\xi^2 + \eta^2 < t^2\} \quad \text{and} \quad dudv = d\xi d\eta
$$

since a rotation does not deform the areas. Thus,

$$
I_1' = \int_{D_t(0,0)} \frac{\cos\sqrt{2}\eta}{\sqrt{t^2 - \eta^2 - \xi^2}} d\xi d\eta.
$$

Since the integrand is even in ξ and η

$$
\begin{aligned}
I_1' &= 4\int_0^t \cos\sqrt{2}\eta\Big(\int_0^{\sqrt{t^2-\eta^2}} \frac{d\xi}{\sqrt{(t^2-\eta^2)-\xi^2}}\Big)d\eta \\
&= 4\int_0^t \cos\sqrt{2}\eta\Big(\int_0^{\frac{\pi}{2}} d\alpha\Big)d\eta \\
&= 2\pi\int_0^t \cos\sqrt{2}\eta d\eta = \frac{2\pi}{\sqrt{2}}\sin\sqrt{2}t.
\end{aligned}
$$

Combining these calculations, the solution has the form

$$u(x,y,t) = \cos x\cos y\frac{d}{dt}\frac{\sin\sqrt{2}t}{\sqrt{2}}.$$

8.7. Solve the problem

$$u_{xy} = xy \ \text{ in } \ y > x, \quad u(s,s) = 0, \ u(s,-s) = s^3$$

$$\nabla u(s,s)\cdot(-1,1) = \begin{cases} \sqrt{2}s(1-s) & \text{if } \ s\in(0,1) \\ 0 & \text{otherwise.} \end{cases}$$

10c The Reflection Technique

10.1. Find the solution of the boundary value problem

$$
\begin{aligned}
u_{tt} - u_{xx} &= 0 && \text{in } \ \mathbb{R}^+\times\mathbb{R}^+ \\
u(0,\cdot) &= h && \text{for } \ t > 0 \\
u(\cdot,0) &= \varphi && \text{for } \ x > 0 \\
u_t(\cdot,0) &= \psi && \text{for } \ x > 0
\end{aligned}
$$

where the data h, φ, ψ are smooth and satisfy the compatibility conditions

$$h(0) = \varphi(0), \quad h'(0) = \psi(0), \quad h''(0) = c^2\varphi''(0).$$

10.2. Transform the problem

$$
\begin{aligned}
u_{tt} - u_{xx} &= 0, \quad \text{in } \ [x>0]\times[t>0] \\
u(0,t) &= 0 \quad \text{for } \ t > 0 \\
u(x,0) &= \begin{cases} 0 & \text{in } \ [0\le x\le 1]\cup[x\ge 2] \\ (2-x)(x-1) & \text{in } \ [1\le x\le 2]. \end{cases} \\
u_t(x,0) &= 0 \quad \text{in } \ x \ge 0
\end{aligned}
$$

into another one in the whole of $\mathbb{R}$. Find the times t such that $u(3,t)\neq 0$. Find the extrema of $x\to u(x,10)$.

11c Problems in Bounded Domains

11.1c Uniqueness

Let $E \subset \mathbb{R}^N$ be bounded, open, and with boundary ∂E of class C^1. Prove that there exists at most one solution of the boundary value problem

$$\begin{aligned} u_{tt} - \Delta u + k(x,t)u &= 0, \quad \text{in } \ E \times \mathbb{R} \\ \frac{\partial}{\partial \mathbf{n}} u + q(x,t)u &= 0, \quad \text{on } \ \partial E \times \mathbb{R} \\ u(\cdot,0) &= \varphi, \quad \text{in } \ E \\ u_t(\cdot,0) &= \psi, \quad \text{in } \ E \end{aligned}$$

where φ and ψ are smooth and $k(\cdot,\cdot)$ and $q(\cdot,\cdot)$ are bounded and non-negative in their domain of definition.

Hint: Multiply the PDE by u_t and integrate by parts over $E \times (0,t)$ to get

$$\mathcal{E}(t) + 2\int_0^t \int_E kuu_t \, dxd\tau + 2\int_0^t \int_{\partial E} quu_t \, d\sigma d\tau = \mathcal{E}_o, \tag{11.1c}$$

where

$$\mathcal{E}(t) = \int_E [u_t^2 + |\nabla u|^2](\cdot,t)dx \quad \text{and} \quad \mathcal{E}_o = \int_E (\psi^2 + |\nabla\varphi|^2)dx. \tag{11.2c}$$

Next, multiply the PDE by u and integrate by parts over $E \times (0,t)$ to get

$$\begin{aligned} \int_E uu_t(\cdot,t)dx &+ \int_0^t \int_E \big[|\nabla u|^2] - u_t^2\big]dxd\tau \\ &+ \int_0^t \int_E ku^2 dxd\tau + \int_0^t \int_{\partial E} qu^2 d\sigma d\tau = \int_E \varphi\psi dx. \end{aligned} \tag{11.3c}$$

Integrate (11.1c) in $d\tau$ over $(0,t)$, multiply by 2 and add it to (11.3c) to get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_E u^2(\cdot,t)\,dx &+ \int_0^t \mathcal{E}(\tau)d\tau + \int_0^t \int_E ku^2 dxd\tau + \int_0^t \int_{\partial E} qu^2 \, d\sigma d\tau \\ &= 2t\mathcal{E}_o + \mathcal{E}_1 - 4\int_0^t \int_0^\tau \int_E kuu_\tau \, dxdsd\tau - 4\int_0^t \int_0^\tau \int_{\partial E} quu_\tau \, d\sigma dsd\tau. \end{aligned} \tag{11.4c}$$

By the Cauchy–Schwarz inequality and the assumptions on k

$$\left| 4\int_0^t \int_0^\tau \int_E kuu_t \, dxdsd\tau \right| \le 2\gamma t \int_0^t \int_E ku^2 \, dxd\tau + 2t \int_0^t \mathcal{E}(\tau)\,d\tau,$$

where $\gamma = \sup k$. Stipulate taking t so small that

$$1 - 2t\max\{1;\gamma\} \ge \tfrac{1}{2}. \tag{11.5c}$$

Then, inserting this estimation in (11.4c) and integrating in $d\tau$ over $(0,t)$ gives

$$\begin{aligned}\int_E u^2(\cdot,t)\,dx + \int_0^t\int_0^\tau \mathcal{E}(s)\,dsd\tau + \int_0^t\int_0^\tau\int_E ku^2\,dxdsd\tau&\\ +2\int_0^t\int_0^\tau\int_{\partial E} qu^2 d\sigma dsd\tau&\\ = 4t^2\mathcal{E}_o + 2t\mathcal{E}_1 + \int_E \varphi^2 dx - 8\int_0^t\int_0^\tau\int_0^s\int_{\partial E} quu_\tau\,d\sigma d\ell dsd\tau&\end{aligned}\tag{11.6c}$$

To establish uniqueness assume $\varphi=\psi=0$ and infer from (11.6c)

$$\begin{aligned}\int_E u^2(\cdot,t)dx + 2\int_0^t\int_0^\tau\int_{\partial E} qu^2\,d\sigma dsd\tau&\\ \le -8\int_0^t\int_0^\tau\int_0^s\int_{\partial E} quu_\tau\,d\sigma d\ell dsd\tau&\end{aligned}\tag{11.7c}$$

Formulate assumptions for uniqueness to hold. Possible assumptions are:

$$q_t \le Cq^m \qquad \text{for given non-negative constants } C \text{ and } m.$$

11.2c Separation of Variables

11.2. Let $R=[0,1]\times[0,\pi]$, and consider the problem

$$\begin{aligned}u_{tt}-\Delta u &= 0, \quad \text{in } R\times\mathbb{R}^+\\ u(\cdot,t)\big|_{\partial R} &= 0\\ u(x,0) &= 0\\ u_t(x,0) &= f(x_1)g(x_2),\end{aligned}$$

where $f(0)=f(1)=g(0)=g(\pi)=0$. Solve by the separation of variables. In particular, write down the explicit solution for the data

$$f(x)=\begin{cases} x & \text{if } 0\le x\le \frac34\\ -3(x-1) & \text{if } \frac34 < x\le 1,\end{cases}\qquad g(y)=\sum_{n=1}^\infty \sin ny.$$

11.3. Solve (11.1) for $E=[0,1]^3$ in terms of the eigenvalues and eigenfunctions (λ_i^2, v_i) of the Laplacean in E

$$\Delta v_i = \lambda_i^2 v_i. \tag{11.8c}$$

11.3-(i). Solve (11.8c) by the separation of variables. Denote by x,y,z the coordinates in $\mathbb{R}^3$ and seek a solution of the form $v_i = X_i(x)W_i(y,z)$. Then,

$$X_i'' = -\xi_i^2\,X_i \qquad \text{and} \qquad \Delta_{(y,z)}W_i = -\nu_i^2\,W_i$$

where ξ_i and ν_i are positive numbers linked by $\xi_i^2+\nu_i^2=\lambda_i^2$.

11.3-(ii). Find W_i of the form $W_i(y,z) = Y_i(y)Z_i(z)$. Then,

$$Y_i'' = -\eta_i^2 Y_i \quad \text{and} \quad Z_i'' = -\zeta_i^2 Z_i,$$

where η_i and ζ_i are positive numbers linked by $\eta_i^2 + \zeta_i^2 = \nu_i^2$.

11.3-(iii). Verify that for all triples (m,n,ℓ) of positive integers, the pairs

$$\lambda_i^2 = \pi^2\left(m^2 + n^2 + \ell^2\right), \quad v_i = \sin \pi m \sin \pi n \sin \pi \ell$$

are eigenvalues and eigenfunctions of (11.8c). Prove that these are all the eigenvalues and eigenfunctions of (11.8c).

12c Hyperbolic Equations in Two Variables

12.1c The General Telegraph Equation

Let $u(s,t)$ be the intensity of electric current in a conductor, considered as a function of t and the distance s from a fixed point of the conductor. Let α denote the capacity and β the induction coefficients. Then,

$$u_{tt} - c^2 u_{ss} + (\alpha+\beta)u_t + \alpha\beta u = 0 \;\text{ in }\; \mathbb{R}\times\mathbb{R}.$$

Setting

$$e^{1/2(\alpha+\beta)t}u(s,t) = v(x,y), \qquad x = s+ct,\; y = s-ct$$

transforms the equation into

$$v_{xy} + \lambda v = 0, \qquad \lambda = \left(\frac{\alpha-\beta}{4c}\right)^2.$$

14c Goursat Problems

14.1. Prove that (14.5) is the unique solution of (14.1)

14.2. Prove that $(14.1)'$ has a unique solution and give a representation formula in terms of $\mathcal{R}(\cdot;\cdot)$.

14.3. Give a representation formula for the characteristic Goursat problems (13.5) and (13.6) in terms of $\mathcal{R}(\cdot;\cdot)$.

14.4. Use the method of successive approximations of Section 13, to find the Riemann function for the operator

$$\mathcal{L}(v) = \frac{\partial^2}{\partial x_1 \partial x_2}v + \mathbf{b}\cdot\nabla v + cv,$$

where b_1, b_2, and c are constants.

14.1c The Riemann Function and the Fundamental Solution of the Heat Equation

The fundamental solution of the heat equation $u_y = u_{xx}$ can be recovered as the limit, as $\varepsilon \to 0$, of the Riemann function for the hyperbolic equation ([110], 145–147).

$$u_{xx} + \varepsilon u_{xy} - u_y = 0.$$

The change of variables $\xi = y$ and $\eta = x - \frac{1}{\varepsilon} y$ transforms this equation into

$$\varepsilon u_{\xi\eta} + \frac{1}{\varepsilon} u_\eta - u_\xi = 0.$$

Using the previous problem, show that the Riemann function, with pole at the origin, for such an equation is given by

$$\mathcal{R}\big[(\xi,\eta);(0,0)\big] = e^{\frac{\xi}{\varepsilon^2} - \frac{\eta}{\varepsilon}} J_o\left(2\sqrt{\frac{\xi\eta}{\varepsilon^3}}\right)$$

where $J_o(\cdot)$ is the Bessel function of order zero. Returning to the original coordinates

$$\mathcal{R}\big[(x,y);(0,0)\big] = e^{\frac{2y}{\varepsilon^2} - \frac{x}{\varepsilon}} J_o\left(\frac{2}{\varepsilon^2}\sqrt{y(\varepsilon x - y)}\right).$$

Let $\varepsilon \to 0$ to recover the fundamental solution of the heat equation in one space dimension with pole at the origin. For the asymptotic behavior of $J_o(s)$ as $s \to \infty$, see Bowman [19].

7

QUASI-LINEAR EQUATIONS OF FIRST ORDER

1 Quasi-Linear Equations

A first-order quasi-linear PDE is an expression of the form

$$a_i(x, u(x))u_{x_i} = a_o(x, u(x)) \tag{1.1}$$

where x ranges over a region $E \subset \mathbb{R}^N$, the function u is in $C^1(E)$, and $(x, z) \to a_i(x, z)$ are given smooth functions of their arguments. Introduce the vector $\mathbf{a} = (a_1, \dots, a_N)$, and rewrite (1.1) as

$$(\mathbf{a}, a_o) \cdot (\nabla u, -1) = 0. \tag*{(1.1)$'$}$$

Thus if u is a solution of (1.1), the vector $(\mathbf{a}, a_o)$ is tangent to the graph of u at each of its points. For this reason, the graph of u is called an *integral surface* for (1.1). More generally, an N-dimensional surface Σ of class C^1 is an integral surface for (1.1) if for every point $P = (x, z) \in \Sigma$, the vector $\big(\mathbf{a}(P), a_o(P)\big)$ is tangent to Σ at P. The curves

$$\begin{gathered}(-\delta, \delta) \ni t \to \begin{cases} \dot{x}_i(t) = a_i(x(t), z(t)), \quad i = 1, \dots, N \\ \dot{z}(t) = a_o(x(t), z(t)) \end{cases} \\ (x(0), z(0)) = (x_o, z_o) \in E \times \mathbb{R} \end{gathered} \tag{1.2}$$

defined for some $\delta > 0$, are the characteristics associated to (1.1), originating at (x_o, z_o). The solution of (1.2) is local in t, and the number δ that defines the interval of existence might depend upon (x_o, z_o). For simplicity we assume that there exists some $\delta > 0$ such that the range of the parameter t is $(-\delta, \delta)$, for all $(x_o, z_o) \in E \times \mathbb{R}$.

Proposition 1.1 *An N-dimensional hypersurface Σ is an integral surface for (1.1) if and only if it is the union of characteristics.*

Proof. Up to possibly relabeling the coordinate variables and the components a_i, represent Σ, locally as $z = u(x)$ for some u of class C^1. For $(x_o, z_o) \in \Sigma$, let $t \to (x(t), z(t))$ be the characteristic trough (x_o, z_o), set

© Springer Nature Switzerland AG 2023

E. DiBenedetto, U. Gianazza, *Partial Differential Equations*,
Cornerstones, https://doi.org/10.1007/978-3-031-46618-2_8

$$w(t) = z(t) - u(x(t))$$

and compute

$$\begin{aligned}\dot{w} = \dot{z} - u_{x_i}\dot{x}_i &= a_o(x,z) - a_i(x,z)u_{x_i}(x)\\ &= a_o(x,u(x)+w) - a_i(x,u(x)+w)u_{x_i}(x).\end{aligned}$$

Since $(x_o, z_o) \in \Sigma$, $w(0) = 0$. Therefore w satisfies the initial value problem

$$\begin{aligned}\dot{w} &= a_o(x,u(x)+w) - a_i(x,u(x)+w)u_{x_i}(x)\\ w(0) &= 0.\end{aligned} \tag{1.3}$$

This problem has a unique solution. If Σ, represented as $z = u(x)$, is an integral surface, then $w = 0$ is a solution of (1.3) and therefore is its only solution. Thus $z(t) = u(x(t), t)$, and Σ is the union of characteristics. Conversely, if Σ is the union of characteristics, then $w = 0$, and (1.3) implies that Σ is an integral surface. ■

2 The Cauchy Problem

Let $s = (s_1, \dots, s_{N-1})$ be an $(N-1)$-dimensional parameter ranging over the cube $Q_\delta = (-\delta, \delta)^{N-1}$. The Cauchy problem associated with (1.1) consists in assigning an $(N-1)$-dimensional hypersurface $\Gamma \subset \mathbb{R}^{N+1}$ of parametric equations

$$Q_\delta \ni s \to \begin{cases} x = \xi(s) = \big(\xi_1(s), \dots, \xi_N(s)\big) \\ z = \zeta(s), \quad \big(\xi(s), \zeta(s)\big) \in E \times \mathbb{R} \end{cases} \tag{2.1}$$

and seeking a function $u \in C^1(E)$ such that $\zeta(s) = u(\xi(s))$ for $s \in Q_\delta$ and the graph $z = u(x)$ is an integral surface of (1.1).

2.1 The Case of Two Independent Variables

If $N = 2$, then s is a scalar parameter and Γ is a curve in $\mathbb{R}^3$, say for example

$$(-\delta, \delta) \ni s \to \mathbf{r}(s) = (\xi_1, \xi_2, \zeta)(s).$$

Any such a curve is noncharacteristic if the two vectors $(a_1, a_2, \zeta)(\mathbf{r}(s))$ and $(\xi_1', \xi_2', \zeta')(s)$ are not parallel for all $s \in (-\delta, \delta)$. The projection of $s \to \mathbf{r}(s)$ into the plane $[z = 0]$ is the planar curve

$$(-\delta, \delta) \ni s \to \mathbf{r}_o(s) = (\xi_1, \xi_2)(s)$$

of tangent vector $(\xi_1', \xi_2')(s)$. The projections of the characteristics through $\mathbf{r}(s)$ into the plane $[z = 0]$ are called *characteristic projections*, and have tangent vector $(a_1, a_2)(\mathbf{r}(s))$. We impose on $s \to \mathbf{r}(s)$ that its projection into the plane $[z = 0]$ be nowhere parallel to the characteristic projections, that is, the two vectors $(a_1, a_2)(\mathbf{r}(s))$ and $(\xi_1', \xi_2')(s)$ are required to be linearly independent for all $s \in (-\delta, \delta)$.

2.2 The Case of N Independent Variables

Returning to Γ as given in (2.1), one may freeze all the components of s but the i^{th}, and consider the map

$$s_i \to (\xi_1, \dots, \xi_N, \zeta)(s_1, \dots, s_i, \dots, s_{N-1}).$$

This is a curve traced on Γ with the tangent vector

$$\left(\frac{\partial \xi}{\partial s_i}, \frac{\partial \zeta}{\partial s_i}\right) = \left(\frac{\partial \xi_1}{\partial s_i}, \dots, \frac{\partial \xi_N}{\partial s_i}, \frac{\partial \zeta}{\partial s_i}\right).$$

Introduce the $(N-1) \times (N+1)$ matrix

$$\left(\frac{\partial \xi(s)}{\partial s}\ \frac{\partial \zeta(s)}{\partial s}\right) = \left(\frac{\partial \xi_j(s)}{\partial s_i}\ \frac{\partial \zeta(s)}{\partial s_i}\right).$$

The $(N-1)$-dimensional surface Γ is noncharacteristic if the vectors

$$(\mathbf{a}, a_o)(\xi(s), \zeta(s)), \qquad \left(\frac{\partial \xi}{\partial s_i}, \frac{\partial \zeta}{\partial s_i}\right)(s)$$

are linearly independent for all $s \in Q_\delta$, equivalently, if the $N \times (N+1)$ matrix

$$\begin{pmatrix} \mathbf{a}(\xi(s), \zeta(s)) & a_o(\xi(s), \zeta(s)) \\ \dfrac{\partial \xi(s)}{\partial s} & \dfrac{\partial \zeta(s)}{\partial s} \end{pmatrix} \tag{2.2}$$

has rank N. We impose that the characteristic projections be nowhere parallel to $s \to \xi(s)$, that is

$$\det \begin{pmatrix} \mathbf{a}(\xi(s), \zeta(s)) \\ \dfrac{\partial \xi(s)}{\partial s} \end{pmatrix} \neq 0 \quad \text{for all } \ s \in Q_\delta. \tag{2.3}$$

Thus we require that the first $N \times N$ minor of the matrix (2.2) be nontrivial.

3 Solving the Cauchy Problem

In view of Proposition 1.1, the integral surface Σ is constructed as the union of the characteristics drawn from points $(\xi, \zeta)(s) \in \Gamma$, that is, Σ is the surface

$$(-\delta, \delta) \times Q_\delta \ni (t, s) \to \big(x(t, s), z(t, s)\big)$$

given by

$$\begin{aligned} \frac{d}{dt} x(t, s) &= \mathbf{a}(x(t, s), z(t, s)), \qquad x(0, s) = \xi(s) \\ \frac{d}{dt} z(t, s) &= a_o(x(t, s), z(t, s)), \qquad z(0, s) = \zeta(s). \end{aligned} \tag{3.1}$$

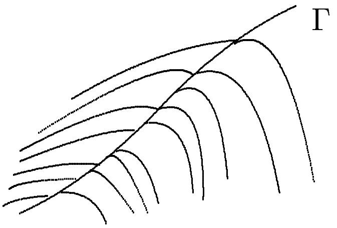

Fig. 3.1

The solutions of (3.1) are local in t. That is, for each $s \in Q_\delta$, (3.1) is solvable for t ranging in some interval $(-t(s), t(s))$. By taking δ smaller if necessary, we may assume that $t(s) = \delta$ for all $s \in Q_\delta$. If the map

$$\mathcal{M} : (-\delta, \delta) \times Q_\delta \ni (t, s) \to x(t, s)$$

is invertible, then there exist functions $\mathbf{S} : E \to Q_\delta$ and $T : E \to (-\delta, \delta)$ such that $s = \mathbf{S}(x)$ and $t = T(x)$ and the unique solution of the Cauchy problem (1.1), (2.1) is given by

$$u(x) = z(t, s) \overset{\text{def}}{=} z(T(x), \mathbf{S}(x)).$$

The invertibility of $\mathcal{M}$ must be realized in particular at $\varGamma$, so that the determinant of the Jacobian matrix

$$J = \begin{pmatrix} \dfrac{dx(0,s)}{dt} & \dfrac{\partial \xi(s)}{\partial s} \end{pmatrix}^t$$

must not vanish, that is, $\varGamma$ cannot contain characteristics. In view of (3.1) for $t = 0$, this is precisely condition (2.3).

The actual computation of the solution involves solving (3.1), calculating the expressions of s and t in terms of x, and substituting them into the expression of $z(t, s)$. The method is best illustrated by some specific examples.

3.1 Constant Coefficients

In (1.1), assume that the coefficients a_i for $i = 0, \dots, N$ are constant. The characteristics are lines of parametric equations

$$x(t) = x_o + \mathbf{a}t, \qquad z(t) = z_o + a_o, t \qquad t \in \mathbb{R}.$$

The first N of these are the characteristic projections. It follows from (1.1)′ that the function $f(x, z) = u(x) - z$ is constant along such lines. If $\varGamma$ is given as in (2.1), the integral surface is

$$\begin{aligned} x(t,s) &= \xi(s) + \mathbf{a}t \\ z(t,s) &= \zeta(s) + a_o t \end{aligned} \qquad s = (s_1, \dots, s_{N-1}). \tag{3.2}$$

The solution $z = u(x)$ is obtained from the last of these upon substitution of s and t calculated from the first N. As an example, let $N = 2$ and let Γ be a curve in the plane $x_2 = 0$, say

$$\Gamma = \{\xi_1(s) = s;\ \xi_2(s) = 0;\ \zeta(s) \in C^1(\mathbb{R})\}.$$

The characteristics are the lines of symmetric equations

$$\frac{x_1 - x_{1,o}}{a_1} = \frac{x_2 - x_{2,o}}{a_2} = \frac{z - z_o}{a_o}, \qquad (x_{1,o}, x_{2,o}, z_o) \in \mathbb{R}^3$$

with the obvious modifications if some of the a_i are zero. The characteristic projections are the lines

$$a_2 x_1 = a_1 x_2 + \text{const}.$$

These are not parallel to the projection of Γ on the plane $z = 0$, provided $a_2 \neq 0$, which we assume. Then (3.2) implies

$$x_2 = a_2 t, \quad \xi_1(s) = s = x_1 - \frac{a_1}{a_2} x_2$$

and the solution is given by

$$u(x_1, x_2) = \zeta\Big(x_1 - \frac{a_1}{a_2} x_2\Big) + \frac{a_o}{a_2} x_2.$$

3.2 Solutions in Implicit Form

Consider the quasi-linear equation

$$\mathbf{a}(u) \cdot \nabla u = 0, \qquad \mathbf{a} = (a_1, a_2, \dots, a_N) \tag{3.3}$$

where $a_i \in C(\mathbb{R})$, and $a_N \neq 0$. The characteristics through points $(x_o, z_o) \in \mathbb{R}^{N+1}$ are the lines

$$x(t) = x_o + \mathbf{a}(z_o) t \quad \text{lying on the hyperplane } z = z_o.$$

A solution u of (3.3) is constant along these lines. Consider the Cauchy problem with data on the hyperplane $x_N = 0$, i.e.,

$$u(x_1, \dots, x_{N-1}, 0) = \zeta(x_1, \dots, x_{N-1}) \in C^1(\mathbb{R}^{N-1}).$$

In such a case the hypersurface Γ is given by

$$\mathbb{R}^{N-1} \ni s \to \begin{cases} x_i = s_i, & i = 1, \dots, N-1 \\ x_N = 0 \\ z(s) = \zeta(s). \end{cases} \tag{3.4}$$

Setting $\bar{x} = (x_1, \dots, x_{N-1})$, and $\bar{\mathbf{a}} = (a_1, \dots, a_{N-1})$, the integral surface associated with (3.3) and Γ as in (3.4), is

$$\begin{aligned}\bar{x}(t,s) &= s + \bar{\mathbf{a}}(\zeta(s))t \\ x_N(t,s) &= a_N(\zeta(s))t \\ z(t,s) &= \zeta(s).\end{aligned} \tag{3.5}$$

From the first two compute

$$s = \bar{x} - \frac{\bar{\mathbf{a}}(\zeta(s))}{a_N(\zeta(s))} x_N.$$

Since a solution u of (3.3) must be constant along $z(s,t) = \zeta(s)$, we have $\zeta(s) = u(x)$. Substitute this in the expression of s, and substitute the resulting s into the third of (3.5). This gives the solution of the Cauchy problem (3.3)–(3.4) in the implicit form

$$u(x) = \zeta\Big(\bar{x} - \frac{\bar{\mathbf{a}}(u(x))}{a_N(u(x))} x_N\Big) \tag{3.6}$$

as long as this defines a function u of class C^1. By the implicit function theorem this is the case in a neighborhood of $x_N = 0$. In general, however, (3.6) fails to give a solution global in x_N.

4 Equations in Divergence Form and Weak Solutions

Let $(x,u) \to \mathbf{F}(x,u)$ be a measurable vector-valued function in $\mathbb{R}^N \times \mathbb{R}$ and consider formally, equations of the type

$$\operatorname{div} \mathbf{F}(x,u) = 0 \qquad \text{in } \mathbb{R}^N. \tag{4.1}$$

The equation (3.3) can be written in this form for $\mathbf{F}(u) = \int^u \mathbf{a}(\sigma)d\sigma$. A measurable function u is a *weak solution* of (4.1) if $\mathbf{F}(\cdot,u) \in [L^1_{\text{loc}}(\mathbb{R}^N)]^N$, and

$$\int_{\mathbb{R}^N} \mathbf{F}(x,u) \cdot \nabla\varphi \, dx = 0 \quad \text{ for all } \varphi \in C_o^\infty(\mathbb{R}^N). \tag{4.2}$$

This is formally obtained from (4.1) by multiplying by φ and integrating by parts. Every classical solution is a weak solution. Every weak solution such that $\mathbf{F}(\cdot,u)$ is of class C^1 in some open set $E \subset \mathbb{R}^N$ is a classical solution of (4.1) in E. Indeed, writing (4.2) for all $\varphi \in C_o^\infty(E)$ implies that (4.1) holds in the classical sense within E. Weak solutions could be classical in sub-domains of $\mathbb{R}^N$. In general, however, weak solutions fail to be classical in the whole of $\mathbb{R}^N$ as shown by the following example. Denote by (x,y) the coordinates in $\mathbb{R}^2$ and consider the Burgers equation ([24, 25]

$$\frac{\partial}{\partial y}u + \frac{1}{2}\frac{\partial}{\partial x}u^2 = 0. \tag{4.3}$$

One verifies that the function

$$u(x,y) = \begin{cases} -\frac{2}{3}\bigl(y + \sqrt{3x+y^2}\bigr) & \text{for } \ 4x+y^2 > 0 \\ 0 & \text{for } \ 4x+y^2 < 0 \end{cases}$$

solves the PDE in the weak form

$$\int_{\mathbb{R}^2} \bigl\{ u\varphi_y + \tfrac{1}{2}u^2\varphi_x \bigr\} dxdy = 0 \qquad \text{for all } \ \varphi \in C_o^\infty(\mathbb{R}^2).$$

The solution is discontinuous across the parabola $3x + y^2 = 0$.

4.1 Surfaces of Discontinuity

Let $\mathbb{R}^N$ be divided into two parts, E_1 and E_2, by a smooth surface Γ of unit normal $\boldsymbol{\nu}$ oriented, say, toward E_2. Let $u \in C^1(\bar{E}_i)$ for $i = 1, 2$ be a weak solution of (4.1), discontinuous across Γ. Assume also that $\mathbf{F}(\cdot, u) \in C^1(\bar{E}_i)$, so that

$$\operatorname{div} \mathbf{F}(x,u) = 0 \qquad \text{in } \ E_i \ \text{ for } \ i = 1, 2$$

in the classical sense. Let $[\mathbf{F}(\cdot, u)]$ denote the jump of $\mathbf{F}(\cdot, u)$ across Γ, i.e.,

$$[\mathbf{F}(x,u)] = \lim_{E_1 \ni x \to \Gamma} \mathbf{F}(x,u) - \lim_{E_2 \ni x \to \Gamma} \mathbf{F}(x,u).$$

Rewrite (4.2) as

$$\int_{E_1} \mathbf{F}(x,u) \cdot \nabla\varphi dx + \int_{E_2} \mathbf{F}(x,u) \cdot \nabla\varphi dx = 0 \qquad \text{for all } \ \varphi \in C_o^\infty(\mathbb{R}^N).$$

Integrating by parts with the aid of Green's theorem gives

$$\int_{\Gamma} \varphi [\mathbf{F}(x,u)] \cdot \boldsymbol{\nu} d\sigma = 0 \qquad \text{for all } \ \varphi \in C_o^\infty(\mathbb{R}^N).$$

Thus if a weak solution suffers a discontinuity across a smooth surface Γ, then

$$[\mathbf{F}(x,u)] \cdot \boldsymbol{\nu} = 0 \qquad \text{on } \ \Gamma. \tag{4.4}$$

Even though this equation has been derived globally, it has a local thrust, and it can be used to find possible local discontinuities of weak solutions.

4.2 The Shock Line

Consider the PDE in two independent variables

$$u_y + a(u)u_x = 0 \qquad \text{for some } \ a \in C(\mathbb{R}) \tag{4.5}$$

and rewrite it as

$$\frac{\partial}{\partial y} R(u) + \frac{\partial}{\partial x} S(u) = 0 \qquad \text{in } \ \mathbb{R}^2$$

where

$$R(u) = u \quad \text{and} \quad S(u) = \int^u a(s)ds.$$

More generally, $R(\cdot)$ and $S(\cdot)$ could be any two functions satisfying

$$S'(u) = a(u)R'(u).$$

Let u be a weak solution of (4.5) in $\mathbb{R}^2$, discontinuous across a smooth curve of parametric equations $\Gamma = \{x = x(t), y = y(t)\}$. Then, according to (4.4), Γ must satisfy the shock condition[1]

$$[R(u)]x' - [S(u)]y' = 0. \tag{4.6}$$

In particular, if Γ is the graph of a function $y = y(x)$, then $y(\cdot)$ satisfies the differential equation

$$y' = \frac{[R(u)]}{[S(u)]}.$$

As an example, consider the case of the Burgers equation (4.3). Let u be a weak solution of (4.3), discontinuous across a smooth curve Γ parametrized locally as $\Gamma = \{x = x(t), y = t\}$. Then (4.6) gives the differential equation of the shock line[2]

$$x'(t) = \frac{[u^+(x(t),t) + u^-(x(t),t)]}{2}, \qquad u^\pm = \lim_{x \to x(t)^\pm} u(x,t). \tag{4.7}$$

5 The Initial Value Problem

Denote by (x,t) points in $\mathbb{R}^N \times \mathbb{R}^+$, and consider the quasi-linear equation in $N+1$ variables

$$u_t + a_i(x,t,u)u_{x_i} = a_o(x,t,u) \tag{5.1}$$

with data prescribed on the N-dimensional surface $[t = 0]$, say

$$u(x,0) = u_o(x) \in C^1(\mathbb{R}^N). \tag{5.2}$$

Using the $(N+1)$st variable t as a parameter, the characteristic projections are

$$\begin{aligned} &x_i'(t) = a_i(x(t),t,z(t)) && \text{for } i = 1,\dots,N \\ &x_{N+1} = t && \text{for } t \in (-\delta,\delta) \text{ for some } \delta > 0 \\ &x(0) = x_o \in \mathbb{R}^N. \end{aligned}$$

Therefore $t = 0$ is noncharacteristic and the Cauchy problem (5.1)–(5.2) is solvable. If the coefficients a_i are constant, the integral surface is given by

---

[1] This is a special case of the Rankine–Hugoniot shock condition ([211, 129]).

[2] The notion of *shock* will be made more precise in Section 13.3.

$$x(t,s) = s + \mathbf{a}t, \quad z(t,s) = u_o(s) + a_o t$$

and the solution is

$$u(x,t) = u_o(x - \mathbf{a}t) + a_o t.$$

In the case $N = 1$ and $a_o = 0$, this is a traveling wave in the sense that the graph of u_o travels with velocity a_1 in the positive direction of the x-axis, keeping the same shape.

5.1 Conservation Laws

Let $(x,t,u) \to \mathbf{F}(x,t,u)$ be a measurable vector-valued function in $\mathbb{R}^N \times \mathbb{R}^+ \times \mathbb{R}$ and consider formally homogeneous, initial value problems of the type

$$\begin{aligned} u_t + \operatorname{div} \mathbf{F}(x,t,u) = 0 \ \text{ in } \ \mathbb{R}^N \times \mathbb{R}^+ \\ u(\cdot, 0) = u_o \in L^1(\mathbb{R}^N). \end{aligned} \tag{5.3}$$

These are called conservation laws. The variable t represents the time, and u is prescribed at some initial time $t = 0$.

Remark 5.1 The method of integral surfaces outlined in Section 2, gives solutions near the noncharacteristic surface $t = 0$. Because of the physics underlying these problems we are interested in solutions defined only for positive times, that is defined only on one side of the surface carrying the data.

A function u is a weak solution of the initial value problem (5.3) if

- **(a)** $u(\cdot, t) \in L^1_{\text{loc}}(\mathbb{R}^N)$ for all $t \ge 0$, and for all $i = 1, \dots, N$ we have $F_i(\cdot, \cdot, u) \in L^1_{\text{loc}}(\mathbb{R}^N \times \mathbb{R}^+)$,
- **(b)** the PDE is satisfied in the sense
$$\int_0^\infty \int_{\mathbb{R}^N} \big[u\varphi_t + \mathbf{F}(x,t,u) \cdot D\varphi \big] dxdt = 0 \tag{5.4}$$
for all $\varphi \in C_o^\infty(\mathbb{R}^N \times \mathbb{R}^+)$, and where D denotes the gradient with respect to the space variables only,
- **(c)** the initial datum is taken in the sense of $L^1_{\text{loc}}(\mathbb{R}^N)$, that is
$$\lim_{\mathbb{R}^+ \ni t \to 0} \| u(\cdot, t) - u_o \|_{1,K} = 0 \tag{5.5}$$
for all compact sets $K \subset \mathbb{R}^N$.

6 Conservation Laws in One Space Dimension

Let $a(\cdot)$ be a continuous function in $\mathbb{R}$ and consider the initial value problem

$$u_t + a(u) u_x = 0 \ \text{ in } \ \mathbb{R} \times \mathbb{R}^+, \quad u(\cdot, 0) = u_o \in C(\mathbb{R}). \tag{6.1}$$

The characteristic through $(x_o, 0, z_o)$, using t as a parameter, is

$$z(t) = z_o, \quad x(t) = x_o + a(z_o)t$$

and the integral surface is

$$x(s,t) = s + a(u_o(s))t, \quad z(s,t) = u_o(s).$$

Therefore the solution, whenever it is well defined, can be written implicitly as

$$u(x,t) = u_o(x - a(u)t). \tag{6.2}$$

The characteristic projections through points $(s, 0)$ of the x-axis are the lines

$$x = s + a(u_o(s))t$$

and u remains constant along such lines. Two of these characteristic projections, say

$$x = s_i + a(u_o(s_i))t \quad i = 1, 2, \ \text{ such that } \ a(u_o(s_1)) \neq a(u_o(s_2)) \tag{γ_i}$$

intersect at (ξ, η) given by

$$\begin{aligned} \xi &= \frac{a(u_o(s_1))s_2 - a(u_o(s_2))s_1}{a(u_o(s_1)) - a(u_o(s_2))} \\ \eta &= -\frac{s_1 - s_2}{a(u_o(s_1)) - a(u_o(s_2))}. \end{aligned} \tag{6.3}$$

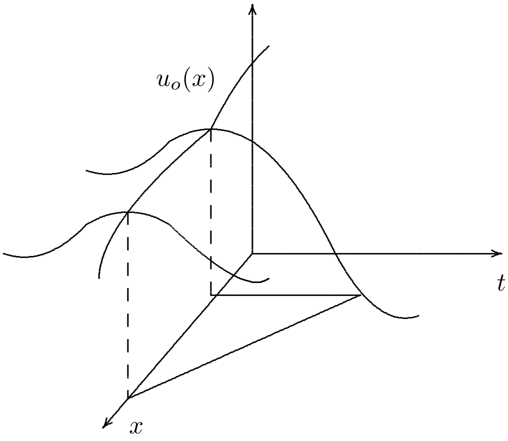

Fig. 6.1

Since u is constant along each of the γ_i, it must be discontinuous at (ξ, η), unless $u_o(s) = \text{const}$. Therefore the solution exists only in a neighborhood of the x-axis. It follows from (6.3) and Remark 5.1 that the solution exists for all $t > 0$ if the function $s \to a(u_o(s))$ is increasing. Indeed, in such a case, the intersection point of the characteristic lines γ_1 and γ_2 occurs in the half-plane $t < 0$. If $a(\cdot)$ and $u_o(\cdot)$ are differentiable, compute from (6.2)

$$\begin{aligned} u_t &= -\frac{u_o'(x - a(u)t)a(u)}{1 + u_o'(x - a(u)t)a'(u)t} \\ u_x &= \frac{u_o'(x - a(u)t)}{1 + u_o'(x - a(u)t)a'(u)t}. \end{aligned}$$

These are implicitly well defined if $a(\cdot)$ and $u_o(\cdot)$ are increasing functions, and when substituted into (6.1) satisfy the PDE for all $t > 0$. Rewrite the initial value problem (6.1) as

$$\begin{aligned} u_t + F(u)_x &= 0 \text{ in } \mathbb{R} \times \mathbb{R}^+ \\ u(\cdot, 0) &= u_o \end{aligned} \qquad \text{where } F(u) = \int_0^u a(s)ds. \tag{6.4}$$

Proposition 6.1 *Let $F(\cdot)$ be convex and of class C^2, and assume that the initial datum $u_o(\cdot)$ is nondecreasing and of class C^1. Then the initial value problem (6.4) has a unique classical solution in $\mathbb{R} \times \mathbb{R}^+$.*

6.1 Weak Solutions and Shocks

If the initial datum u_o is decreasing, then a solution global in time is necessarily a weak solution. The shock condition (4.6) might be used to construct weak solutions, as shown by the following example. The initial value problem

$$u_t + \tfrac{1}{2}(u^2)_x = 0 \text{ in } \mathbb{R} \times \mathbb{R}^+$$

$$u(x, 0) = \begin{cases} 1 & \text{for } x < 0 \\ 1 - x & \text{for } 0 \le x \le 1 \\ 0 & \text{for } x \ge 1 \end{cases} \tag{6.5}$$

has a unique weak solution for $0 < t < 1$, given by

$$u = \begin{cases} 1 & \text{for } x < t \\ \dfrac{x - 1}{t - 1} & \text{for } t < x < 1 \\ 0 & \text{for } x \ge 1. \end{cases} \tag{6.6}$$

For $t > 1$ the geometric construction of (6.2) fails for the sector $1 < x < t$.

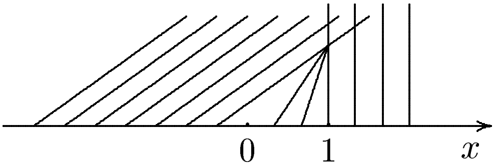

Fig. 6.2

The jump discontinuity across the lines $x = 1$ and $x = t$ is 1. Therefore, starting at $(1, 1)$ we draw a curve satisfying (4.7). This gives the shock line $2x = t + 1$, and we define the weak solution u for $t > 1$ as

$$u = \begin{cases} 1 & \text{for } 2x < t + 1 \\ 0 & \text{for } 2x > t + 1. \end{cases} \tag{6.7}$$

Remark 6.1 For $t > 1$ fixed, the solution $x \to u(x, t)$ drops from 1 to 0 as the increasing variable x crosses the shock line.

6.2 Lack of Uniqueness

If u_o is nondecreasing and somewhere discontinuous, then (6.4) has, in general, more than one weak solution. This is shown by the following Riemann problem:

$$\begin{gathered} u_t + \frac{1}{2}(u^2)_x = 0 \ \text{ in } \ \mathbb{R} \times \mathbb{R}^+ \\ \\ u(x, 0) = \begin{cases} 0 & \text{for } x \le 0 \\ 1 & \text{for } x > 0. \end{cases} \end{gathered} \tag{6.8}$$

No points of the sector $0 < x < t$ can be reached by characteristics originating from the x-axis and carrying the data (Figure 6.3). The solution is zero for $x < 0$, and it is 1 for $x > t$. Enforcing the shock condition (4.7) gives

$$u(x, t) = \begin{cases} 0 & \text{for } 2x < t \\ 1 & \text{for } 2x > t. \end{cases} \tag{6.9}$$

However, the continuous function

$$u(x, t) = \begin{cases} 0 & \text{for } x < 0 \\ \dfrac{x}{t} & \text{for } 0 \le x \le t \\ 1 & \text{for } x > t \end{cases} \tag{6.10}$$

is also a weak solution of (6.8).

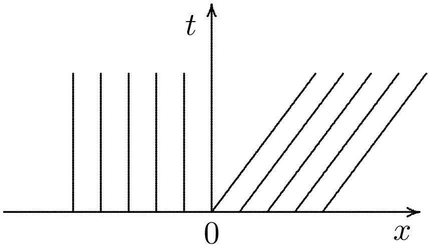

Fig. 6.3

7 Hopf Solution of The Burgers Equation

Insight into the solvability of the initial value problem (6.4) is gained by considering first the special case of the Burgers equation, for which $F(u) = \frac{1}{2}u^2$. Hopf's method [122] consists in solving first the regularized parabolic problems

$$\begin{aligned} u_{n,t} - \frac{1}{n}u_{n,xx} &= -u_n u_{n,x} \text{ in } \mathbb{R}\times\mathbb{R}^+ \\ u_n(\cdot,0) &= u_o \end{aligned} \tag{7.1}$$

and then letting $n \to \infty$ in a suitable topology. Setting

$$U(x,t) = \int_{x_o}^x u_n(y,t)dy$$

for some arbitrary but fixed $x_o \in \mathbb{R}$ transforms the Cauchy problem (7.1) into

$$\begin{aligned} U_t - \frac{1}{n}U_{xx} &= -\frac{1}{2}(U_x)^2 \text{ in } \mathbb{R}\times\mathbb{R}^+ \\ U(x,0) &= \int_{x_o}^x u_o(s)ds. \end{aligned}$$

Next, one introduces the new unknown function $w = e^{-\frac{n}{2}U}$ and verifies that w is a positive solution of the Cauchy problem

$$\begin{aligned} w_t - \frac{1}{n}w_{xx} &= 0 \text{ in } \mathbb{R}\times\mathbb{R}^+ \\ w(x,0) &= e^{-\frac{n}{2}\int_{x_o}^x u_o(s)ds}. \end{aligned} \tag{7.2}$$

Such a positive solution is uniquely determined by the representation formula

$$w(x,t) = \frac{1}{\sqrt{4\pi t}}\int_{\mathbb{R}} e^{-\frac{n}{2}\int_{x_o}^y u_o(s)ds} e^{-n\frac{|x-y|^2}{4t}}dy$$

provided u_o satisfies the growth condition[3]

---

[3] See (2.7) and Theorem 2.1 of Chapter V, and Section 14 of the same Chapter.

$$|u_o(s)| \le C_o|s|^{1-\varepsilon_o} \quad \text{for all } |s| \ge r_o$$

for some given positive constants C_o, r_o, and ε_o. The unique solutions u_n of (7.1) are then given explicitly by

$$u_n(x,t) = \int_{\mathbb{R}} \frac{(x-y)}{t} d\lambda_n(y)$$

where $d\lambda_n(y)$ are the probability measures

$$d\lambda_n(y) = \frac{e^{-\frac{n}{2}\left(\int_{x_o}^{y} u_o(s)ds + \frac{|x-y|^2}{2t}\right)}}{\int_{\mathbb{R}} e^{-\frac{n}{2}\left(\int_{x_o}^{y} u_o(s)ds + \frac{|x-y|^2}{2t}\right)} dy} dy.$$

The a priori estimates needed to pass to the limit can be derived either from the parabolic problems (7.1)–(7.2) or from the explicit representation of u_n and the corresponding probability measures $\lambda_n(y)$. In either case they depend on the fact that $F(\cdot)$ is convex and $F' = a(\cdot)$ is strictly increasing.[4]

Because of the parabolic regularization (7.1), it is reasonable to expect that those solutions of (6.4) constructed in this way satisfy some form of the maximum principle.[5] It turns out that Hopf's approach, and in particular the explicit representation of the approximating solutions u_n and the corresponding probability measures $\lambda_n(y)$, continues to hold for the more general initial value problem (6.4). It has been observed that these problems fail, in general, to have a unique solution. It turns out that those solutions of (6.4) that satisfy the maximum principle, form a special subclass of solutions within which uniqueness holds. These are called *entropy* solutions.

8 Weak Solutions to (6.4) When $a(\cdot)$ is Strictly Increasing

We let $a(\cdot)$ be continuous and strictly increasing in $\mathbb{R}$, that is, there exists a positive constant L such that

$$a'(s) \ge \frac{1}{L} \quad \text{a.e. } s \in \mathbb{R}. \tag{8.1}$$

Assume that the initial datum u_o satisfies

$$\begin{aligned} &u_o \in L^\infty(\mathbb{R}) \cap L^1(-\infty, x) \ \text{ for all } x \in \mathbb{R} \\ &\limsup_{x\to-\infty} |u_o(x)| = 0 \\ &\inf_{x\in\mathbb{R}} \int_{-\infty}^{x} u_o(s)ds \ge -C \ \text{ for some } C > 0. \end{aligned} \tag{8.2}$$

---

[4] Some cases of nonconvex F are in [133].

[5] By **3.2.** of the Complements of Chapter 5, the presence of the term $u_n u_{n,x}$ in (7.1) is immaterial for a maximum principle to hold.

For example, the datum of the Riemann problem (6.8) satisfies such a condition. The initial datum is not required to be increasing, nor in $L^1(\mathbb{R})$. Since $F(\cdot)$ is convex ([50], Chapter IV, Section 13)

$$F(u) - F(v) \ge a(v)(u - v) \quad u, v \in \mathbb{R}, \quad a(v) = F'(v) \tag{8.3}$$

and since F' is strictly increasing, equality holds only if $u = v$. This inequality permits one to solve (6.4) in a weak sense and to identify a class of solutions, called *entropy* solutions, within which uniqueness holds ([157, 158]).

8.1 Lax Variational Solution

To illustrate the method assume first that $F(\cdot)$ is of class C^2 and that u_o is regular, increasing and satisfies

$$u_o(x) = 0 \text{ for all } x < b \text{ for some } b < 0.$$

The geometric construction of (6.2) guarantees that a solution must vanish for $x < b$ for all $t > 0$. Therefore the function

$$U(x,t) = \int_{-\infty}^{x} u(s,t)ds$$

is well defined in $\mathbb{R} \times \mathbb{R}^+$. Integrating (6.4) in dx over $(-\infty, x)$ shows that U satisfies the initial value problem

$$U_t + F(U_x) = 0 \text{ in } \mathbb{R} \times \mathbb{R}^+, \quad U(x,0) = \int_{-\infty}^{x} u_o(s)ds.$$

It follows from (8.3), with $u = U_x$ and all $v \in \mathbb{R}$, that

$$U_t + a(v)U_x \le a(v)v - F(v) \tag{8.4}$$

and equality holds only if $v = u(x,t)$. For $(x,t) \in \mathbb{R} \times \mathbb{R}^+$ fixed, consider the line of slope $1/a(v)$ through (x,t). Denoting by (ξ, τ) the variables, such a line has equation $x - \xi = a(v)(t - \tau)$, and it intersects the axis $\tau = 0$ at the abscissa

$$\eta = x - a(v)t. \tag{8.5}$$

The left-hand side of (8.4) is the derivative of U along such a line. Therefore

$$\frac{d}{d\tau} U\big(x - a(v)(t - \tau), \tau\big) = U_t + a(v)U_x \le a(v)v - F(v).$$

Integrating this over $\tau \in (0,t)$ gives

$$U(x,t) \le \int_{-\infty}^{\eta} u_o(s)ds + t\big[a(v)v - F(v)\big]$$

valid for all $v \in \mathbb{R}$, and equality holds only for $v = u(x,t)$. From (8.5) compute

$$v = a^{-1}\Big(\frac{x-\eta}{t}\Big) \tag{8.6}$$

and rewrite the previous inequality for $U(x,t)$ in terms of η only, that is

$$U(x,t) \le \Psi(x,t;\eta) \qquad \text{for all } \ \eta \in \mathbb{R} \tag{8.7}$$

where

$$\begin{aligned}\Psi(x,t;\eta) = &\int_{-\infty}^{\eta} u_o(s)ds \\ &+ t\Big\{\Big(\frac{x-\eta}{t}\Big)a^{-1}\Big(\frac{x-\eta}{t}\Big) - F\Big[a^{-1}\Big(\frac{x-\eta}{t}\Big)\Big]\Big\}.\end{aligned} \tag{8.8}$$

Therefore, having fixed (x,t), for that value of $\eta = \eta(x,t)$ for which v in (8.6) equals $u(x,t)$, equality must hold in (8.7). Returning now to $F(\cdot)$ convex and u_o satisfying (8.1)–(8.2), the arguments leading to (8.7) suggest the construction of the weak solution of (6.4) in the following two steps:

Step 1: For (x,t) fixed, minimize the function $\Psi(x,t;\eta)$, i.e., find $\eta = \eta(x,t)$ such that

$$\Psi(x,t;\eta(x,t)) \le \Psi(x,t;s) \qquad \text{for all } \ s \in \mathbb{R}. \tag{8.9}$$

Step 2: Compute $u(x,t)$ from (8.6), that is

$$u(x,t) = a^{-1}\Big(\frac{x-\eta(x,t)}{t}\Big). \tag{8.10}$$

9 Constructing Variational Solutions I

Proposition 9.1 *For fixed $t > 0$ and a.e. $x \in \mathbb{R}$ there exists a unique $\eta = \eta(x,t)$ that minimizes $\Psi(x,t;\cdot)$. The function $x \to u(x,t)$ defined by (8.10) is a.e. differentiable in $\mathbb{R}$ and satisfies*

$$\frac{u(x_2,t)-u(x_1,t)}{x_2-x_1} \le \frac{L}{t} \qquad \textit{for a.e. } \ x_1 < x_2 \in \mathbb{R}. \tag{9.1}$$

Moreover, for a.e. $(x,t) \in \mathbb{R}\times\mathbb{R}^+$

$$|u(x,t)| \le \sqrt{\frac{2L}{t}}\Big(\int_{-\infty}^{x-a(o)t} u_o(s)ds - \inf_{y\in\mathbb{R}}\int_{-\infty}^{y} u_o(s)ds\Big)^{1/2}. \tag{9.2}$$

Proof. The function $\eta \to \Psi(x,t;\eta)$ is bounded below. Indeed, by the expressions (8.6) and (8.8) and the assumptions (8.1)–(8.2)

$$\Psi\left(x,t;\eta\right)\geq\inf_{y\in\mathbb{R}}\int_{-\infty}^{y}u_{o}(s)ds+t[va(v)-F(v)]\geq-C+\frac{t}{2L}v^{2}\tag{9.3}$$

for $\eta=x-a(v)t$. A minimizer can be found by a minimizing sequences $\{\eta_n\}$, that is one for which

$$\Psi\left(x,t;\eta_{n}\right)>\Psi\left(x,t;\eta_{n+1}\right)\qquad\text{and}\qquad\lim\Psi\left(x,t;\eta_{n}\right)=\inf_{\eta}\Psi\left(x,t;\eta\right).$$

By (9.3), the sequence $\{\eta_n\}$ is bounded. Therefore, a subsequence can be selected and relabeled with n such that $\{\eta_n\}\to\eta(x,t)$. Since $\Psi\left(x,t;\cdot\right)$ is continuous in $\mathbb{R}$

$$\lim_{n\to\infty}\Psi\left(x,t;\eta_{n}\right)=\Psi\left(x,t;\eta(x,t)\right)\leq\Psi\left(x,t;\eta\right),\qquad\text{for all }\ \eta\in\mathbb{R}.$$

This process guarantees the existence of at least one minimizer for every fixed $x\in\mathbb{R}$. Next we prove that such a minimizer is unique, for a.e. $x\in\mathbb{R}$.

Let $H(x)$ denote the set of all the minimizers of $\Psi\left(x,t;\cdot\right)$, and define a function $x\to\eta(x,t)$ as an arbitrary selection out of $H(x)$.

Lemma 9.1 *If $x_1<x_2$, then $\eta(x_1,t)<\eta(x_2,t)$.*

Proof (of Proposition 9.1 assuming Lemma 9.1). Since $x\to\eta(x,t)$ is increasing, it is continuous in $\mathbb{R}$ except possibly for countably many points. Therefore, $\eta(x,t)$ is uniquely defined for a.e. $x\in\mathbb{R}$. From (8.10) it follows that for a.e. $x_1<x_2$ and some $\xi\in\mathbb{R}$

$$\begin{aligned}u(x_{2},t)-u(x_{1},t)&\leq\frac{a^{-1'}(\xi)}{t}\{(x_{2}-x_{1})-[\eta(x_{2},t)-\eta(x_{1},t)]\}\\&\leq\frac{a^{-1'}(\xi)}{t}(x_{2}-x_{1})\leq\frac{L}{t}(x_{2}-x_{1}).\end{aligned}$$

This proves (9.1). To prove (9.2), write (9.3) for $\eta=\eta(x,t)$, the unique minimizer of $\Psi(x,t;\cdot)$. For such a choice, by (8.10), $v=u$. Therefore

$$\begin{aligned}\frac{t}{2L}u^{2}(x,t)&\leq\Psi\big(x,t;\eta(x,t)\big)-\inf_{y\in\mathbb{R}}\int_{-\infty}^{y}u_{o}(s)ds\\&\leq\Psi\left(x,t;\eta\right)-\inf_{y\in\mathbb{R}}\int_{-\infty}^{y}u_{o}(s)ds\end{aligned}$$

for all $\eta\in\mathbb{R}$, since $\eta(x,t)$ is a minimizer. Taking $\eta=x-a(0)t$ and recalling the definitions (8.8) of $\Psi\left(x,t;\cdot\right)$ proves (9.2). ■

9.1 Proof of Lemma 9.1

Let $\eta_i=\eta(x_i,t)$ for $i=1,2$. It will suffice to prove that

$$\Psi(x_2,t;\eta_1) < \Psi(x_2,t;\eta) \quad \text{ for all } \eta < \eta_1. \tag{9.4}$$

By minimality, $\Psi(x_1,t;\eta_1) \le \Psi(x_1,t;\eta)$ for all $\eta < \eta_1$. From this

$$\begin{aligned} \Psi(x_2,t;\eta_1) + &[\Psi(x_1,t;\eta_1) - \Psi(x_2,t;\eta_1)] \\ &\le \Psi(x_2,t;\eta) + [\Psi(x_1,t;\eta) - \Psi(x_2,t;\eta)]. \end{aligned}$$

Therefore inequality (9.4) will follow if the function

$$\eta \to L(\eta) = \Psi(x_1,t;\eta) - \Psi(x_2,t;\eta)$$

is increasing. Rewrite $L(\eta)$ in terms of $v_i = v_i(\eta)$ given by (8.6) with $x = x_i$ for $i = 1,2$. This gives

$$L(\eta) = t[v_1 a(v_1) - F(v_1)] - t[v_2 a(v_2) - F(v_2)] = t\int_{v_2}^{v_1} s a'(s) ds.$$

From this one computes

$$\begin{aligned} L'(\eta) &= t\left[v_1 a'(v_1)\frac{\partial v_1}{\partial \eta} - v_2 a'(v_2)\frac{\partial v_2}{\partial \eta}\right] \\ &= a^{-1}\left(\frac{x_2-\eta}{t}\right) - a^{-1}\left(\frac{x_1-\eta}{t}\right) > 0. \end{aligned}$$

■

10 Constructing Variational Solutions II

For fixed $t > 0$, the minimizer $\eta(x,t)$ of $\Psi(x,t;\cdot)$ exists and is unique for a.e. $x \in \mathbb{R}$. We will establish that for all such (x,t)

$$a^{-1}\left(\frac{x-\eta(x,t)}{t}\right) = \lim_{n\to\infty} \int_{\mathbb{R}} a^{-1}\left(\frac{x-\eta}{t}\right) d\lambda_n(\eta) \tag{10.1}$$

where $d\lambda_n(\eta)$ are the probability measures on $\mathbb{R}$

$$d\lambda_n(\eta) = \frac{e^{-n\Psi(x,t;\eta)}}{\int_{\mathbb{R}} e^{-n\Psi(x,t;\eta)} d\eta} d\eta \qquad \text{ for } n \in \mathbb{N}. \tag{10.2}$$

Therefore the expected solution

$$u(x,t) = a^{-1}\left(\frac{x-\eta(x,t)}{t}\right)$$

can be constructed by the limiting process (10.1). More generally, we will establish the following

Lemma 10.1 *Let f be a continuous function in $\mathbb{R}$ satisfying the growth condition*

$$|f(v)| \le C_o|v|e^{c_o\int_o^v sa'(s)ds} \qquad \text{for all } |v| \ge \gamma_o \tag{10.3}$$

for given positive constants C_o, c_o and γ_o. Then for fixed $t > 0$ and a.e. $x \in \mathbb{R}$

$$f[u(x,t)] = \lim_{n\to\infty}\int_{\mathbb{R}} f\left[a^{-1}\left(\frac{x-\eta}{t}\right)\right]d\lambda_n(\eta).$$

Proof. Introduce the change of variables

$$v = a^{-1}\left(\frac{x-\eta}{t}\right), \qquad v_o = a^{-1}\left(\frac{x-\eta(x,t)}{t}\right) \tag{10.4}$$

and rewrite $\Psi(x,t;\eta)$ as

$$\Psi(v) = \int_{-\infty}^{x-a(v)t} u_o(s)ds + t[va(v) - F(v)]. \tag{10.5}$$

The probability measures $d\lambda_n(\eta)$ are transformed into the probability measures

$$d\mu_n(v) = \frac{e^{-n\Psi(v)}a'(v)}{\int_{\mathbb{R}} e^{-n\Psi(v)}a'(v)dv}dv = \frac{e^{-n[\Psi(v)-\Psi(v_o)]}a'(v)}{\int_{\mathbb{R}} e^{-n[\Psi(v)-\Psi(v_o)]}a'(v)dv}dv \tag{10.6}$$

and the statement of the lemma is equivalent to

$$f(v_o) = \lim_{n\to\infty}\int_{\mathbb{R}} f(v)d\mu_n(v)$$

where v_o is the unique minimizer of $v \to \Psi(v)$. For this it suffices to show that

$$I_n \overset{\text{def}}{=} \int_{\mathbb{R}} |f(v) - f(v_o)|d\mu_n(v) \to 0 \qquad \text{as } n \to \infty.$$

By (9.3) the function $\Psi(\cdot)$ grows to infinity as $|v| \to \infty$. Since v_o is the only minimizer, for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\Psi(v) > \Psi(v_o) + \delta \qquad \text{for all } |v - v_o| > \varepsilon. \tag{10.7}$$

Moreover, the numbers ε and δ being fixed, there exists some positive number σ such that

$$\Psi(v) \le \Psi(v_o) + \tfrac{1}{2}\delta \qquad \text{for all } |v - v_o| < \sigma.$$

From this we estimate from below

$$\int_{\mathbb{R}} e^{-n[\Psi(v)-\Psi(v_o)]}a'(v)dv \ge \frac{1}{L}\int_{|v-v_o|<\sigma} e^{-n[\Psi(v)-\Psi(v_o)]}dv \ge \frac{2\sigma}{L}e^{-\frac{1}{2}n\delta}.$$

Therefore

$$d\mu_n(v) \le \frac{L}{2\sigma} e^{\frac{1}{2}n\delta} e^{-n[\Psi(v)-\Psi(v_o)]} a'(v)dv.$$

Next estimate I_n by using these remarks as

$$\begin{aligned} I_n \le & \int_{|v-v_o|<\varepsilon} |f(v)-f(v_o)| d\mu_n(v) \\ & + \sup_{|v-v_o|<2\gamma} |f(v)| \frac{L}{2\sigma} e^{\frac{1}{2}n\delta} \int_{\varepsilon<|v-v_o|<2\gamma} e^{-n[\Psi(v)-\Psi(v_o)]} a'(v)dv \\ & + \frac{L}{2\sigma} e^{\frac{1}{2}n\delta + n\Psi(v_o)} \int_{|v-v_o|>2\gamma} (|f(v)|+|f(v_o)|) e^{-n\Psi(v)} a'(v)dv \\ = & I_n^{(1)} + I_n^{(2)} + I_n^{(3)} \end{aligned}$$

where γ is a positive number to be chosen. Denoting by $\omega(\cdot)$ the modulus of continuity of f, estimate $I_n^{(1)} \le \omega(\varepsilon)$, since $d\mu_n(v)$ is a probability measure. The second term $I_n^{(2)}$ is estimated by means of (10.7) as

$$\begin{aligned} I_n^{(2)} \le & \sup_{|v-v_o|<2\gamma} |f(v)| a'(v) \frac{L}{2\sigma} e^{\frac{1}{2}n\delta} \int_{\varepsilon<|v-v_o|<2\gamma} e^{-n[\Psi(v)-\Psi(v_o)]} dv \\ & \le C(\gamma,\sigma,L) e^{-\frac{1}{2}n\delta} \end{aligned}$$

for a constant C depending only on the indicated quantities. Thus $I_n^{(2)} \to 0$ as $n \to \infty$. The last term $I_n^{(3)}$ is estimated using the lower bound

$$\Psi(v) \ge -C + t[va(v) - F(v)] \ge -C + t\int_0^v sa'(s)ds.$$

Also choose $\gamma \ge \gamma_o$, where γ_o is the constant in the growth condition (10.3). By choosing γ even larger if necessary, we may ensure that $[|v-v_o| > 2\gamma] \subset [|v| > \gamma]$. For this choice

$$I_n^{(3)} \le \frac{C_o L}{\sigma} e^{n[\delta + C + \Psi(v_o)]} \int_{|v|>\gamma} e^{-(nt-c_o)\int_o^v sa'(s)ds} |v| a'(v)dv. \tag{10.8}$$

If n is so large that $nt - c_o > 0$, the integral on the right-hand side of (10.8) can be computed explicitly, and estimated as follows

$$\begin{aligned} \int_{|v|>\gamma} e^{-(nt-c_o)\int_o^v sa'(s)ds} |v| a'(v)dv &= \int_\gamma^\infty \cdots dv + \int_{-\infty}^{-\gamma} \cdots dv \\ = \frac{2}{nt-c_o} e^{-(nt-c_o)\int_o^\gamma sa'(s)ds} &\le \frac{2}{nt-c_o} e^{-(nt-c_o)\frac{\gamma^2}{2L}}. \end{aligned}$$

This in (10.8) gives

$$I_n^{(3)} \le \frac{2C_o L e^{\frac{c_o\gamma^2}{2L}}}{\sigma(nt-c_o)} e^{-n\left(-\delta - C - \Psi(v_o) + \frac{t\gamma^2}{2L}\right)}.$$

The number $t > 0$ being fixed, choose γ large enough that

$$-\delta - C - \Psi(v_o) + \frac{t\gamma^2}{2L} > 0.$$

Then let $n \to \infty$ to conclude that $\lim_{n\to\infty} I_n \le \omega(\varepsilon)$ for all $\varepsilon > 0$. ■

11 The Theorems of Existence and Stability

11.1 Existence of Variational Solutions

Theorem 11.1 (Existence). *Let the assumptions (8.1)–(8.2) hold, and let $u(\cdot,t)$ denote the function constructed in Sections 8–10. Then*

$$\|u(\cdot,t)\|_{\infty,\mathbb{R}} \le \|u_o\|_{\infty,\mathbb{R}} \quad \text{for all } t > 0. \tag{11.1}$$

The function u solves the initial value problem (6.4) in the weak sense

$$\int_0^t \int_{\mathbb{R}} [u\varphi_t + F(u)\varphi_x] dx d\tau = \int_{\mathbb{R}} u(x,t)\varphi(x,t)dx - \int_{\mathbb{R}} u_o(x)\varphi(x,0)dx \tag{11.2}$$

for all $\varphi \in C^1[\mathbb{R}^+; C_o^\infty(\mathbb{R})]$ and a.e. $t > 0$. Moreover, u takes the initial datum u_o in the sense of $L^1_{\rm loc}(\mathbb{R})$, that is, for every compact subset $K \subset \mathbb{R}$

$$\lim_{t\to 0} \|u(\cdot,t) - u_o\|_{1,K} = 0. \tag{11.3}$$

Finally, if $u_o(\cdot)$ is continuous, then for all $t > 0$

$$u(x,t) = u_o(x - a[u(x,t)]t) \quad \text{for a.e. } x \in \mathbb{R}. \tag{11.4}$$

11.2 Stability of Variational Solutions

Assuming the existence theorem for the moment, we establish that the solutions constructed by the method of Sections 8–10 are stable in $L^1_{\rm loc}(\mathbb{R})$. Let $\{u_{o,m}\}$ be a sequence of functions satisfying (8.2) and in addition

$$\begin{cases} \|u_{o,m}\|_{\infty,\mathbb{R}} \le \gamma\|u_o\|_{\infty,\mathbb{R}} & \text{for all } m, \text{ for some } \gamma > 0 \\ u_{o,m} \to u_o \quad \text{weakly in } L^1(-\infty,x) & \text{for all } x \in \mathbb{R}. \end{cases} \tag{11.5}$$

Denote by u_m the functions constructed by the methods of Sections 8–10, corresponding to the initial datum $u_{o,m}$. Specifically, first consider the functions $\Psi_m(x,t;\cdot)$ defined as in (8.8), with u_o replaced by $u_{o,m}$. For fixed $t > 0$, let $\eta_m(x,t)$ be a minimizer of $\Psi_m(x,t;\cdot)$. Such a minimizer is unique for almost all $x \in \mathbb{R}$. Then set

$$u_m(x,t) = a^{-1}\left(\frac{x - \eta_m(x,t)}{t}\right).$$

Theorem 11.2 (Stability in $L^1_{\text{loc}}(\mathbb{R})$). *For fixed $t > 0$ and all compact subsets $K \subset \mathbb{R}$*

$$\|u_m(\cdot,t) - u(\cdot,t)\|_{1,K} \to 0 \qquad \textit{as } \ m \to \infty.$$

Proof. Denote by $\mathcal{E}_o$ and $\mathcal{E}_m$ the subsets of $\mathbb{R}$ where $u(\cdot,t)$ and $u_m(\cdot,t)$ are not uniquely defined. The set $\mathcal{E} = \bigcup \mathcal{E}_m$ has measure zero and $\{u_m(\cdot,t), u(\cdot,t)\}$ are all uniquely well defined in $\mathbb{R} - \mathcal{E}$. We claim that

$$\lim_{m\to\infty} u_m(x,t) = u(x,t) \qquad \text{and} \qquad \lim_{m\to\infty} \eta_m(x,t) = \eta(x,t)$$

for all $x \in \mathbb{R} - \mathcal{E}$, where $\eta(x,t)$ is the unique minimizer of $\Psi(x,t;\cdot)$. By (11.1) and the first of (11.5), $\{u_m(x,t)\}$ is bounded. Therefore also $\{\eta_m(x,t)\}$ is bounded, and a subsequence $\{\eta_{m'}(x,t)\}$ contains in turn a convergent subsequence, say for example $\{\eta_{m''}(x,t)\} \to \eta_o(x,t)$. By minimality

$$\Psi_{m''}(x,t;\eta_{m''}(x,t)) \le \Psi_{m''}(x,t;\eta(x,t)).$$

Letting $m'' \to \infty$

$$\Psi(x,t;\eta_o(x,t)) \le \Psi(x,t;\eta(x,t)).$$

Therefore $\eta_o(x,t) = \eta(x,t)$, since the minimizer of $\Psi(x,t;\cdot)$ is unique. Therefore any subsequence out of $\{\eta_m(x,t)\}$ contains in turn a subsequence convergent to the same limit $\eta(x,t)$. Thus the entire sequence converges to $\eta(x,t)$. Such a convergence holds for all $x \in \mathbb{R} - \mathcal{E}$, i.e., $\{u_m(\cdot,t)\} \to u(\cdot,t)$ a.e. in $\mathbb{R}$. Since $\{u_m(\cdot,t)\}$ is uniformly bounded in $\mathbb{R}$, the stability theorem in $L^1_{\text{loc}}(\mathbb{R})$ follows from the Lebesgue dominated convergence theorem. ■

12 Proof of Theorem 11.1

12.1 The Representation Formula (11.4)

Let $d\lambda_n(\eta)$ and $d\mu_n(v)$ be the probability measures introduced in (10.2) and (10.6), and set

$$\begin{aligned} u_n(x,t) &= \int_{\mathbb{R}} a^{-1}\Big(\frac{x-\eta}{t}\Big) d\lambda_n(\eta) = \int_{\mathbb{R}} v d\mu_n(v) \\ F_n(x,t) &= \int_{\mathbb{R}} F\left[a^{-1}\Big(\frac{x-\eta}{t}\Big)\right] d\lambda_n(v) = \int_{\mathbb{R}} F(v) d\mu_n(v) \\ H_n(x,t) &= \ln \int_{\mathbb{R}} e^{-n\Psi(x,t;\eta)} d\eta = \ln \int_{\mathbb{R}} e^{-n\Psi(v)} a'(v) dv \end{aligned} \tag{12.1}$$

where the integrals on the right are computed from those on the left by the change of variables (10.4)–(10.5). From the definitions (8.8) and (10.5) of $\Psi(x,t;\eta)$ and $\Psi(v)$, and recalling that $F' = a(\cdot)$

$$\begin{aligned}\Psi_x(x,t;\eta) &= a^{-1}\left(\frac{x-\eta}{t}\right) = v \\ \Psi_t(x,t;\eta) &= -F\left[a^{-1}\left(\frac{x-\eta}{t}\right)\right] = -F(v).\end{aligned} \tag{12.2}$$

Then compute

$$\begin{aligned}\frac{\partial}{\partial x}H_n(x,t) &= -n\int_{\mathbb{R}}\Psi_x(x,t;\eta)d\lambda_n(\eta) = -n\int_{\mathbb{R}} v d\mu_n(v) \\ &= -n\int_{\mathbb{R}} u_o(x-a(v)t)d\mu_n(v) \\ \frac{\partial}{\partial t}H_n(x,t) &= -n\int_{\mathbb{R}}\Psi_t(x,t;\eta)d\lambda_n(\eta) = n\int_{\mathbb{R}} F(v)d\mu_n(v).\end{aligned}$$

Therefore

$$u_n(x,t) = -\frac{1}{n}\frac{\partial}{\partial x}H_n(x,t), \qquad F_n(x,t) = \frac{1}{n}\frac{\partial}{\partial t}H_n(x,t).$$

These imply

$$\frac{\partial}{\partial t}u_n + \frac{\partial}{\partial x}F_n = 0 \quad \text{in } \mathbb{R}\times\mathbb{R}^+ \tag{12.3}$$

and

$$u_n(x,t) = \int_{\mathbb{R}} u_o(x-a(v)t)d\mu_n(v). \tag{12.4}$$

Since $u_o \in L^\infty(\mathbb{R})$ and $d\mu_n(v)$ is a probability measure

$$\|u_n(\cdot,t)\|_{\infty,\mathbb{R}} \le \|u_o\|_{\infty,\mathbb{R}} \quad \text{for all } t>0.$$

Therefore by Lemma 10.1 and Lebesgue's dominated convergence theorem, $\{u_n(\cdot,t)\}$ and $\{F_n(\cdot,t)\}$ converge to $u(\cdot,t)$ and $F[u(\cdot,t)]$ respectively in $L^1_{\text{loc}}(\mathbb{R})$, for all $t>0$. Moreover $\{u_n\}$ and $\{F_n\}$ converge to u and $F(u)$ respectively, in $L^1_{\text{loc}}(\mathbb{R}\times\mathbb{R}^+)$. This also proves (11.1).

If u_o is continuous, the representation formula (11.4) follows from (12.4) and Lemma 10.1, upon letting $n\to\infty$. ■

12.2 Initial Datum in the Sense of $L^1_{\text{loc}}(\mathbb{R})$

Assume first $u_o \in C(\mathbb{R})$. Then by the representation formula (11.4)

$$\lim_{t\to 0}\|u(\cdot,t)-u_o\|_{1,K} = \lim_{t\to 0}\int_K |u_o(x-a[u(x,t)]t)-u_o(x)|dx = 0$$

since u is uniformly bounded in K for all $t>0$. If u_o merely satisfies (8.2), construct a sequence of smooth functions $u_{o,m}$ satisfying (11.5), and in addition $\{u_{o,m}\}\to u_o$ in $L^1_{\text{loc}}(\mathbb{R})$. Such a construction may be realized through a mollification kernel $J_{1/m}$, by setting $u_{o,m} = J_{1/m} * u_o$. By the stability Theorem 11.2

$$\|u(\cdot,t) - u_m(\cdot,t)\|_{1,K} \to \qquad \text{for all } \ t > 0.$$

Moreover, since $u_{o,m}$ are continuous

$$\|u_m(\cdot,t) - u_{o,m}\|_{1,K} \to \qquad \text{as } \ t \to 0.$$

This last limit is actually uniform in m. Indeed

$$\begin{aligned}\int_K |u_m(x,t) - u_{o,m}(x)|dx &= \int_K |u_{o,m}(x - a[u_m(x,t)]t) - u_{o,m}(x)|dx \\ &= \int_K |J_{1/m} * [u_o(x - a[u_m(x,t)]t) - u_o(x)]|dx \\ &\le \int_K |u_o(x - a[u_m(x,t)]t) - u_o(x)|dx.\end{aligned}$$

Since $a[u_m(x,t)]$ is uniformly bounded in K for all $t > 0$, the right-hand side tends to zero as $t \to 0$, uniformly in m.

Fix a compact subset $K \subset \mathbb{R}$ and $\varepsilon > 0$. Then choose $t > 0$ such that

$$\|u_m(\cdot,t) - u_{o,m}\|_{1,K} \le \varepsilon.$$

Such a time t can be chosen independent of m, in view of the indicated uniform convergence. Then we write

$$\|u(t) - u_o\|_{1,K} \le \|u(t) - u_m(t)\|_{1,K} + \|u_m(t) - u_{o,m}\|_{1,K} + \|u_{o,m} - u_o\|_{1,K}.$$

Letting $m \to \infty$ gives $\|u(\cdot,t) - u_o\|_{1,K} \le \varepsilon$. ■

12.3 Weak Forms of the PDE

Multiply (12.3) by $\varphi \in C^1[\mathbb{R}^+; C_o^\infty(\mathbb{R})]$ and integrate over $(\varepsilon,t) \times \mathbb{R}$ for some fixed $\varepsilon > 0$. Integrating by parts and letting $n \to \infty$ gives

$$\int_\varepsilon^t \int_{\mathbb{R}} [u\varphi_t + F(u)\varphi_x]dxd\tau = \int_{\mathbb{R}} u(x,t)\varphi(x,t)dx - \int_{\mathbb{R}} u(x,\varepsilon)\varphi(x,\varepsilon)dx.$$

Now (11.2) follows, since $u(\cdot,t) \to u_o$ in $L^1_{\text{loc}}(\mathbb{R})$ as $t \to 0$. ■

The following proposition provides another weak form of the PDE.

Proposition 12.1 *For all* $t > 0$ *and a.e.* $x \in \mathbb{R}$

$$\begin{aligned}\int_{-\infty}^x u(s,t)ds &= \Psi[x,t;u(x,t)] \\ &= \int_{-\infty}^{x - a[u(x,t)]t} u_o(s)ds + t[ua(u) - F(u)](x,t).\end{aligned} \tag{12.5}$$

Proof. Integrate (12.3) in $d\tau$ over (ε, t) and then in ds over (k, x), where k is a negative integer, and in the resulting expression let $n \to \infty$. Taking into account the expression (12.1) of F_n and the second of (12.2), compute

$$\begin{aligned}\int_k^x u(s,t)ds - \int_k^x u(s,\varepsilon)ds &= \lim_{n\to\infty} \int_\varepsilon^t \int_{\mathbb{R}} [\Psi_\tau(x,\tau;\eta) - \Psi_\tau(k,\tau;\eta)]\, d\lambda_n(\eta) d\tau \\ &= \Psi[x,t;u(x,t)] - \Psi[k,\varepsilon;u(k,\varepsilon)]\end{aligned}$$

by virtue of Lemma 10.1. To prove the proposition first let $\varepsilon \to 0$ and then $k \to -\infty$. ∎

13 The Entropy Condition

A consequence of (9.1) is that the variational solution claimed by Theorem 11.1 satisfies the *entropy condition*

$$\limsup_{0<h\to 0} [u(x+h,t) - u(x,t)] \le 0 \tag{13.1}$$

for all fixed $t > 0$ and a.e. $x \in \mathbb{R}$. The notion of a weak solution introduced in Section 5.1 does not require that (13.1) be satisfied. However, as shown by the examples in Section 6.2, weak solutions need not be unique. We will prove that weak solutions of the initial value problem (6.4) that in addition satisfy the entropy condition (13.1), are unique. The method, due to Kruzhkov [141], is N-dimensional and uses a notion of entropy condition more general than (13.1).

13.1 Entropy Solutions

Consider the initial value problem

$$\begin{aligned} u_t + \operatorname{div} \mathbf{F}(u) = 0 \ \text{ in } \ S_T = \mathbb{R}^N \times (0,T] \\ u(\cdot,0) = u_o \in L^1_{\text{loc}}(\mathbb{R}^N) \end{aligned} \tag{13.2}$$

where $\mathbf{F} \in [C^1(\mathbb{R})]^N$. A weak solution of (13.2), in the sense of (5.4)–(5.5), is an *entropy* solution if

$$\iint_{S_T} \operatorname{sign}(u-k)\{(u-k)\varphi_t + [\mathbf{F}(u) - \mathbf{F}(k)] \cdot D\varphi\} dxdt \ge 0 \tag{13.3}$$

for all non-negative $\varphi \in C^1_o(S_T)$ and all $k \in \mathbb{R}$, where D denotes the gradient with respect to the space variables only.

The first notion of entropy solution is due to Lax [157, 158], and it amounts to (13.1). A more general notion, that would cover some cases of nonconvex $F(\cdot)$, and would ensure stability, still in one space dimension, was introduced by Oleinik [197, 198]. A formal derivation and a motivation of Kruzhkov notion of entropy solution (13.3) is in Section 13c of the Complements. When $N = 1$ the Kruzhkov and Lax notions are equivalent, as we show next.

13.2 Variational Solutions of (6.4) Are Entropy Solutions

Proposition 13.1 *Let u be the weak variational solution claimed by Theorem 11.1. Then for every convex function $\Phi \in C^2(\mathbb{R})$ and all non-negative $\varphi \in C_o^\infty(\mathbb{R} \times \mathbb{R}^+)$*

$$\iint_{\mathbb{R}\times\mathbb{R}^+} \left[\Phi(u)\varphi_t + \left(\int_k^u F'(s)\Phi'(s)ds\right)\varphi_x\right]dxdt \ge 0 \quad \textit{for all } \ k \in \mathbb{R}.$$

Corollary 13.1 *The variational solutions claimed by Theorem 11.1 are entropy solutions.*

Proof. Apply the proposition with $\Phi(s) = |s-k|$, modulo an approximation procedure. Then $\Phi'(s) = \mathrm{sign}(s-k)$ for $s \ne k$. ■

The proof of Proposition 13.1 uses the notion of Steklov averages of a function $f \in L^1_{\mathrm{loc}}(\mathbb{R} \times \mathbb{R}^+)$. These are defined as

$$f_h(x,t) = \frac{1}{h}\int_x^{x+h} f(s,t)ds, \qquad f_\ell(x,t) = \frac{1}{\ell}\int_t^{t+\ell} f(x,\tau)d\tau$$
$$f_{h\ell}(x,t) = \frac{1}{\ell}\int_t^{t+\ell} \frac{1}{h}\int_x^{x+h} f(s,\tau)dsd\tau$$

for all $h \in \mathbb{R}$ and all $\ell \in \mathbb{R}$ such that $t+\ell > 0$. One verifies that as $h,\ell \to 0$

$$\begin{array}{llll} f_h(\cdot,t) \to f(\cdot,t) & \text{in } L^1_{\mathrm{loc}}(\mathbb{R}) & \text{a.e. } t \in \mathbb{R}^+ \\ f_\ell(x,\cdot) \to f(x,\cdot) & \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^+) & \text{a.e. } x \in \mathbb{R} \\ f_{h\ell} \quad\;\; \to f & \text{in } L^1_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R}^+). \end{array}$$

Lemma 13.1 *The variational solutions of Theorem 11.1 satisfy the weak formulation*

$$\begin{aligned} \frac{\partial}{\partial t}u_{h\ell} + \frac{\partial}{\partial x}F_{h\ell}(u) &= 0 \quad \textit{in } \ \mathbb{R}\times\mathbb{R}^+ \\ u_{h\ell}(\cdot,0) &= \frac{1}{\ell}\int_0^\ell u_h(\cdot,\tau)d\tau. \end{aligned} \tag{13.4}$$

Moreover, $u_{h\ell}(\cdot,0) \to u_o$ in $L^1_{\mathrm{loc}}(\mathbb{R})$ as $h,\ell \to 0$.

Proof. Fix $(x,t) \in \mathbb{R}\times\mathbb{R}^+$ and $h \in \mathbb{R}$ and $\ell > 0$. Integrate (12.3) in $d\tau$ over $(t,t+\ell)$ and in ds over $(x,x+h)$, and divide by $h\ell$. Letting $n \to \infty$ proves the lemma. ■

Proof (of Proposition 13.1). Let $\Phi \in C^2(\mathbb{R})$ be convex and let $\varphi \in C_o^\infty(\mathbb{R} \times \mathbb{R}^+)$ be non-negative. Multiplying the first of (13.4) by $\Phi'(u_{h\ell})\varphi$ and integrating over $\mathbb{R}\times\mathbb{R}^+$, gives

$$
\begin{aligned}
-\iint_{\mathbb{R}\times\mathbb{R}^+} [\Phi(u_{h\ell})\varphi_t - F'(u_{h\ell})\Phi'(u_{h\ell})u_{h\ell x}\varphi]\, dxdt \\
= \iint_{\mathbb{R}\times\mathbb{R}^+} [F_{h,\ell}(u) - F(u_{h\ell})]\Phi''(u_{h\ell})u_{h\ell x}\varphi dxdt \\
+ \iint_{\mathbb{R}\times\mathbb{R}^+} [F_{h,\ell}(u) - F(u_{h\ell})]\Phi'(u_{h\ell})\varphi_x dxdt.
\end{aligned}
$$

The second term on the left-hand side is transformed by an integration by parts and equals

$$
\iint_{\mathbb{R}\times\mathbb{R}^+} \left(\int_k^{u_{h\ell}} F'(s)\Phi'(s)ds\right)\varphi_x dxdt
$$

where k is an arbitrary constant. Then let $\ell \to 0$ and $h \to 0$ in the indicated order to obtain

$$
\begin{aligned}
\iint_{\mathbb{R}\times\mathbb{R}^+} \left[\Phi(u)\varphi_t + \left(\int_k^u F'(s)\Phi'(s)ds\right)\varphi_x\right] dxdt \\
= -\lim_{h\to 0}\iint_{\mathbb{R}\times\mathbb{R}^+} [F_h(u) - F(u_h)]\Phi''(u_h)(u_h)_x\varphi dxdt.
\end{aligned}
$$

It remains to show that the right-hand side is non-negative. Since $F(\cdot)$ is convex, by Jensen's inequality $F_h(u) \ge F(u_h)$. By (9.1), for a.e. $(x,t) \in \mathbb{R}\times\mathbb{R}^+$

$$
(u_h)_x = \frac{\partial}{\partial x}\fint_x^{x+h} u(s,t)ds = \frac{u(x+h,t) - u(x,t)}{h} \le \frac{L}{t}.
$$

$$
\begin{aligned}
-\lim_{h\to 0}\iint_{\mathbb{R}\times\mathbb{R}^+} [F_h(u) - F(u_h)]\Phi''(u_h)(u_h)_x\varphi dxdt \\
\ge \lim_{h\to 0}\iint_{\mathbb{R}\times\mathbb{R}^+} \Phi''(u_h)[F_h(u) - F(u_h)]\varphi\frac{L}{\tau} dxdt = 0.
\end{aligned}
$$

■

13.3 Remarks on the Shock and the Entropy Conditions

Let u be an entropy solution of (13.2), discontinuous across a smooth hypersurface Γ. The notion (13.3) contains information on the nature of the discontinuities of u across Γ. In particular, it does include the shock condition (4.4) and a weak form of the entropy condition (13.1).

If $P \in \Gamma$, the ball $B_\rho(P)$ centered at P with radius ρ is divided by Γ, at least for small ρ, into B_ρ^+ and B_ρ^- as in Figure 13.1. Let $\boldsymbol{\nu} = (\nu_t; \nu_{x_1}, \dots, \nu_{x_N}) = (\nu_t; \boldsymbol{\nu}_x)$ denote the unit normal oriented toward B_ρ^+.

We assume that $u \in C^1(\bar{B}_\rho^\pm)$ and that it satisfies the equation in (13.2) in the classical sense in $B_\rho^\pm$. In (13.3) take a non-negative test function $\varphi \in$

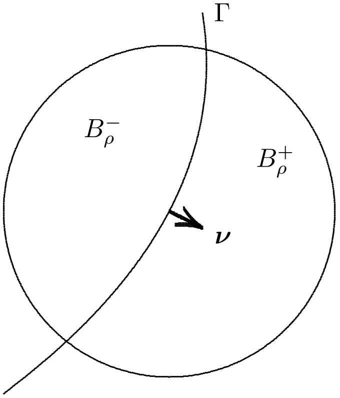

Fig. 13.1

$C_o^\infty(B_\rho(P))$ and integrate by parts by means of Green's theorem. This gives, for all $k \in \mathbb{R}$

$$\begin{aligned}\int_\Gamma &\operatorname{sign}(u^+ - k)\{(u^+ - k)\nu_t + [\mathbf{F}(u^+) - \mathbf{F}(k)] \cdot \boldsymbol{\nu}_x\}\varphi d\sigma \\ &\le \int_\Gamma \operatorname{sign}(u^- - k)\{(u^- - k)\nu_t + [\mathbf{F}(u^-) - \mathbf{F}(k)] \cdot \boldsymbol{\nu}_x\}\varphi d\sigma\end{aligned}$$

where $d\sigma$ is the surface measure on Γ and $u^\pm$ are the limits of $u(x,t)$ as (x,t) tends to Γ from $B_\rho^\pm$. Since $\varphi \ge 0$ is arbitrary, this gives the pointwise inequality

$$\begin{aligned}\operatorname{sign}(u^+ - k)&\{(u^+ - k)\nu_t + [\mathbf{F}(u^+) - \mathbf{F}(k)] \cdot \boldsymbol{\nu}_x\} \\ &\le \operatorname{sign}(u^- - k)\{(u^- - k)\nu_t + [\mathbf{F}(u^-) - \mathbf{F}(k)] \cdot \boldsymbol{\nu}_x\}\end{aligned} \tag{13.5}$$

on Γ. If $k > \max\{u^+, u^-\}$, (13.5) implies

$$([u^+ - u^-], [\mathbf{F}(u^+) - \mathbf{F}(u^-)]) \cdot \boldsymbol{\nu} \ge 0$$

and if $k < \min\{u^+, u^-\}$

$$([u^+ - u^-], [\mathbf{F}(u^+) - \mathbf{F}(u^-)]) \cdot \boldsymbol{\nu} \le 0.$$

Therefore, the surface of discontinuity Γ must satisfy the shock condition (4.4). Next, in (13.5), take $k = \frac{1}{2}[u^+ + u^-]$, to obtain

$$\operatorname{sign}[u^+ - u^-]\big[\mathbf{F}(u^+) + \mathbf{F}(u^-) - 2\mathbf{F}(k)\big]\boldsymbol{\nu}_x \le 0. \tag{13.6}$$

This is an N-dimensional generalized version of the entropy condition (13.1).

Lemma 13.2 *If $N = 1$ and $F(\cdot)$ is convex, then (13.6) implies (13.1).*

Proof. If $N = 1$, Γ is a curve in $\mathbb{R}^2$, and we may orient it, locally, so that $\boldsymbol{\nu} = (\nu_t, \nu_x)$, and $\nu_x \ge 0$. Since $F(\cdot)$ is convex, (8.3) implies that

$$F(u^{\pm}) - F(k) \ge F'(k)(u^{\pm} - k).$$

Adding these two inequalities gives

$$[F(u^+) + F(u^-) - 2F(k)]\nu_x \ge 0.$$

This in (13.6) implies $\text{sign}[u^+ - u^-] \le 0$. ∎

14 The Kruzhkov Uniqueness Theorem

Theorem 14.1. *Let u and v be two entropy solutions of (13.2) satisfying in addition*

$$\left\| \frac{\mathbf{F}(u) - \mathbf{F}(v)}{u - v} \right\|_{\infty, S_T} \le M \qquad \textit{for some } \ M > 0. \tag{14.1}$$

Then $u = v$.

Remark 14.1 The assumption (14.1) is satisfied if $\mathbf{F} \in C^1(\mathbb{R})$ and the solutions are bounded. In particular

Corollary 14.1 *There exists at most one bounded entropy solution to the initial value problem (6.4).*

14.1 Proof of the Uniqueness Theorem I

Lemma 14.1 *Let u and v be any two entropy solutions of (13.2). Then for every non-negative $\varphi \in C_o^\infty(S_T)$*

$$\iint_{S_T} \text{sign}(u - v)\{(u - v)\varphi_t + [\mathbf{F}(u) - \mathbf{F}(v)] \cdot D\varphi\} dxdt \ge 0. \tag{14.2}$$

Proof. For $\varepsilon > 0$, let J_ε be the Friedrichs mollifying kernels, and set

$$\delta_\varepsilon\left(\frac{x - y}{2}, \frac{t - \tau}{2}\right) = J_\varepsilon\left(\frac{t - \tau}{2}\right) J_\varepsilon\left(\frac{|x - y|}{2}\right).$$

Let $\varphi \in C_o^\infty(S_T)$ be non-negative and assume that its support is contained in the cylinder $B_R \times (s_1, s_2)$ for some $R > 0$ and $\varepsilon < s_1 < s_2 < T - \varepsilon$. Set

$$\lambda(x, t; y, \tau) = \varphi\left(\frac{x + y}{2}, \frac{t + \tau}{2}\right) \delta_\varepsilon\left(\frac{x - y}{2}, \frac{t - \tau}{2}\right). \tag{14.3}$$

The function λ is compactly supported in $S_T \times S_T$, with support contained in

$$\left[\frac{|x+y|}{2} < R\right] \cap \left[\frac{|x-y|}{2} < \varepsilon\right] \;;\; \left[s_1 < \frac{|t+\tau|}{2} < s_2\right] \cap \left[\frac{|t-\tau|}{2} < \varepsilon\right].$$

The variables of integration in (13.3) are x and t. We take $k = v(y, \tau)$ for a.e. $(y, \tau) \in S_T$ and integrate in $dyd\tau$ over S_T. This gives

$$\iint_{S_T} \iint_{S_T} \operatorname{sign}[u(x,t) - v(y,\tau)]\{[u(x,t) - v(y,\tau)]\lambda_t \\ + [\mathbf{F}(u(x,t)) - \mathbf{F}(v(y,\tau))] \cdot \nabla_x \lambda\} dxdtdyd\tau \ge 0.$$

Analogously, one may write (13.3) for v in the variables of integration y, τ, and take $k = u(x, t)$. Integrating in $dxdt$ over S_T gives an analogous inequality with λ_t and $D_x\lambda$ replaced by λ_τ and $D_y\lambda$. Adding these two inequalities gives

$$\iint_{S_T} \iint_{S_T} \{|u(x,t) - v(y,\tau)|(\lambda_t + \lambda_\tau) + \operatorname{sign}[u(x,t) - v(y,\tau)] \\ + [\mathbf{F}(u(x,t) - \mathbf{F}(v(y,\tau)] \cdot (D_x\lambda + D_y\lambda)\} dxdtdyd\tau \ge 0. \tag{14.4}$$

To transform this integral, compute from (14.3)

$$\lambda_t + \lambda_\tau = \varphi_t\left(\frac{x+y}{2}, \frac{t+\tau}{2}\right)\delta_\varepsilon\left(\frac{x-y}{2}, \frac{t-\tau}{2}\right)$$
$$D_x\lambda + D_y\lambda = D\varphi\left(\frac{x+y}{2}, \frac{t+\tau}{2}\right)\delta_\varepsilon\left(\frac{x-y}{2}, \frac{t-\tau}{2}\right).$$

Then, in the resulting integral, make the change of variables

$$\frac{x+y}{2} = \xi, \quad \frac{t+\tau}{2} = s; \qquad \frac{x-y}{2} = \eta, \quad \frac{t-\tau}{2} = \sigma.$$

The domain of integration is mapped into

$$\{[|\xi| < R] \times [s_1, s_2]\} \times \{\, [|\eta| < \varepsilon] \times [|\sigma| < \varepsilon]\}$$

and (14.4) is transformed into

$$\iint_{S_T} \varphi_t(\xi, s)\Big\{ \iint_{S_T} |u(\xi+\eta, s+\sigma) - v(\xi-\eta, s-\sigma)|\delta_\varepsilon(\eta,\sigma)d\eta d\sigma\Big\} d\xi ds \\ + \iint_{S_T} D\varphi(\xi, s) \cdot \Big\{ \iint_{S_T} \operatorname{sign}[u(\xi+\eta, s+\sigma) - v(\xi-\eta, s-\sigma)] \\ \times [\mathbf{F}(u(\xi+\eta, s+\sigma) - \mathbf{F}(v(\xi-\eta, s-\sigma)]\delta_\varepsilon(\eta,\sigma)d\eta d\sigma\Big\} d\xi ds \ge 0.$$

By the properties of mollifiers the integrals in $\{\cdots\}$ converge respectively to

$$|u(\xi,s) - v(\xi,s)| \quad \text{and} \quad \operatorname{sign}[u(\xi,s) - v(\xi,s)][\mathbf{F}(u(\xi,s) - \mathbf{F}(v(\xi,s))]$$

for a.e. $(\xi, s) \in S_T$. Moreover, they are uniformly bounded in ε, for a.e. $(\xi, s) \in \operatorname{supp}(\varphi)$. Therefore (14.2) follows by letting $\varepsilon \to 0$ in the previous expression and passing to the limit under the integrals. ■

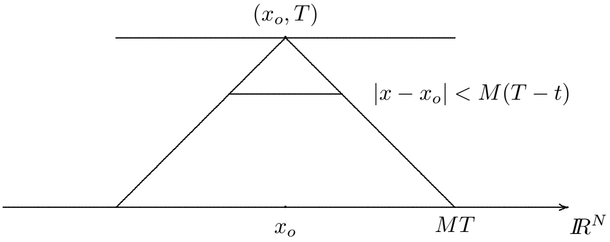

Fig. 14.1

14.2 Proof of the Uniqueness Theorem II

Fix $x_o \in \mathbb{R}^N$ and $R > 0$ and construct the backward characteristic cone of "slope" M

$$[|x - x_o| < M(T-t)] \times [0 < t < T].$$

The cross section of this cone with the hyperplane $t = \text{const}$, for $0 < t < T$, is the ball $|x - x_o| < M(T-t)$. The uniqueness theorem is a consequence of the following

Proposition 14.1 *For all $x_o \in \mathbb{R}^N$ and for almost all $0 < \tau < t < T$*

$$\int_{|x-x_o|<M(T-t)} |u - v|(x,t)dx \le \int_{|x-x_o|<M(T-\tau)} |u-v|(x,\tau)dx. \tag{14.5}$$

Proof. Assume $x_o = 0$ and in (14.2) take

$$\varphi(x,t) = \int_{t-t_1}^{t-\tau} J_\varepsilon(s)ds \int_{|x|-M(T-t)+\varepsilon}^{\infty} J_\varepsilon(s)ds \tag{14.6}$$

where $\varepsilon < \tau < t_1 < T - \varepsilon$ are arbitrary but fixed. Such a φ is admissible since it is non-negative, is in $C^\infty(S_T)$, and vanishes outside the truncated backward cone

$$\big[|x| < M(T-t)\big] \times \big[\tau - \varepsilon < t < t_1 + \varepsilon\big].$$

Compute

$$\begin{aligned}\varphi_t &= [J_\varepsilon(t-\tau) - J_\varepsilon(t-t_1)] \int_{|x|-M(T-t)+\varepsilon}^{\infty} J_\varepsilon(s)ds \\ &\quad - J_\varepsilon(|x| - M(T-t) + \varepsilon)M \int_{t-t_1}^{t-\tau} J_\varepsilon(s)ds \\ D\varphi &= -J_\varepsilon(|x| - M(T-t)+\varepsilon)\frac{x}{|x|}\int_{t-t_1}^{t-\tau} J_\varepsilon(s)ds.\end{aligned}$$

Put this in (14.2) and change the sign to obtain

$$
\begin{aligned}
\iint_{S_T} & [J_\varepsilon(t-t_1) - J_\varepsilon(t-\tau)]|u-v| \int_{|x|-M(T-t)+\varepsilon}^{\infty} J_\varepsilon(s)dsdxdt \\
&\le \iint_{S_T} \int_{t-t_1}^{t-\tau} J_\varepsilon(s)dsJ_\varepsilon(|x|-M(T-t)+\varepsilon) \\
&\quad \cdot \{|\mathbf{F}(u)-\mathbf{F}(v)| - M|u-v|\}dxdt.
\end{aligned}
$$

By virtue of (14.1), the right-hand side is nonpositive. Letting $\varepsilon \to 0$, by the properties of the mollifiers, the left-hand side converges to

$$
\int_{|x|<M(T-t_1)} |u-v|(x,t_1)dx - \int_{|x|<M(T-\tau)} |u-v|(x,\tau)dx. \qquad \blacksquare
$$

14.3 Stability in $L^1(\mathbb{R}^N)$

Let u and v be entropy solutions of (13.2) defined in the whole of S_∞. Fix $T > 0$ and rewrite (14.5) as

$$
\int_{|y|<M(T-t)} |u-v|(x_o-y,t)dy \le \int_{|y|<M(T-\tau)} |u-v|(x_o-y,\tau)dy.
$$

Integrating this in dx_o over $\mathbb{R}^N$ gives

$$
\|u-v\|_{1,\mathbb{R}^N}(t) \le \left(\frac{T-\tau}{T-t}\right)^N \|u-v\|_{1,\mathbb{R}^N}(\tau).
$$

Since the solutions u and v are global in time, let $T \to \infty$, and deduce that the function $t \to \|u-v\|_{1,\mathbb{R}^N}(t)$ is nondecreasing.

Theorem 14.2. *Let u and v be, global-in-time, entropy solutions of (13.2) originating from initial data u_o and v_o in $L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, and let (14.1) hold. Then*

$$
\|u-v\|_{1,\mathbb{R}^N}(t) \le \|u_o-v_o\|_{1,\mathbb{R}^N} \quad \textit{for a.e. } t > 0.
$$

15 The Maximum Principle for Entropy Solutions

Proposition 15.1 *Let u and v be any two weak entropy solutions of (13.2). Then for all $x_o \in \mathbb{R}^N$*

$$
\int_{|x-x_o|<M(T-t)} (u-v)_+(x,t)dx \le \int_{|x-x_o|<M(T-\tau)} (u-v)_+(x,\tau)dx
$$

for a.e. $0 < \tau < t < T$.

Proof. Since u and v are weak solutions of (13.2) in the sense of (5.4)–(5.5), starting from these, we may arrive at an analogue of (14.4) with equality and without the extra factor $\operatorname{sign}[u(x,t) - v(y,\tau)]$. Precisely, starting from (5.4) written for

$$u - v(y,\tau) \quad \text{and} \quad [\mathbf{F}(u(x,t)) - \mathbf{F}(v(y,\tau))]$$

choose φ and λ as in (14.3) and proceed as before to arrive at

$$\iint_{S_T}\iint_{S_T} \{[u(x,t) - v(y,\tau)](\lambda_t + \lambda_\tau) \\ + [\mathbf{F}(u(x,t)) - \mathbf{F}(v(y,\tau))]\cdot(D_x\lambda + D_y\lambda)\} dxdtdyd\tau = 0.$$

Add this to (14.4) and observe that

$$|u - v| + (u - v) = 2(u - v)_+$$

to obtain

$$\iint_{S_T}\iint_{S_T} \big\{2[u(x,t) - v(y,\tau)]_+(\lambda_t + \lambda_\tau) + \big(1 + \operatorname{sign}[u(x,t) - v(y,\tau)]\big) \\ \times [\mathbf{F}(u(x,t)) - \mathbf{F}(v(y,\tau))]\cdot(D_x\lambda + D_y\lambda)\big\} dxdtdyd\tau \ge 0.$$

Proceeding as in the proof of Lemma 14.1, we arrive at the inequality

$$\iint_{S_T} \{2(u - v)_+\varphi_t + (1 + \operatorname{sign}[u - v])[\mathbf{F}(u) - \mathbf{F}(v)]\cdot D\varphi\} dxdt \ge 0$$

for all non-negative $\varphi \in C_o^\infty(S_T)$. Take φ as in (14.6) and let $\varepsilon \to 0$. ■

Corollary 15.1 *Let u and v be two entropy solutions of (13.2), and let (14.1) hold. Then $u_o \ge v_o$ implies $u \ge v$ in S_T.*

Corollary 15.2 *Let u be an entropy solution of (13.2), and let (14.1) hold. Then*

$$\|u(\cdot,t)\|_{\infty,\mathbb{R}^N} \le \|u_o\|_{\infty,\mathbb{R}^N} \quad \textit{for a.e. } 0 < t < T.$$

Corollary 15.3 *Assume that $u_o \in L^\infty(\mathbb{R}^N)$ and let $\mathbf{F} \in C^1(\mathbb{R})$. Then there exists at most one bounded, weak entropy solution to the initial value problem (13.2).*

Problems and Complements

3c Solving the Cauchy Problem

3.1. Solve $x \cdot \nabla u = \alpha$ with Cauchy data $h(\cdot)$ on the hyperplane $x_N = 1$, that is

$$\begin{gathered} \Gamma = \{\xi_i = s_i\}, \quad i = 1, \dots, N-1 \\ \xi_N = 1, \quad \zeta(s) = h(s) \in C^1(\mathbb{R}^{N-1}). \end{gathered}$$

Denote by $\bar{x} = (x_1, \dots, x_{N-1})$ points in $\mathbb{R}^{N-1}$ and by $(\bar{x}, x_N)$, points in $\mathbb{R}^N$. An integral surface is given by

$$\bar{x}(s,t) = se^t, \quad x_N(s,t) = e^t, \quad z(s,t) = h(s) + \alpha t.$$

For $x_N > 0$, we have $s = \bar{x}/x_N$, and the solution is given by

$$u(x) = h\left(\frac{\bar{x}}{x_N}\right) + \ln x_N^\alpha.$$

3.2. Let u solve the linear equation $a_i(x) u_{x_i} = \gamma u$ for some $\gamma \in \mathbb{R}$. Show that the general solution is given by $x \to u_o(x) u(x)$, where u_o is a solution of the associated homogeneous equation.

3.3. Show that the characteristic projections of

$$y u_x + x u_y = \gamma u, \quad \gamma > 0 \tag{3.1c}$$

are the curves

$$\begin{aligned} x(t) &= x_o \cosh t + y_o \sinh t \\ y(t) &= x_o \sinh t + y_o \cosh t \end{aligned} \qquad (x_o, y_o) \in \mathbb{R}^2$$

and observe in particular that the lines $x = \pm y$ are characteristic.

3.4. Show that the general solution of the homogeneous equation associated with (3.1c) is $f(x^2 - y^2)$, for any $f \in C^1(\mathbb{R})$.

3.5. Solve (3.1c) for Cauchy data $u(\cdot, 0) = h \in C^1(\mathbb{R})$. The integral surfaces are

$$x(s,t) = s \cosh t, \quad y(s,t) = s \sinh t, \quad z(s,t) = h(s) e^{\gamma t}.$$

The solution exists in the sector $|x| > |y|$, and it is given by

$$u(x,t) = h(\sqrt{x^2 - y^2}) \left(\frac{x+y}{x-y}\right)^{\gamma/2}.$$

This is discontinuous at $x = y$ and continuous but not of class C^1 at $x = -y$. Explain in terms of characteristics.

3.6. Consider (3.1c) with data on the characteristic $x = y$, that is

$$u(x, x) = h(x) \in C^1(\mathbb{R}).$$

In general, the problem is not solvable. Following the method of Section 3, we find the integral surfaces

$$\begin{aligned} x(s,t) &= s(\cosh t + \sinh t) \\ y(s,t) &= s(\cosh t + \sinh t) \\ z(s,t) &= h(s)e^{\gamma t}. \end{aligned}$$

From these compute

$$se^t = \frac{x+y}{2}, \qquad u(x,y) = \left(\frac{x+y}{2}\right)^{\gamma} \frac{h(s)}{s^{\gamma}}.$$

Therefore the problem is solvable only if $h(s) = Cs^{\gamma}$. It follows from **3.4** that $C = f(x^2 - y^2)$, for any $f \in C^1(\mathbb{R})$.

3.7. Show that the characteristic projections of $yu_x - xu_y = \gamma u$, are the curves

$$\begin{aligned} x(t) &= x_o \cos t + y_o \sin t \\ y(t) &= y_o \cos t - x_o \sin t \end{aligned} \qquad (x_o, y_o) \in \mathbb{R}^2.$$

Solve the Cauchy problem with data $u(x, 0) = h(x)$. Show that if $\gamma = 0$ then the Cauchy problem is globally solvable only if $h(\cdot)$ is symmetric.

6c Explicit Solutions to the Burgers Equation

6.1. Verify that for $\lambda > 0$, the following are families of weak solutions to the Burgers equations in $\mathbb{R} \times \mathbb{R}^+$.

$$W^+(x,t) = \begin{cases} 0 & \text{for } x < 0 \\ \dfrac{x}{t} & \text{for } 0 \le x \le \sqrt{2\lambda t} \\ 0 & \text{for } x > \sqrt{2\lambda t}. \end{cases} \tag{6.1c}$$

$$W^-(x,t) = \begin{cases} 0 & \text{for } x < -\sqrt{2\lambda t} \\ \dfrac{x}{t} & \text{for } -\sqrt{2\lambda t} \le x \le 0 \\ 0 & \text{for } x > 0. \end{cases} \tag{6.2c}$$

$$U(x,t) = \begin{cases} 0 & \text{for } x < 0 \\ \dfrac{x}{t} & \text{for } 0 \le x \le \dfrac{\lambda}{2} \\ \dfrac{x-\lambda}{t} & \text{for } \dfrac{\lambda}{2} < x \le \lambda \\ 0 & \text{for } x > \lambda. \end{cases} \tag{6.3c}$$

6.2c Invariance of Burgers Equations by Some Transformation of Variables

Let φ be a solution of Burgers equation in $\mathbb{R}\times\mathbb{R}^+$. Verify that for all $a, b, c \in \mathbb{R}$ the following transformed functions are also solutions of Burgers equation:

$$
\begin{aligned}
u(x,t) &= \varphi(x+a, t+b) \quad \text{ for } \ t > -b && \text{(i)}\\
u(x,t) &= a + \varphi(x - at, t) && \text{(ii)}\\
u(x,t) &= a\varphi(bx, abt) \overset{\text{def}}{=} T_{a,b}\varphi && \text{(iii)}\\
u(x,t) &= \frac{x}{t} + \frac{a}{t}\varphi\Big(\frac{bx}{t}, c - \frac{ab}{t}\Big) \quad \text{ for } \ t > \frac{ab}{c}. && \text{(iv)}
\end{aligned}
$$

6.2. Assume that a weak solution φ is known of the initial value problem

$$\varphi_t + \frac{1}{2}(\varphi^2)_x = 0 \ \text{ in } \ \mathbb{R}\times\mathbb{R}^+, \quad \varphi(\cdot,0) = \varphi_o.$$

where φ_o is subject to proper assumptions that would ensure existence of such a φ. Find a solutions of the initial value problems

$$
\begin{aligned}
u_t + \frac{1}{2}(u^2)_x &= 0 \ \text{ in } \ \mathbb{R}\times\mathbb{R}^+ \qquad & u_t + \frac{1}{2}(u^2)_x &= 0 \ \text{ in } \ \mathbb{R}\times\mathbb{R}^+\\
u(\cdot,0) &= a + \varphi_o; & u(\cdot,0) &= \gamma x + \varphi_o
\end{aligned}
$$

A solution of the first is

$$u(x,t) = a + \varphi(x - at, t)$$

and a solution of the second is

$$u(x,t) = \frac{\gamma x}{1+\gamma t} + \frac{1}{1+\gamma t}\varphi\Big(\frac{x}{1+\gamma t}, \frac{t}{1+\gamma t}\Big).$$

Note that the initial values of these solutions do not satisfy the assumptions (8.2).

6.3. Prove that those solutions of Burgers equations for which $\varphi = T_{1,b}\varphi$, are of the form $f(x/t)$.

6.4. Prove that those solutions of Burgers equations for which $\varphi = T_{b,b}\varphi$, are of the form $f(\sqrt{x/t})/\sqrt{t}$.

6.3c The Generalized Riemann Problem

Consider the initial value problem

$$
\begin{aligned}
u_t + \frac{1}{2}(u^2)_x &= 0 \ \text{ in } \ \mathbb{R}\times\mathbb{R}^+\\
u(x,0) &= \begin{cases} \alpha + px & \text{for } \ x < 0\\ \beta + qx & \text{for } \ x > 0 \end{cases}
\end{aligned}
\tag{6.4c}
$$

where α, β, p, q are given constants. Verify that if $\alpha \le \beta$, then the solution to (6.4c) is

$$u(x,t) = \begin{cases} \dfrac{\alpha + px}{1+pt} & \text{for } \ x \le \alpha t \\[2ex] \dfrac{x}{t} & \text{for } \ \alpha t \le x \le \beta t \\[2ex] \dfrac{\beta + qx}{1+qt} & \text{for } \ x \ge \beta t \end{cases} \qquad (\alpha \le \beta)$$

for all times $1 + (\alpha \wedge \beta)t > 0$. If $\alpha > \beta$, the characteristics from the left of $x = 0$ intersect the characteristics from the right. Let $x = x(t)$ be the line of discontinuity and verify that a weak solution is given by

$$u(x,t) = \begin{cases} \dfrac{\alpha + px}{1+pt} & \text{for } \ x < x(t) \\[2ex] \dfrac{\beta + qx}{1+qt} & \text{for } \ x > x(t) \end{cases} \qquad (\alpha > \beta)$$

where $x = x(t)$ satisfies the shock condition (4.7). Enforcing it gives

$$x'(t) = \frac{1}{2}\Big(\frac{\alpha + px(t)}{1+pt} + \frac{\beta + qx(t)}{1+qt}\Big).$$

Solve this ODE to find

$$x(t) = \frac{\alpha\sqrt{1+qt} + \beta\sqrt{1+qt}}{\sqrt{1+pt} + \sqrt{1+qt}}t.$$

13c The Entropy Condition

Solutions of (13.2) can be constructed by solving first the Cauchy problems

$$\begin{aligned} u_{\varepsilon,t} - \varepsilon \Delta u_\varepsilon + \operatorname{div} \mathbf{F}(u_\varepsilon) = 0 \ \text{ in } \ S_T \\ u_\varepsilon(\cdot, 0) = u_o \end{aligned}$$

and then letting $\varepsilon \to 0$. Roughly speaking, as $\varepsilon \to 0$, the term $\varepsilon \Delta u_\varepsilon$ "disappears" and the solution is found as the limit, in a suitable topology, of the net $\{u_\varepsilon\}$. The method can be made rigorous by estimating $\{u_\varepsilon\}$, uniformly in ε, in the class of functions of *bounded variation* ([265]).

In what follows we assume that a priori estimates have been derived that ensure that $\{u_\varepsilon\} \to u$ in $L^1_{\text{loc}}(S_T)$. Let $k \in \mathbb{R}$ and write the PDE as

$$\frac{\partial}{\partial t}(u_\varepsilon - k) - \varepsilon \Delta (u_\varepsilon - k) + \operatorname{div}[\mathbf{F}(u_\varepsilon) - \mathbf{F}(k)] = 0.$$

Let $h_\delta(\cdot)$ be the approximation to the Heaviside function introduced in (14.2) of Chapter 5. Multiply the PDE by $h_\delta(u_\varepsilon - k)\varphi$, where $\varphi \in C_o^\infty(S_T)$ is non-negative and integrate by parts over S_T to obtain

$$\begin{aligned}\iint_{S_T} \Big\{\frac{\partial}{\partial t}\Big(\int_0^{u_\varepsilon - k} h_\delta(s)ds\Big)\varphi dxdt + \varepsilon h'_\delta(u_\varepsilon - k)|Du_\varepsilon|^2\varphi \\ + \varepsilon h_\delta(u_\varepsilon - k)D(u_\varepsilon - k)\cdot D\varphi \\ + h_\delta(u_\varepsilon - k)[\mathbf{F}(u_\varepsilon) - \mathbf{F}(k)]\cdot D\varphi \\ + h'_\delta(u_\varepsilon - k)[\mathbf{F}(u_\varepsilon) - \mathbf{F}(k)]\cdot D(u_\varepsilon - k)\varphi\Big\}dxdt = 0.\end{aligned}$$

First let $\delta \to 0$ and then let $\varepsilon \to 0$. The various terms are transformed and estimated as follows.

$$\begin{aligned}\lim_{\varepsilon\to 0}\lim_{\delta\to 0}\iint_{S_T}\frac{\partial}{\partial t}\Big(\int_0^{u_\varepsilon - k} h_\delta(s)ds\Big)\varphi dxdt \\ = -\lim_{\varepsilon\to 0}\lim_{\delta\to 0}\iint_{S_T}\Big(\int_0^{u_\varepsilon - k} h_\delta(s)ds\Big)\varphi_t dxdt \\ = -\iint_{S_T}|u - k|\varphi_t dxdt.\end{aligned}$$

The second term on the left-hand side is non-negative and is discarded. Next

$$\begin{aligned}\lim_{\varepsilon\to 0}\lim_{\delta\to 0}\iint_{S_T}\varepsilon h_\delta(u_\varepsilon - k)D(u_\varepsilon - k)\cdot D\varphi dxdt \\ = \lim_{\varepsilon\to 0}\lim_{\delta\to 0}\iint_{S_T}\varepsilon D\Big(\int_0^{u_\varepsilon - k} h_\delta(s)ds\Big)\cdot D\varphi dxdt \\ = -\lim_{\varepsilon\to 0}\lim_{\delta\to 0}\iint_{S_T}\varepsilon\Big(\int_0^{u_\varepsilon - k} h_\delta(s)ds\Big)\Delta\varphi dxdt = 0.\end{aligned}$$

$$\begin{aligned}\lim_{\varepsilon\to 0}\lim_{\delta\to 0}\iint_{S_T} h_\delta(u_\varepsilon - k)[\mathbf{F}(u_\varepsilon) - \mathbf{F}(k)]\cdot D\varphi dxdt \\ = \iint_{S_T}\operatorname{sign}(u - k)[\mathbf{F}(u) - \mathbf{F}(k)]\cdot D\varphi dxdt.\end{aligned}$$

The last term is transformed and estimated as

$$\begin{aligned}\iint_{S_T}\operatorname{div}\Big(\int_0^{u_\varepsilon} h'_\delta(s - k)[\mathbf{F}(s) - \mathbf{F}(k)]ds\Big)\varphi dxdt \\ = -\iint_{S_T}\Big(\int_0^{u_\varepsilon} h'_\delta(s - k)[\mathbf{F}(s) - \mathbf{F}(k)]ds\Big)\cdot D\varphi dxdt.\end{aligned}$$

For $\varepsilon > 0$ fixed

$$\lim_{\delta\to 0} h'_\delta(s-k)[\mathbf{F}(s)-\mathbf{F}(k)] = 0 \qquad \text{a.e. } s\in(0,u_\varepsilon).$$

Moreover, by (14.1)

$$0 \le h'_\delta(s-k)[\mathbf{F}(s)-\mathbf{F}(k)] \le M.$$

Therefore by dominated convergence

$$\lim_{\delta\to 0}\iint_{S_T} h'_\delta(u_\varepsilon-k)[\mathbf{F}(u_\varepsilon)-\mathbf{F}(k)]\cdot D(u_\varepsilon-k)\varphi dxdt = 0.$$

Combining these remarks yields (13.3).

14c The Kruzhkov Uniqueness Theorem

The theorem of Kruzhkov holds for the following general initial value problem

$$\begin{aligned} u_t - \operatorname{div}\mathbf{F}(x,t,u) &= g(x,t,u) \ \text{ in } \ S_T \\ u(\cdot,0) &= u_o \in L^1_{\text{loc}}(\mathbb{R}^N). \end{aligned} \tag{14.1c}$$

A function $u\in L^\infty_{\text{loc}}(S_T)$ is an entropy solution of (14.1c) if for all $k\in\mathbb{R}$

$$\begin{aligned} \iint_{S_T} \operatorname{sign}(u-k)\big\{&(u-k)\varphi_t + [\mathbf{F}(x,t,u)-\mathbf{F}(x,t,k)]\cdot D\varphi \\ &+ [F_{i,x_i}(x,t,u)+g(x,t,u)]\varphi\big\}dxdt \ge 0 \end{aligned} \tag{14.2c}$$

provided the various integrals are well defined. Assume

$$g, F_i \in C^1(S_T\times\mathbb{R}) \quad i=1,\dots,N. \tag{14.3c}$$

Moreover

$$\begin{aligned} \Big\|\frac{\mathbf{F}(x,t,u)-\mathbf{F}(x,t,v)}{u-v}\Big\|_{\infty,S_T\times\mathbb{R}} &\le M_o \\ \sum_{i=1}^N \Big\|\frac{F_{i,x_i}(x,t,u)-F_{i,x_i}(x,t,v)}{u-v}\Big\|_{\infty,S_T\times\mathbb{R}} &\le M_1 \\ \Big\|\frac{\mathbf{F}(x,t,u)-\mathbf{F}(x,t,v)}{u-v}\Big\|_{\infty,S_T\times\mathbb{R}} &\le M_2 \\ \Big\|\frac{g(x,t,u)-g(x,t,v)}{u-v}\Big\|_{\infty,S_T\times\mathbb{R}} &\le M_3 \end{aligned} \tag{14.4c}$$

for given positive constants M_i, $i=0,1,2,3$. The initial datum is taken in the sense of $L^1_{\text{loc}}(\mathbb{R}^N)$. Set $M=\max\{M_o,M_1,M_2,M_3\}$.

Theorem 14.1c. *Let u and v be two entropy solutions of (14.1c) and let (14.3c)–(14.4c) hold. There exists a constant γ dependent only on N and the numbers M_i, $i=0,1,2,3$, such that for all $T>0$ and all $x_o\in\mathbb{R}^N$*

$$\int_{|x-x_o|<M(T-t)} |u-v|(x,t)dx \le e^{\gamma t}\int_{|x-x_o|<MT} |u_o-v_o|dx$$

for a.e. $0<t<T$.

8

NONLINEAR EQUATIONS OF FIRST ORDER

1 Integral Surfaces and Monge's Cones

A first-order nonlinear PDE is an expression of the form

$$F(x,u,\nabla u)=0 \tag{1.1}$$

where x ranges over a given region $E\subset\mathbb{R}^N$, the function u is in $C^1(E)$ and F is a given smooth real-valued function of its arguments. If u is a solution of (1.1), then its graph $\Sigma(u)$ is an *integral surface* for (1.1). Conversely, a surface Σ is an integral surface for (1.1) if it is the graph of a smooth function u solution of (1.1). For a fixed $(x,z)\in E\times\mathbb{R}$, consider the associated equation $F(x,z,p)=0$ and introduce the set

$$\mathbf{P}(x,z)=\{\text{the set of all } p\in\mathbb{R}^N \text{ satisfying } \ F(x,z,p)=0\}.$$

If $\Sigma(u)$ is an integral surface for (1.1), then for every $(x,z)\in\Sigma(u)$

$$z=u(x)\quad \text{ and }\quad p=\nabla u(x). \tag{1.2}$$

Therefore solving (1.1) amounts to finding a function $u\in C^1(E)$ such that for all $x\in E$, among the pairs (x,z) there is one for which (1.2) holds. Let Σ be an integral surface for (1.1). For $(x_o,z_o)\in\Sigma$ consider the family of planes

$$z-z_o=p\cdot(x-x_o),\qquad p\in\mathbf{P}(x,z). \tag{1.3}$$

Since Σ is an integral surface, among these there must be one tangent Σ at (x_o,z_o). The envelope of such a family of planes is a cone $C(x_o,z_o)$, called Monge's cone with vertex at (x_o,z_o). Thus the integral surface Σ is tangent, at each of its points, to the Monge's cone with vertex at that point.[1]

---

[1] Gaspard Monge, Beaune, France 1746–Paris 1818, combined equally well his scientific vocation with his political aspirations. He took part in the French Rev-

© Springer Nature Switzerland AG 2023

E. DiBenedetto, U. Gianazza, *Partial Differential Equations*,
Cornerstones, https://doi.org/10.1007/978-3-031-46618-2_9

1.1 Constructing Monge's Cones

The envelope of the family of planes in (1.3) is that surface $\mathbb{S}$ tangent, at each of its points, to one of the planes of the family (1.3). Thus for each $(x, z) \in \mathbb{S}$, there exists $p = p(x)$ such that the corresponding plane in (1.3), for such a choice of p, has the same normal as $\mathbb{S}$. These remarks imply that the equation of $\mathbb{S}$ is

$$z - z_o = p(x) \cdot (x - x_o). \tag{1.4}$$

The tangency requirement can be written as

$$\underbrace{p_j(x) + p_{i,x_j}(x_i - x_{o,i})}_{\substack{j\text{th component of the}\\ \text{normal to } \mathbb{S} \text{ at } x}} \quad = \quad \underbrace{\qquad p_j(x) \qquad}_{\substack{j\text{th component of the normal}\\ \text{to the tangent plane at } x}} .$$

This gives the N equations

$$p_{x_j} \cdot (x - x_o) = 0, \qquad i = 1, \ldots, N.$$

Since $p(x) \in \mathbf{P}_o$, the vector-valued function $x \to p(x)$ must also satisfy

$$D_pF\big(x_o, z_o, p(x)\big) \cdot p_{x_j} = 0, \qquad j = 1, \ldots, N$$

where $D_pF = (F_{p_1}, \ldots, F_{p_N})$. It follows that for each x fixed in a neighborhood of x_o, the vectors D_pF and $x - x_o$ are parallel, and there exists $\lambda(x)$ such that

$$\begin{aligned} D_pF\big(x_o, z_o, p(x)\big) &= \lambda(x)(x - x_o) \\ F\big(x_o, z_o, p(x)\big) &= 0. \end{aligned} \tag{1.5}$$

This is a nonlinear system of $N + 1$ equations in the $N + 1$ unknowns $p_1(x), \ldots, p_N(x), \lambda(x)$. Solving it and putting the functions $x \to p(x)$ so obtained in (1.4) gives the equation of the envelope.

1.2 The Symmetric Equation of Monge's Cones

Eliminating λ from (1.4) and the first of (1.5) gives the *symmetric* equation of the cone

$$\frac{z - z_o}{p \cdot D_pF} = \frac{x_i - x_{o,i}}{F_{p_i}}, \qquad i = 1, \ldots, N. \tag{1.6}$$

This implies the Cartesian form of the Monge's cone $C(x_o, z_o)$

$$|z - z_o|^2 = \left(\frac{p \cdot D_pF}{|D_pF|}\right)^2 |x - x_o|^2. \tag{1.7}$$

---

olution and became minister of the navy in the Robespierre government (1792). Mathematician and physicist of diverse interests, he contributed with Lavoisier to the chemical synthesis of water (1785), and with Bertholet and Vandermonde in identifying various metallurgical states of iron (1794). The indicated construction is in *Feuilles d'Analyse appliquée à la Géométrie*, lectures delivered at the École Polytechnique in 1801, and published by J. Liouville in 1850.

Remark 1.1 If $F(x,z,p)$ is such that the "coefficient" of $|x-x_o|^2$ is constant, then the cone in (1.7) is circular and its axis is normal to the hyperplane $z=0$. This is occurs for the first-order nonlinear PDE $|\nabla u| = \text{const}$, which arises in geometric optics.

We stress however that the indicated "coefficient" depends on x via the functions $x \to p_j(x)$, and therefore $C(x_o, z_o)$ is not, in general, a circular cone, nor is its axis normal to the hyperplane $z=0$.

2 Characteristic Curves and Characteristic Strips

Let ℓ denote the line of intersection between the cone in (1.6) and the hyperplane tangent to the integral surface Σ at (x_o, z_o). For $(x,z) \in \ell$, the vector $p(x)$ remains constant. Therefore, for infinitesimal increments dz and dx_i, along ℓ

$$\frac{dz}{p \cdot D_pF} = \frac{dx_1}{F_{p_1}} = \dots = \frac{dx_N}{F_{p_N}}$$

where p and F_{p_i} are computed at $p(x)$, constant along ℓ. We conclude that ℓ has directions

$$\big(D_pF(x_o, z_o, p(x)), p \cdot D_pF(x_o, z_o, p(x))\big)$$

where $p(x)$ is computed on ℓ. Following these directions, starting from (x_o, z_o), trace a curve on the surface Σ. Such a curve, described in terms of a parameter $t \in (-\delta, \delta)$, for some $\delta > 0$, takes the form

$$\begin{aligned} \dot{x}(t) &= D_pF\big(x(t), z(t), p(t)\big) \qquad & x(0) &= x_o \\ \dot{z}(t) &= p \cdot D_pF\big(x(t), z(t), p(t)\big) \qquad & z(0) &= z_o. \end{aligned} \tag{2.1}$$

Here $p(t)$ is the solution of the system (1.5) with x_o and z_o replaced by $x(t)$ and $z(t)$, and computed at points x on the tangency line of the integral surface Σ with the Monge's cone with vertex at $\big(x(t), z(t)\big)$. The system (2.1) is not well defined, because the functions $t \to p_i(t)$ are in general not known. For quasi-linear equations, $F(x,z,p) = a_i(x,z)p_i$. In such a case $F_{p_i} = a_i(x,z)$ are independent of p and (2.1) are the characteristics originating at P_o (Section 1 of Chapter 7). Because of this analogy, we call the curves (2.1), *characteristics*. To render such a system well defined, observe that if Σ is an integral surface, then $p_i = u_{x_i}(x)$. From this and the first of (2.1)

$$\dot{p}_i = u_{x_i x_j} \dot{x}_j = u_{x_i x_j} F_{p_j}.$$

Also, from the PDE (1.1), by differentiation

$$F_{x_i} + F_u u_{x_i} + F_{p_j} u_{x_i x_j} = 0.$$

Therefore

$$\dot{p}_i = -F_{x_i} - F_u u_{x_i}, \qquad i = 1, \dots, N.$$

Thus, the characteristics for the nonlinear equation (1.1) are the curves

$$(-\delta,\delta)\ni t\to\Gamma(t)=\begin{cases}\dot{x}(t)=D_pF\big(x(t),z(t),p(t)\big)\\ \dot{z}(t)=p\cdot D_pF\big(x(t),z(t),p(t)\big)\\ \dot{p}(t)=-D_xF\big(x(t),z(t),p(t)\big)\\ \qquad\qquad -F_z\big(x(t),z(t),p(t)\big)p(t)\end{cases}\tag{2.2}$$

where $D_xF=(F_{x_1},\dots,F_{x_N})$. For every choice of "initial" data

$$\big(x(0),z(0),p(0)\big)=(x_o,z_o,p_o)\in E\times\mathbb{R}\times\mathbb{R}^N$$

the system (2.2) has a unique solution, local it t, with the interval of existence depending, in general, on the initial datum. To simplify the presentation we assume that the interval of unique solvability is $(-\delta,\delta)$, for every choice of data (x_o,z_o,p_o).

2.1 Characteristic Strips

A solution of (2.2) can be thought of as a curve $t\to\big(x(t),z(t)\big)\in\mathbb{R}^{N+1}$ whose points are associated to an infinitesimal portion of the hyperplane trough them and normal $p(t)$. Putting together these portions along $t\to\big(x(t),z(t)\big)$, the function $t\to\Gamma(t)$ can be regarded as a strip of infinitesimal width, called a *characteristic strip.* These remarks suggest that integral surfaces are union of characteristic strips. Let Σ be a hypersurface in $\mathbb{R}^{N+1}$ given as the graph of $z=u(x)\in C^1(E)$, and for $(x_o,z_o)\in\Sigma$, let $t\to\Gamma_{(x_o,z_o)}(t)$ be the characteristic strip originating at (x_o,z_o), that is, the unique solution of (2.2) with data

$$x(0)=x_o,\quad z(0)=z_o=u(x_o),\quad p(0)=\nabla u(x_o).\tag{2.3}$$

The surface Σ is a union of characteristic strips if for every $(x_o,z_o)\in\Sigma$, the strip $t\to\Gamma_{(x_o,z_o)}(t)$ is contained in Σ, in the sense that

$$z(t)=u(x(t))\quad\text{ and }\quad p(t)=\nabla u(x(t))\quad\text{ for all }\ t\in(-\delta,\delta).\tag{2.4}$$

Proposition 2.1 *An integral surface for (1.1) is union of characteristic strips.*

Proof. Let Σ be the graph of a solution $u\in C^1(E)$ of (1.1). Having fixed $x_o\in E$, let $t\to x(t)$ be the unique solution of

$$\dot{x}(t)=D_pF\big(x(t),u(x(t)),\nabla u(x(t))\big),\qquad x(0)=x_o.$$

One verifies that the $2N+1$ functions

$$(-\delta,\delta)\ni t\to\quad x(t),\quad z(t)=u(x(t)),\quad p(x(t))=\nabla u(x(t))$$

solve (2.2), with initial data (2.3). These are then characteristic strips. ∎

Remark 2.1 Unlike the case of quasi-linear equations, the converse does not hold, as (2.4) are not sufficient for one to conclude that Σ is an integral surface. Indeed, even though F is constant along $t\to\big(x(t),z(t)\big)$, the PDE (1.1) need not hold identically.

3 The Cauchy Problem

Let $s = (s_1, \dots, s_{N-1})$ be an $(N-1)$-dimensional parameter ranging over the cube $Q_\delta = (-\delta, \delta)^{N-1}$. The Cauchy problem associated with (1.1) consists in assigning an $(N-1)$-dimensional hypersurface $\varGamma \subset \mathbb{R}^{N+1}$ of parametric equations

$$Q_\delta \ni s \to \varGamma(s) = \begin{cases} x = \xi(s) = \big(\xi_1(s), \dots, \xi_N(s)\big) \\ z = \zeta(s), \quad \big(\xi(s), \zeta(s)\big) \in E \times \mathbb{R} \end{cases} \tag{3.1}$$

and seeking a function $u \in C^1(E)$ such that $\zeta(s) = u(\xi(s))$ for $s \in Q_\delta$ and such that the graph $z = u(x)$ is an integral surface of (1.1).

An integral surface for the Cauchy problem, must be a union of characteristic strips, and it must contain $\varGamma$. Therefore, one might attempt to construct it by drawing, from each point $(\xi(s), \zeta(s)) \in \varGamma$, a characteristic strip, a solution of (2.2), starting from the initial data

$$x(0) = \xi(s), \quad z(0) = \zeta(s), \quad p(0, s) = p(s) \in \mathbb{R}^N.$$

However, a surface that is a union of characteristic strips need not be an integral surface. Moreover, starting from a point on $\varGamma$, one may construct ∞^N characteristic strips, each corresponding to a choice of the initial vector $p(0, s) = p(s)$. A geometric construction of a solution to the Cauchy problem for (1.1), hinges on a criterion that would identify, for each $(\xi(s), \zeta(s)) \in \varGamma$, those initial data $p(0, s)$ for which the union of the corresponding characteristic strips, is indeed an integral surface.

3.1 Identifying the Initial Data $p(0, s)$

Set $(\xi(0), \zeta(0)) = (\xi_o, \zeta_o) \in \varGamma$, and assume that there exists a vector $\mathbf{p}_o$ such that

$$F(\xi_o, \zeta_o, p_o) = 0, \qquad D\zeta(0) = p_o \cdot \nabla\xi(0) \tag{3.2}$$

and in addition[2]

$$\det \begin{pmatrix} \nabla\xi(0) \\ D_p F(\xi_o, \zeta_o, p_o) \end{pmatrix} \ne 0.$$

Consider now the N-valued function

$$Q_\delta \times \mathbb{R}^N \ni (s, p) \to \varPsi(s, p) = \begin{pmatrix} \nabla\zeta(s) - p \cdot \nabla\xi(s) \\ F\big(\xi(s), \zeta(s), p\big) \end{pmatrix}.$$

By (3.2) such a function vanishes for $(s, p) = (0, p_o)$. More generally, for $s \in Q_\delta$, we seek those vectors $p(s)$ for which $\varPsi$ vanishes, that is $\varPsi(s, p(s)) = 0$. By the implicit function theorem, this defines, locally, a smooth N-valued function $Q_\delta \ni s \to p(s)$ such that

---

[2] See Section 2.2 of Chapter 7 for symbolism and motivation.

$$\left.\begin{aligned} &\nabla\zeta(s) = p(\mathbf{s})\cdot\nabla\xi(s) \\ &F\big(\xi(s),\zeta(s),p(s)\big) = 0 \end{aligned}\right\} \quad \text{for all } \ s\in Q_\delta. \tag{3.3}$$

Such a representation holds locally in a neighborhood of $s=0$, which might be taken as Q_δ by possibly reducing δ. The vector $p(s)$, so identified, is the set of initial data $p(0,s)=p(s)$ to be taken in the construction of the characteristic strips.

3.2 Constructing the Characteristic Strips

The characteristic strips may now be constructed as the solutions of the system of ODE's

$$\begin{aligned} \frac{d}{dt}x(t,s) &= D_pF\big(x(t,s),z(t,s),p(t,s)\big) \\ \frac{d}{dt}z(t,s) &= p(t,s)\cdot D_pF\big(x(t,s),z(t,s),p(t,s)\big) \\ \frac{d}{dt}p(t,s) &= -D_xF\big(x(t,s),z(t,s),p(t,s)\big) \\ &\qquad\quad -F_z\big(x(t,s),z(t,s),p(t,s)\big)p(t,s) \end{aligned} \tag{3.4}$$

with initial data given at each $s\in Q_\delta$

$$x(0,s)=\xi(s), \quad z(0,s)=\zeta(s), \quad p(0,s)=p(s). \tag{3.5}$$

The solution of (3.4)–(3.5) is local in t, that is, it exists in a time interval that depends on the initial data, or equivalently on the parameter $s\in Q_\delta$. By further reducing δ if needed, we may assume that (3.4)–(3.5) is uniquely solvable for $(t,s)\in(-\delta,\delta)\times Q_\delta$. Having solved such a system, consider the map

$$(-\delta,\delta)\times Q_\delta \ni (t,s) \to \big(x(t,s),z(t,s)\big).$$

This represents a surface $\Sigma\subset\mathbb{R}^{N+1}$, which by construction contains a local portion of Γ about (ξ_o,ζ_o).

Proposition 3.1 *The surface Σ is an integral surface for the Cauchy problem (1.1), (3.1).*

4 Solving the Cauchy Problem

To prove the proposition, we construct a function $x\to u(x)$ whose graph is Σ and that solves (1.1) in a neighborhood of (ξ_o,ζ_o). Observe first that by continuity, (3.2) continues to hold in a neighborhood of $s=0$, i.e.,

$$\det\begin{pmatrix} \nabla\xi(s) \\ D_pF\big(\xi(s),\zeta(s),p(s)\big)\end{pmatrix} \neq 0 \qquad \text{for } \ s\in Q_\delta$$

where δ is further reduced if needed. Next consider the map

$$M \,:\, (-\delta,\delta)\times Q_\delta \ni (t,s) \to x(t,s).$$

From (3.4)–(3.5) and the previous remarks

$$\det\begin{pmatrix} D_s x(0,s) \\ x_t(0,s) \end{pmatrix} = \det\begin{pmatrix} \nabla\xi(s) \\ D_p F\big(\xi(s),\zeta(s),p(s)\big) \end{pmatrix} \neq 0$$

for all $s \in Q_\delta$. By continuity this continues to hold for $t \in (-\delta,\delta)$, where δ is further reduced if necessary. Therefore

$$\det\begin{pmatrix} x_t(t,s) \\ D_s x(t,s) \end{pmatrix} \neq 0 \quad \text{for all } (t,s) \in (-\delta,\delta)\times Q_\delta. \tag{4.1}$$

Therefore M is locally invertible in a neighborhood of ξ_o. In particular, there exist $\varepsilon = \varepsilon(\delta)$, a cube $Q_\varepsilon(\xi_o)$, and smooth functions T and S, defined in $Q_\varepsilon(\xi_o)$, such that $t = T(x)$ and $s = S(x)$ for $x \in Q_\varepsilon(\xi_o)$.

The function $x \to u(x)$ is constructed by setting

$$u(x) = z\big(T(x),S(x)\big) \quad \text{for } x \in Q_\varepsilon(\xi_o).$$

By construction, $u\big(\xi(s)\big) = \zeta(s)$ and $t \to F\big(x(t,s),z(t,s),p(t,s)\big)$ is constant for all $s \in Q_\delta$. Moreover, by (3.3), F is also constant along Γ. Therefore

$$F\big(x(t,s),z(t,s),p(t,s)\big) = 0 \quad \text{for all } (t,s) \in (-\delta,\delta)\times Q_\delta. \tag{4.2}$$

It remains to prove that

$$p\big(T(x),S(x)\big) = \nabla u(x) \quad \text{for all } x \in Q_\varepsilon(\xi_o). \tag{4.3}$$

From the definition of $u(\cdot)$

$$z_t(t,s) = \nabla u \cdot x_t(t,s), \qquad D_s z(t,s) = \nabla u \cdot D_s x(t,s). \tag{4.4}$$

These and the equations of the characteristic strips, yield

$$[\nabla u - p(t,s)]\cdot x_t(t,s) = 0 \quad \text{for all } (t,s) \in (-\delta,\delta)\times Q_\delta. \tag{4.5}$$

4.1 Verifying (4.3)

Lemma 4.1 *The relation (4.3) would follow from*

$$D_s z(t,s) = p(t,s)\cdot D_s x(t,s) \quad \textit{for all } (t,s) \in (-\delta,\delta)\times Q_\delta. \tag{4.6}$$

Proof. Assuming (4.6) holds true, rewrite it as

$$\nabla u \cdot D_s x(t,s) = p(t,s) D_s x(t,s)$$

which follows by making use of the second of (4.4). Combining this with (4.5) gives the following linear homogeneous algebraic system in the unknowns $\nabla u - p(t,s)$

$$\begin{gathered}\big(\nabla u - p(t,s)\big)\cdot x_t(t,s) = 0\\ \big(\nabla u - p(t,s)\big)\cdot D_s x(t,s) = 0.\end{gathered}$$

By (4.1), this admits only the trivial solution for all $(t,s) \in (-\delta,\delta)\times Q_\delta$. ■

To establish (4.6), set

$$M(t,s) = D_s z(t,s) - p(t,s)\cdot D_s x(t,s)$$

and verify that by the first of (3.3), $M(0,s) = 0$ for all $s \in Q_\delta$. From (4.2)

$$D_p F \cdot D_s p + D_x F \cdot D_s x = -F_z D_s z.$$

Using this identity in s and the equations (3.4) of the characteristic strips, compute

$$\begin{aligned} M_t &= D_s z_t - p_t \cdot D_s x - p\cdot D_s x_t\\ &= D_s p D_p F + p\cdot D_p D_s F + D_x F\cdot D_s x + F_z p\cdot D_s x - p\cdot D_p D_s F\\ &= F_z p\cdot D_s x - F_z D_s z\\ &= -F_z(D_s z - p\cdot D_s x) = -F_z M.\end{aligned}$$

This has the explicit integral

$$M(t,s) = M(0,s)\exp\left(-\int_0^t F_z d\tau\right)$$

and gives $(t,s)\to M(t,s) = 0$, since $M(0,s) = 0$.

4.2 A Quasi-Linear Example in $\mathbb{R}^2$

Denote by (x,y) the coordinates in $\mathbb{R}^2$, and given two positive numbers A and ρ, consider the Cauchy problem

$$xuu_x - Au_y = 0, \qquad u(\rho,y) = y.$$

The surface Γ in (3.1) is the line $z = y$ in the plane $x = \rho$, which can be written in the parametric form

$$\xi_1(s) = \rho, \quad \xi_2(s) = s, \quad \zeta(s) = s, \quad s\in\mathbb{R}.$$

We solve the Cauchy problem in a neighborhood of $(\rho,0,0)$. The vector p_o satisfying (3.2) is $p_o = (0,1)$. The system (3.4) takes the form

$$
\begin{aligned}
x_t(t,s) &= x(t,s)z(t,s)\\
y_t(t,s) &= -A\\
z_t(t,s) &= x(t,s)z(t,s)p_1(t,s) - Ap_2(t,s)\\
p_{1,t}(t,s) &= -z(t,s)p_1(t,s) - x(t,s)p_1^2(t,s)\\
p_{2,t}(t,s) &= x(t,s)p_1(t,s)p_2(t,s)
\end{aligned}
$$

with initial conditions

$$x(0,s)=\rho, \quad y(0,s)=s, \quad z(0,s)=s, \quad p_1(0,s)=0, \quad p_2(0,s)=1.$$

The solution is

$$z(t,s)=s, \quad y(t,s)=-At+s, \quad \ln\frac{x}{\rho}=st.$$

Eliminate the parameters s and t to obtain the solution in implicit form

$$u^2(x,t) - yu(x,y) = A\ln\frac{x}{\rho}.$$

The solution is analytic in the region $y^2+4A\ln(x/\rho)>0$.

5 The Cauchy Problem for the Equation of Geometrical Optics

Let Φ_o be a surface in $\mathbb{R}^N$ with parametric equations $x=\xi(s)$ where s is a $(N-1)$-parameter ranging over some cube $Q_\delta \subset \mathbb{R}^{N-1}$. Consider the Cauchy problem for the *eikonal* equation ([51] Chapter 9, Section 8)

$$|\nabla u| = 1 \qquad u\,\big|_{\Phi_o} = 0. \tag{5.1}$$

The function $x \to u(x)$ is the time it takes a light ray to reach x starting from a point source at the origin. The level sets $\Phi_t = [u=t]$ are the wave fronts of the light propagation, and the light rays are normal to these fronts. Thus Φ_o is an initial wave front, and the Cauchy problem seeks to determine the fronts Φ_t at later times t. The Monge's cones are circular, with vertical axis and their equation is (Section 1.2)

$$|z-t| = |x-y| \qquad \text{for every } \ y\in\Phi_t.$$

The characteristic strips are constructed from (3.4)–(3.5) as

$$
\begin{array}{lll}
x_t(t,s)=p(t,s) & x(0,s)=\xi(s) & s\in Q_\delta\\
z_t(t,s)=1 & z(0,t)=0 & t\in(-\delta,\delta)\\
p_t(t,s)=0 & p(0,s)=p(s). &
\end{array}
\tag{5.2}
$$

Computing the initial vectors $p(s)$ from (3.3) gives

$$p(s) \cdot \nabla\xi(s) = 0, \qquad |p(s)| = 1, \quad \text{for all } \ s \in Q_\delta.$$

Thus $p(s)$ is a unit vector normal to the front Φ_o. By the third of (5.2) such a vector is constant along characteristics, and the characteristic system has the explicit integral

$$x(t,s) = tp(s) + \xi(s), \quad z(t,s) = t, \quad p(t,s) = p(s). \tag{5.3}$$

Therefore after a time t, the front Φ_o evolves into the front Φ_t, obtained by transporting each point $\xi(s) \in \Phi_o$, along the normal $p(s)$ with unitary speed, for a time t.

5.1 Wave Fronts, Light Rays, Local Solutions and Caustics

For a fixed $s \in Q_\delta$ the first of (5.2) are the parametric equations of a straight line in $\mathbb{R}^N$, which we denote by $\ell(t;s)$. Since $p(s)$ is normal to the front Φ_o, such a line can be identified with the light ray through $\xi(s) \in \Phi_o$. By construction such a ray is always normal to the wave front Φ_t that it crosses.

This geometrical interpretation is suggestive on the one hand of the underlying physics, and on the other, it highlights the local nature of the Cauchy problem. Indeed the solution, as constructed, becomes meaningless if two of these rays, say for example $\ell(t;s_1)$ and $\ell(t;s_2)$, intersect at some point, for such a point would have to belong to two distinct wave fronts. To avoid such an occurrence, the number δ that limits the range of the parameters s and t has to be taken sufficiently small.

The possible intersection of the light rays $\ell(s;t)$ might depend also on the initial front. If Φ_o is an $(N-1)$-dimensional hyperplane, then all rays are parallel and normal to Φ_o. In such a case the solution exists for all $s \in \mathbb{R}^{N-1}$ and all $t \in \mathbb{R}$. If Φ_o is an $(N-1)$-dimensional sphere of radius R centered at the origin of $\mathbb{R}^N$, all rays $\ell(t;s)$ intersect at the origin after a time $t = R$. The solution exists for all times, and the integral surfaces are right circular cones with vertex at the origin.

The envelope of the family $\ell(t;s)$ as s ranges over Q_δ, if it exists, is called a *caustic* or *focal curve*. By definition of envelope, the caustic is tangent in any of its points to at least one light ray. Therefore, such a tangency point is instantaneously illuminated, and the caustic can be regarded as a light tracer following the parameter t.

If Φ_o is a hyperplane the caustic does not exists, and if Φ_o is a sphere, the caustic degenerates into its origin.

6 The Initial Value Problem for Hamilton–Jacobi Equations

Denote by $(x;x_{N+1})$ points in $\mathbb{R}^{N+1}$, and for a smooth function u defined in a domain of $\mathbb{R}^{N+1}$, set $\nabla u = (D_x u, u_{x_{N+1}})$. Given a smooth nonlinear function

$$(x, x_{N+1}, p) \to \mathcal{H}(x, p; x_{N+1})$$

defined in a domain of $\mathbb{R}^{N+1} \times \mathbb{R}^N$, consider the first-order equation

$$F(x; x_{N+1}, u, D_x u, u_{x_{N+1}}) = u_{x_{N+1}} + \mathcal{H}(x, D_x u; x_{N+1}) = 0. \tag{6.1}$$

The Cauchy problem for (6.1) consists in giving an N-dimensional surface $\varPhi$ and a smooth function u_o defined on $\varPhi$, and seeking a smooth function u that solves (6.1) in a neighborhood of $\varPhi$ and equals u_o on $\varPhi$. If the surface $\varPhi$ is the hyperplane $x_{N+1} = 0$, it has parametric equations $x = s$ and the characteristic system (3.4) takes the form

$$\begin{aligned}
x_t(t,s) &= D_p\mathcal{H}(x(t,s), p(t,s); x_{N+1}(t,s)) \\
x_{N+1,t}(t,s) &= 1 \\
z_t(x,t) &= p(t,s) \cdot D_p\mathcal{H}(x(t,s), p(t,s); x_{N+1}(t,s)) + p_{N+1}(t,s) \\
p_t(t,s) &= -D_x\mathcal{H}(x(t,s), p(t,s), x_{N+1}(t,s)) \\
p_{N+1,t}(t,s) &= -D_{x_{N+1}}\mathcal{H}(x(t,s), p(t,s); x_{N+1}(t,s))
\end{aligned}$$

with the initial conditions

$$\begin{gathered}
x(0,s) = s, \quad x_{N+1}(0,s) = 0, \quad z(0,s) = u_o(s) \\
p(0,s) = p(s), \quad p_{N+1}(0,s) = p_{N+1}(s).
\end{gathered}$$

The second of these and the corresponding initial datum imply $x_{N+1} = t$. Therefore the $(N+1)$st coordinate may be identified with time, and the Cauchy problem for the surface $[t = 0]$ is the *initial value problem* for the Hamilton–Jacobi equation (6.1). The characteristic system can be written concisely as

$$\begin{aligned}
x_t(t,s) &= D_p\mathcal{H}(x(t,s), p(t,s); t) \qquad & x(0,s) &= s \\
p_t(t,s) &= -D_x\mathcal{H}(x(t,s), p(t,s); t) \qquad & p(0,s) &= p(s)
\end{aligned} \tag{6.2}$$

where the initial data $(p(s), p_{N+1}(s))$ are determined from (3.3) as

$$p(s) = D_s u_o(s), \qquad p_{N+1}(0,s) = -\mathcal{H}(s, p(s); 0). \tag{6.3}$$

Moreover, the functions $(t,s) \to p_{N+1}(t,s), z(t,s)$, satisfy

$$\begin{aligned}
p_{N+1,t}(t,s) &= -\mathcal{H}(x(t,s), p(t,s); t) \\
p_{N+1}(0,s) &= -\mathcal{H}(s, p(s); 0) \\
z_t(x,t) &= p(t,s) \cdot D_p\mathcal{H}(x(t,s), p(t,s); t) + p_{N+1}(t,s) \\
z(0,s) &= u_o(s).
\end{aligned}$$

It is apparent that (6.2) is independent of (6.3), and the latter can be integrated as soon as one determines the functions $(t,s) \to x(t,s), p(t,s)$, solutions of (6.2). Therefore (6.2) is the characteristic system associated with the initial value problem for (6.1).

Consider now a mechanical system with N degrees of freedom governed by a Hamiltonian $\mathcal{H}$. The system (6.2) is precisely the canonical Hamiltonian system that describes the motion of the system, through its Lagrangian coordinates $t \to x(t,s)$ and the kinetic momenta $t \to p(t,s)$, starting from its initial configuration. Therefore the characteristics associated with the initial value problem for the Hamilton–Jacobi equation (6.1) are the dynamic trajectories, in phase space, of the underlying mechanical system.

From now on we will restrict the theory to the case $\mathcal{H}(x,t,p) = \mathcal{H}(p)$, that is, the Hamiltonian depends only on the kinetic momenta p. In such a case the initial value problem takes the form

$$u_t + \mathcal{H}(D_x u) = 0, \qquad u(\cdot, 0) = u_o \tag{6.4}$$

where u_o is a bounded continuous function in $\mathbb{R}^N$. The characteristic curves and initial data are

$$\begin{aligned} x_t(t,s) &= D_p\mathcal{H}(p(t,s) & x(0,s) &= s \\ p_t(t,s) &= 0 & p(0,s) &= D_s u_o(s) \\ p_{N+1,t}(t,s) &= 0, & p_{N+1}(0,s) &= -\mathcal{H}(D_s u_o(s)). \end{aligned} \tag{6.5}$$

Moreover

$$\begin{aligned} &z_t(t,s) = p(t,s)\cdot D_p\mathcal{H}(p(t,s)) + p_{N+1}(t,s) \\ &z(0,s) = u_o(s). \end{aligned} \tag{6.6}$$

7 The Cauchy Problem in Terms of the Lagrangian

Assume that $p \to \mathcal{H}(p)$ is convex and coercive, that is[3]

$$\lim_{|p|\to\infty} \frac{\mathcal{H}(p)}{\|p\|} = \infty.$$

The Lagrangian $q \to \mathcal{L}(q)$, corresponding to the Hamiltonian $\mathcal{H}$, is given by the Legendre transform of $\mathcal{H}$, that is[4]

$$\mathcal{L}(q) = \sup_{p\in\mathbb{R}^N} \left[q\cdot p - \mathcal{H}(p)\right].$$

By the coercivity of $\mathcal{H}$ the supremum is achieved at a vector p satisfying

$$q = D_p\mathcal{H}(p) \quad \text{and} \quad \mathcal{L}(q) = q\cdot p - \mathcal{H}(p). \tag{7.1}$$

---

[3] This occurs, for example, for $\mathcal{H}(p) = |p|^{1+\alpha}$, for all $\alpha > 0$. It does not hold for the Hamiltonian $\mathcal{H}(p) = |p|$ corresponding to the eikonal equation. The Cauchy problem for such noncoercive Hamiltonians is investigated in [142, 143, 144].

[4] [51] Chapter 6 Section 5, and [50], Section 13 of the Complements of Chapter IV.

Moreover, $q \to \mathcal{L}(q)$ is itself convex and coercive, and the Hamiltonian $\mathcal{H}$ is the Legendre transform of the Lagrangian $\mathcal{L}$, that is

$$\mathcal{H}(p) = \sup_{q\in\mathbb{R}^N} \left[p \cdot q - \mathcal{L}(q)\right].$$

Since $\mathcal{L}$ is coercive, the supremum is achieved at a vector q, satisfying

$$p = D_q\mathcal{L}(q) \qquad \text{and} \qquad \mathcal{H}(p) = q \cdot p - \mathcal{L}(q). \tag{7.2}$$

The equations for the characteristic curves (6.5)–(6.6) can be written in terms of the Lagrangian as follows. The equations in (7.1), written for $q = x_t(t, s)$, and the first of (6.5) imply that the vector $p(s, t)$ for which the supremum in the Legendre transform of $\mathcal{H}$ is achieved is the solution of the second of (6.5). Therefore

$$\mathcal{L}(x_t(t, s)) = x_t(t, s) \cdot p(t, s) - \mathcal{H}(p(t, s)). \tag{7.3}$$

Taking the gradient of $\mathcal{L}$ with respect to x_t and then the derivative with respect to time t, gives

$$D_{\dot{x}}\mathcal{L}(\dot{x}) = p(t, s) \qquad \text{and} \qquad \frac{d}{dt}\frac{\partial\mathcal{L}(\dot{x})}{\partial\dot{x}_h} = 0, \quad h = 1, \dots, N.$$

These are the Lagrange equations of motion for a mechanical system of Hamiltonian $\mathcal{H}$.

8 The Hopf Variational Solution

Let u be a smooth solution in $\mathbb{R}^N \times \mathbb{R}^+$ of the Cauchy problem (6.4) for a smooth initial datum u_o. Then for every $x \in \mathbb{R}^N$ and every time $t > 0$, there exists some $s \in \mathbb{R}^N$ such that $x = x(t, s)$, that is the position x is reached in time t by the characteristic $\ell = \{x(t, s)\}$ originating at s. Therefore

$$u(x, t) = u(x(t, s), t) \qquad \text{and} \qquad D_x u(x(t, s), t) = p(t, s).$$

Equivalently, taking into account that u is a solution of (6.4)

$$\begin{aligned} u(x, t) - u_o(s) &= \int_\ell \frac{\partial u}{\partial \ell} d\ell \\ &= \int_0^t \big[D_x u(x(\tau, s), \tau) \cdot x_t(\tau, s) + u_t(x(\tau, s), \tau)\big] d\tau \\ &= \int_0^t \big\{p(\tau, s) \cdot x_t(\tau, s) - \mathcal{H}(p(\tau, s))\big\} d\tau. \end{aligned}$$

Using now (7.1), this implies

$$u(x, t) = \int_0^t \mathcal{L}(x_\tau(\tau, s)) d\tau + u_o(s). \tag{8.1}$$

8.1 The First Hopf Variational Formula

The integral on the right-hand side is the Hamiltonian action of a mechanical system with N degrees of freedom, governed by a Lagrangian $\mathcal{L}$, in its motion from a Lagrangian configuration s at time $t = 0$ to a Lagrangian configuration x at time t. Introduce the class of all smooth *synchronous variations*

$$K^s_{\text{sync}} = \left\{ \begin{array}{l} \text{the collection of all smooth paths } q(\cdot) \\ \text{in } \mathbb{R}^N \text{ such that } q(0) = s \text{ and } q(t) = x \end{array} \right\}.$$

By the least action principle ([51], Chapter IX, Section 2)

$$\int_0^t \mathcal{L}(x_t(\tau, s))d\tau = \min_{q \in K^s_{\text{sync}}} \int_0^t \mathcal{L}(\dot{q}(\tau))d\tau.$$

Therefore

$$\begin{aligned} u(x,t) &= \min_{q \in K^s_{\text{sync}}} \int_0^t \mathcal{L}(\dot{q}(\tau))d\tau + u_o(s) \\ &\geq \inf_{y \in \mathbb{R}^N} \inf_{q \in K^y_{\text{sync}}} \left[\int_0^t \mathcal{L}(\dot{q}(\tau))d\tau + u_o(y) \right]. \end{aligned}$$

Such a formula actually holds with the equality sign, since if $u(x,t)$ is known, by (8.1), for each fixed $x \in \mathbb{R}^N$ and $t > 0$ there exist some $s \in \mathbb{R}^N$ and a smooth curve $\tau \to x(\tau, s)$ of extremities s and x such that the infimum is actually achieved. This establishes the first Hopf variational formula, that is, if $(x,t) \to u(x,t)$ is a solution of the Cauchy problem (6.4), then

$$u(x,t) = \min_{y \in \mathbb{R}^N} \min_{q \in K^y_{\text{sync}}} \left[\int_0^t \mathcal{L}(\dot{q}(\tau))d\tau + u_o(y) \right]. \tag{8.2}$$

8.2 The Second Hopf Variational Formula

A drawback of the first Hopf variational formula is that, given x and t, it requires the knowledge of the classes K^y_{sync} for all $y \in \mathbb{R}^N$. The next variational formula dispenses with such classes ([124, 125]).

Proposition 8.1 *let $(x,t) \to u(x,t)$ be a solution of the minimum problem (8.2). Then for all $x \in \mathbb{R}^N$ and all $t > 0$*

$$u(x,t) = \min_{y \in \mathbb{R}^N} \Big[t\mathcal{L}\Big(\frac{x-y}{t}\Big) + u_o(y) \Big]. \tag{8.3}$$

Proof. For $s \in \mathbb{R}^N$ consider the curve

$$\tau \to q(\tau) = s + \frac{\tau}{t}(x - s) \qquad \tau \in [0, t].$$

If $u(x,t)$ is a solution of (8.2)

$$u(x,t) \le \int_0^t \mathcal{L}(\dot{q}(\tau))d\tau + u_o(s) = t\mathcal{L}\Big(\frac{x-s}{t}\Big) + u_o(s)$$

and since s is arbitrary

$$u(x,t) \le \inf_{y\in\mathbb{R}^N} \Big[t\mathcal{L}\Big(\frac{x-y}{t}\Big) + u_o(y)\Big].$$

Now let $q \in K^s_{\rm sync}$ for some $s \in \mathbb{R}^N$. Since $\mathcal{L}(\cdot)$ is convex, by Jensen's inequality

$$\mathcal{L}\Big(\frac{x-s}{t}\Big) = \mathcal{L}\left(\frac{1}{t}\int_0^t \dot{q}(\tau)d\tau\right) \le \frac{1}{t}\int_0^t \mathcal{L}(\dot{q}(\tau))d\tau.$$

From this

$$t\mathcal{L}\Big(\frac{x-s}{t}\Big) + u_o(s) \le \int_0^t \mathcal{L}(\dot{q}(\tau))d\tau + u_o(s).$$

Since $s \in \mathbb{R}^N$ is arbitrary, by (8.2)

$$\begin{aligned}\inf_{y\in\mathbb{R}^N} \Big[t\mathcal{L}\Big(\frac{x-y}{t}\Big) + u_o(y)\Big] &\le \min_{y\in\mathbb{R}^N} \min_{q\in K^y_{\rm sync}} \Big[\int_0^t \mathcal{L}(\dot{q}(\tau))d\tau + u_o(y)\Big] \\ &= u(x,t).\end{aligned}$$

■

9 Semigroup Property of Hopf Variational Solutions

Proposition 9.1 ([7, 23]) *Let $(x,t) \to u(x,t)$ be a solution of the variational problem (8.3). Then for all $x \in \mathbb{R}^N$ and every pair $0 \le \tau < t$*

$$u(x,t) = \min_{y\in\mathbb{R}^N} \Big[(t-\tau)\mathcal{L}\Big(\frac{x-y}{t-\tau}\Big) + u(y,\tau)\Big]. \tag{9.1}$$

Proof. Write (8.3) for $x = \eta$ at time τ and let $\xi \in \mathbb{R}^N$ be a point where the minimum is achieved. Thus

$$u(\eta,\tau) = \tau\mathcal{L}\Big(\frac{\eta-\xi}{\tau}\Big) + u_o(\xi).$$

Since $\mathcal{L}(\cdot)$ is convex

$$\mathcal{L}\Big(\frac{x-\xi}{t}\Big) = \Big(1-\frac{\tau}{t}\Big)\mathcal{L}\Big(\frac{x-\eta}{t-\tau}\Big) + \frac{\tau}{t}\mathcal{L}\Big(\frac{\eta-\xi}{\tau}\Big).$$

Therefore

$$\begin{aligned} u(x,t) &= \min_{y\in\mathbb{R}^N}\Big[t\mathcal{L}\Big(\frac{x-y}{t}\Big)+u_o(y)\Big] \le t\mathcal{L}\Big(\frac{x-\xi}{t}\Big)+u_o(\xi) \\ &\le (t-\tau)\mathcal{L}\Big(\frac{x-\eta}{t-\tau}\Big)+\tau\mathcal{L}\left(\frac{\eta-\xi}{\tau}\right)+u_o(\xi) \\ &= (t-\tau)\mathcal{L}\Big(\frac{x-\eta}{t-\tau}\Big)+u(\eta,\tau) \\ &\le \min_{\eta\in\mathbb{R}^N}\Big[(t-\tau)\mathcal{L}\Big(\frac{x-\eta}{t-\tau}\Big)+u(\eta,\tau)\Big]. \end{aligned}$$

Now let $\xi\in\mathbb{R}^N$ be a point for which the minimum in (8.3) is achieved, i.e.,

$$u(x,t)=t\mathcal{L}\Big(\frac{x-\xi}{t}\Big)+u_o(\xi).$$

For $\tau\in(0,t)$ write

$$\eta=\frac{\tau}{t}x+\Big(1-\frac{\tau}{t}\Big)\xi \quad\Longrightarrow\quad \frac{x-\eta}{t-\tau}=\frac{x-\xi}{t}=\frac{\eta-\xi}{\tau}.$$

Moreover, by (8.3)

$$u(\eta,\tau)\le\tau\mathcal{L}\Big(\frac{\eta-\xi}{\tau}\Big)+u_o(\xi).$$

Combining these remarks

$$\begin{aligned} (t-\tau)\mathcal{L}\Big(\frac{x-\eta}{t-\tau}\Big)+u(\eta,\tau) &\le (t-\tau)\mathcal{L}\Big(\frac{x-\eta}{t-\tau}\Big)+\tau\mathcal{L}\Big(\frac{\eta-\xi}{\tau}\Big)+u_o(\xi) \\ &= t\mathcal{L}\Big(\frac{x-\xi}{t}\Big)+u_o(\xi)=u(x,t). \end{aligned}$$

From this

$$u(x,t)\ge\min_{\eta\in\mathbb{R}^N}\Big[(t-\tau)\mathcal{L}\Big(\frac{x-\eta}{t-\tau}\Big)+u(\eta,\tau)\Big].$$ ■

10 Regularity of Hopf Variational Solutions

For $(x,t)\to u(x,t)$ to be a solution of the Cauchy problem (6.4), it would have to be differentiable. While this is in general not the case, the next proposition asserts that if the initial datum u_o is Lipschitz continuous, the corresponding Hopf variational solution is Lipschitz continuous. Assume then that there is a positive constant C_o such that

$$|u_o(x)-u_o(y)|\le C_o|x-y| \quad \text{for all } x,y\in\mathbb{R}^N. \tag{10.1}$$

Proposition 10.1 *Let $(x,t)\to u(x,t)$ be a solution of (8.3) for an initial datum u_o satisfying (10.1). Then there exists a positive constant C depending only on C_o and $\mathcal{H}$ such that for all $x,y\in\mathbb{R}^N$ and all $t,\tau\in\mathbb{R}^+$*

$$|u(x,t)-u(y,\tau)|\le C(|x-y|+|t-\tau|). \tag{10.2}$$

Proof. For a fixed $t > 0$, let $\xi \in \mathbb{R}^N$ be a vector for which the minimum in (8.3) is achieved. Then for all $y \in \mathbb{R}^N$,

$$\begin{aligned} u(y,t) - u(x,t) &= \inf_{\eta \in \mathbb{R}^N} \Big[t\mathcal{L}\Big(\frac{y-\eta}{t}\Big) + u_o(\eta)\Big] - t\mathcal{L}\Big(\frac{x-\xi}{t}\Big) - u_o(\xi) \\ &\le t\mathcal{L}\Big(\frac{y-(y-(x-\xi))}{t}\Big) + u_o(y-(x-\xi)) \\ &\quad - t\mathcal{L}\Big(\frac{x-\xi}{t}\Big) - u_o(\xi) \\ &= u_o(y-(x-\xi)) - u_o(\xi) \le C_o|y-x|. \end{aligned}$$

Interchanging the role of x and y gives

$$|u(x,t) - u(y,t)| \le C_o|x-y| \quad \text{ for all } \ x,y \in \mathbb{R}^N.$$

This establishes the Lipschitz continuity of u in the space variables uniformly in time. The variational formula (8.3) implies

$$u(x,t) \le t\mathcal{L}(0) + u_o(x) \quad \text{ for all } \ x \in \mathbb{R}^N$$

and

$$\begin{aligned} u(x,t) &= \min_{y \in \mathbb{R}^N} \Big[t\mathcal{L}\Big(\frac{x-y}{t}\Big) + u_o(y) - u_o(x) + u_o(x)\Big] \\ &\ge u_o(x) + \min_{y \in \mathbb{R}^N} \Big[t\mathcal{L}\Big(\frac{x-y}{t}\Big) - C_o|x-y|\Big] \\ &\ge u_o(x) - \max_{q \in \mathbb{R}^N} [C_o t|q| - t\mathcal{L}(q)] \\ &\ge u_o(x) - t \max_{|p|<C_o} \max_{q \in \mathbb{R}^N} [p \cdot q - \mathcal{L}(q)] \\ &= u_o(x) - t \max_{|p|<C_o} \mathcal{H}(p). \end{aligned}$$

Therefore

$$|u(x,t) - u_o(x)| \le \bar{C}t, \quad \text{ where } \ \bar{C} = \mathcal{L}(0) \wedge \max_{|p|<C_o} \mathcal{H}(p). \tag{10.3}$$

from this

$$|u(x,t) - u(x,\tau)| \le \bar{C}|t-\tau| \quad \text{ for all } \ t,\tau \in \mathbb{R}^+. \qquad \blacksquare$$

Remark 10.1 By the Rademacher theorem $(x,t) \to u(x,t)$ is a.e. differentiable in $\mathbb{R}^N \times \mathbb{R}^+$ ([50], Chapter VII, Section 23).

11 Hopf Variational Solutions (8.3) Are Weak Solutions of the Cauchy Problem (6.4)

Assume that u_o is Lipschitz continuous as in (10.1). Then by (10.3), a solution $(x,t) \to u(x,t)$ of the corresponding variational problem (8.3) takes the initial datum u_o in the classical sense. The next proposition asserts that such a variational solution satisfies the Hamilton–Jacobi equation (6.4) at each point (x, t) where it is differentiable.

Proposition 11.1 *Let $(x, t) \to u(x, t)$ be a solution of the minimum problem (8.3). If u is differentiable at $(x, t) \in \mathbb{R}^N \times \mathbb{R}^+$, then*

$$u_t + \mathcal{H}(D_x u) = 0 \qquad \text{at } \ (x, t).$$

Proof. Fix $\eta \in \mathbb{R}^N$ and $h > 0$. By the semigroup property

$$\begin{aligned} u(x + h\eta, t + h) &= \min_{y \in \mathbb{R}^N} \Big[(t + h)\mathcal{L}\Big(\frac{x + h\eta - y}{t + h}\Big) + u_o(y)\Big] \\ &= \min_{y \in \mathbb{R}^N} \Big[h\mathcal{L}\Big(\frac{x + h\eta - y}{h}\Big) + u(y, t)\Big] \le h\mathcal{L}(\eta) + u(x, t). \end{aligned}$$

From this

$$\frac{u(x + h\eta, t + h) - u(x, t)}{h} \le \mathcal{L}(\eta)$$

and letting $h \to 0$

$$\eta \cdot D_x u(x, t) + u_t(x, t) \le \mathcal{L}(\eta).$$

Recalling that $\mathcal{H}$ is the Legendre transform of $\mathcal{L}$

$$u_t(x, t) + \mathcal{H}(D_x u(x, t)) = u_t(x, t) + \max_{q \in \mathbb{R}^N} [q \cdot D_x u(x, t) - \mathcal{L}(q)] \le 0.$$

Let $\xi \in \mathbb{R}^N$ be a point for which the minimum in (8.3) is achieved. Fix $0 < h < t$, set $\tau = t - h$, and let

$$\eta = \frac{\tau}{t} x + \Big(1 - \frac{\tau}{t}\Big)\xi \implies \frac{x - \xi}{t} = \frac{\eta - \xi}{\tau}; \quad \eta = x - \frac{h}{t}(x - \xi).$$

Using these definitions, compute

$$\begin{aligned} u(x, t) - u(\eta, \tau) &= t\mathcal{L}\Big(\frac{x - \xi}{t}\Big) + u_o(\xi) - \min_{y \in \mathbb{R}^N} \Big[\tau\mathcal{L}\Big(\frac{\eta - y}{\tau}\Big) + u_o(y)\Big] \\ &\ge t\mathcal{L}\Big(\frac{x - \xi}{t}\Big) + u_o(\xi) - \tau\mathcal{L}\Big(\frac{\eta - \xi}{\tau}\Big) - u_o(\xi) \\ &= (t - \tau)\mathcal{L}\Big(\frac{x - \xi}{t}\Big) = h\mathcal{L}\Big(\frac{x - \xi}{t}\Big). \end{aligned}$$

From this

$$\frac{1}{h}\Big[u(x, t) - u\Big(x - \frac{h}{t}(x - \xi), t - h\Big)\Big] \ge \mathcal{L}\Big(\frac{x - \xi}{t}\Big)$$

and letting $h \to 0$

$$u_t(x, t) + \frac{x - \xi}{t} \cdot D_x u(x, t) \ge \mathcal{L}\Big(\frac{x - \xi}{t}\Big).$$

Since $\mathcal{H}$ is the Legendre transform of the Lagrangian $\mathcal{L}$

$$\begin{aligned} u_t(x,t) + \mathcal{H}(D_x u(x,t)) &= u_t(x,t) + \max_{q\in\mathbb{R}^N}[q\cdot D_x u(x,t) - \mathcal{L}(q)] \\ &\ge u_t(x,t) + \frac{x-\xi}{t}\cdot D_x u(x,t) - \mathcal{L}\Big(\frac{x-\xi}{t}\Big) \ge 0. \qquad \blacksquare \end{aligned}$$

12 Some Examples

Proposition 11.1 ensures that the variational solutions (8.3), satisfy the Hamilton–Jacobi equation in (6.4) only at points of differentiability, as shown by the examples below.

12.1 Example I

$$u_t + \tfrac{1}{2}|D_x u|^2 = 0 \quad \text{in } \mathbb{R}^N\times\mathbb{R}^+, \qquad u(x,0) = |x|. \tag{12.1}$$

The Lagrangian $\mathcal{L}$ corresponding to the Hamiltonian $\mathcal{H}(p) = \frac{1}{2}|p|^2$ is

$$\mathcal{L}(q) = \max_{p\in\mathbb{R}^N}\big[p\cdot q - \tfrac{1}{2}|p|^2\big] = \tfrac{1}{2}|q|^2$$

and the Hopf variational solution is

$$u(x,t) = \min_{y\in\mathbb{R}^N}\bigg(\frac{|x-y|^2}{2t} + |y|\bigg).$$

The minimum is computed by setting

$$D_y\bigg(\frac{|x-y|^2}{2t} + |y|\bigg) = \frac{y-x}{t} + \frac{y}{|y|} = 0 \quad\Longrightarrow \frac{x}{t} = \bigg(\frac{1}{t} + \frac{1}{|y|}\bigg)y.$$

From this compute, for $|x| > t$

$$y = x - \frac{xt}{|x|} \quad\Longrightarrow\quad u(x,t) = |x| - \tfrac{1}{2}t \quad \text{for } |x| > t.$$

If $|x| \le t$

$$\begin{aligned} \frac{|x-y|^2}{2t} + |y| &= \frac{|x|^2}{2t} - \frac{x\cdot y}{t} + \frac{|y|^2}{2t} + |y| \\ &\ge \frac{|x|^2}{2t} - \frac{|x|}{t}|y| + \frac{|y|^2}{2t} + |y| \ge \frac{|x|^2 + |y|^2}{2\,t}. \end{aligned}$$

This holds for all $y\in\mathbb{R}^N$, and equality holds for $y = 0$. Therefore if $|x| \le t$ the minimum is achieved for $y = 0$, and

$$u(x,t) = \begin{cases} \dfrac{|x|^2}{2t} & \text{for } |x| \le t \\ |x| - \frac{1}{2}t & \text{for } |x| > t. \end{cases}$$

One verifies that u satisfies the Hamilton–Jacobi equation in (12.1) in $\mathbb{R}^N\times\mathbb{R}^+$ except at the cone $|x| = t$.

Remark 12.1 For fixed $t > 0$ the graph of $x \to u(x,t)$ is convex for $|x| < t$ and concave for $|x| > t$. In the region of convexity, the Hessian matrix of u is $\mathbb{I}/t$. Therefore for all $\xi \in \mathbb{R}^N$

$$u_{x_i x_j} \xi_i \xi_j \le \frac{|\xi|^2}{t}. \tag{12.2}$$

In the region of concavity $|x| > t$

$$u_{x_i x_j} \xi_i \xi_j = (|x|^2 \delta_{ij} - x_i x_j) \frac{\xi_i \xi_j}{|x|^3} \le \frac{|\xi|^2}{t}.$$

Therefore (12.2) holds in the whole of $\mathbb{R}^N \times \mathbb{R}^+$ except for $|x| = t$.

12.2 Example II

$$u_t + \tfrac{1}{2}|D_x u|^2 = 0 \quad \text{in } \mathbb{R}^N \times \mathbb{R}^+, \qquad u(x,0) = -|x|. \tag{12.3}$$

As before, $\mathcal{L}(q) = \frac{1}{2}|q|^2$, and the Hopf variational solution is

$$u(x,t) = \min_{y \in \mathbb{R}^N} \left(\frac{|x-y|^2}{2t} - |y| \right).$$

The minimum is computed by setting

$$D_y \left(\frac{|x-y|^2}{2t} - |y| \right) = \frac{y-x}{t} - \frac{y}{|y|} = 0 \quad \Longrightarrow \quad y = (|x| + t)\frac{x}{|x|}.$$

Therefore

$$u(x,t) = -|x| - \tfrac{1}{2}t \quad \text{in } \mathbb{R}^N \times \mathbb{R}^+.$$

One verifies that this function satisfies the Hamilton–Jacobi equation in (12.3) for all $|x| > 0$.

Remark 12.2 For fixed $t > 0$, the graph of $x \to u(x,t)$ is concave, and

$$u_{x_i x_j} \xi_i \xi_j \le 0 \quad \text{for all } \xi \in \mathbb{R}^N \quad \text{in } \mathbb{R}^N \times \mathbb{R}^+ - \{|x| = 0\}. \tag{12.4}$$

12.3 Example III

The Cauchy problem

$$u_t + (u_x)^2 = 0 \quad \text{in } \mathbb{R} \times \mathbb{R}^+, \qquad u(x,0) = 0 \tag{12.5}$$

has the identically zero solution. However the function

$$u(x,t) = \begin{cases} 0 & \text{for} \quad |x| \ge t \\ x - t & \text{for} \quad 0 \le x \le t \\ -x - t & \text{for} \quad -t \le x \le 0. \end{cases}$$

is Lipschitz continuous in $\mathbb{R} \times \mathbb{R}^+$, and it satisfies the equation (12.5) in $\mathbb{R} \times \mathbb{R}^+$ except on the half-lines $x = \pm t$.

Remark 12.3 For fixed $t > 0$, the graph of $x \to u(x,t)$ is convex for $|x| < t$ and concave for $|x| > t$. In the region of concavity, $u_{xx} = 0$, whereas in the region of convexity, $u(\cdot,t)$ is not of class C^2, and whenever it does exist, the second derivative does not satisfy an upper bound of the type of (12.2). This lack of control on the *convex part* of the graph of $u(\cdot,t)$ is responsible for the lack of uniqueness of the solution of (12.5).

This example raises the issue of identifying a class of solutions of the Cauchy problem (6.4) within which uniqueness holds.

13 Uniqueness

Denote by $\mathcal{C}_o$ the class of solutions $(x,t) \to u(x,t)$ to the Cauchy problem (6.4), of class $C^2(\mathbb{R}^N \times \mathbb{R}^+)$, uniformly Lipschitz continuous in $\mathbb{R}^N \times \mathbb{R}^+$ and such that the graph of $x \to u(x,t)$ is concave for all $t > 0$, that is

$$\mathcal{C}_o = \begin{cases} u \in C^2(\mathbb{R}^N \times \mathbb{R}^+) & \\ u_{x_i x_j}\xi_i\xi_j \le 0 & \text{for all } \ \xi \in \mathbb{R}^N \quad \text{in } \ \mathbb{R}^N \times \mathbb{R}^+ \\ |\nabla u| \le C & \text{for some } \ C > 0 \quad \text{in } \ \mathbb{R}^N \times \mathbb{R}^+. \end{cases} \tag{13.1}$$

Proposition 13.1 *Let u_1 and u_2 be two solutions of the Cauchy problem (6.4) in the class $\mathcal{C}_o$. Then $u_1 = u_2$.*

Proof. Setting $w = u_1 - u_2$, compute

$$\begin{aligned} w_t &= \mathcal{H}(D_x u_2) - \mathcal{H}(D_x u_1) = \int_0^1 \frac{d}{ds}\mathcal{H}(sD_x u_2 + (1-s)D_x u_1)ds \\ &= -\left(\int_0^1 \mathcal{H}_{p_j}(sD_x u_2 + (1-s)D_x u_1)ds\right) w_{x_j} = -\mathbf{V}\cdot D_x w \end{aligned} \tag{13.2}$$

where

$$\mathbf{V} = \int_0^1 D_p\mathcal{H}(sD_x u_2 + (1-s)D_x u_1)ds. \tag{13.3}$$

Multiplying (13.2) by $2w$ gives

$$w_t^2 = -\mathbf{V}\cdot D_x w^2 = -\operatorname{div}(\mathbf{V}w^2) + w^2 \operatorname{div}\mathbf{V}. \tag{13.4}$$

Lemma 13.1 $\operatorname{div}\mathbf{V} \le 0$.

Proof. Fix $(x,t) \in \mathbb{R}^N \times \mathbb{R}^+$ and $s \in (0,1)$, set

$$\begin{aligned} p &= sD_x u_2(x,t) + (1-s)D_x u_1(x,t) \\ z_{ij} &= su_{2,x_i x_j}(x,t) + (1-s)u_{1,x_i x_j}(x,t) \end{aligned}$$

and compute

$$\operatorname{div}\mathbf{V} = \int_0^1 \mathcal{H}_{p_ip_j}(p) z_{ji} ds.$$

The integrand is the trace of the product matrix $(\mathcal{H}_{p_ip_j})(z_{ij})$. Since $\mathcal{H}$ is convex, $(\mathcal{H}_{p_ip_j})$ is symmetric and positive semi-definite, and its eigenvalues $\lambda_h = \lambda_h(x,t,s)$ for $h = 1,\dots,N$ are non-negative. Since both matrices (u_{ℓ,x_ix_j}) for $\ell = 1,2$ are negative semi-definite, the same is true for the convex combination

$$(z_{ij}) = s(u_{2,x_ix_j}) + (1-s)(u_{1,x_ix_j}).$$

In particular, its eigenvalues $\mu_h = \mu_h(x,t,s)$ for $h = 1,\dots,N$ are nonpositive. Therefore

$$\mathcal{H}_{p_ip_j} z_{ji} = \operatorname{trace}(\mathcal{H}_{p_ip_j})(z_{ij}) = \lambda_h \mu_h \le 0. \qquad \blacksquare$$

This in (13.4) gives

$$w_t^2 + \operatorname{div}(\mathbf{V}w^2) \le 0 \quad \text{in } \mathbb{R}^N \times \mathbb{R}^+. \tag{13.5}$$

Fix $x_o \in \mathbb{R}^N$ and $T > 0$, and introduce the backward characteristic cone with vertex at (x_o, T)

$$C_M = \big[|x - x_o| \le M(T-\tau);\ 0 \le \tau \le T\big] \tag{13.6}$$

where $M > 0$ is to be chosen. The exterior unit normal to the lateral surface of C_M is

$$\nu = \frac{(x/|x|, M)}{\sqrt{1+M^2}} = (\nu_x, \nu_t).$$

For $t \in (0,T)$ introduce also the backward truncated characteristic cone

$$C_M^t = \big[|x - x_o| \le M(T-\tau);\ 0 \le \tau \le t\big] \tag{13.7}$$

Integrating (13.5) over such a truncated cone gives

$$\begin{aligned} &\int_{|x-x_o|<M(T-t)} w^2(x,t)dx + \frac{M}{\sqrt{1+M^2}} \int_0^t \int_{|x-x_o|=M(T-\tau)} w^2 d\sigma(\tau) d\tau \\ &\le \int_{|x-x_o|<MT} w^2(x,0)dx - \int_0^t \int_{|x-x_o|=M(T-\tau)} w^2 \mathbf{V}\cdot\nu_x d\sigma(\tau) d\tau \end{aligned} \tag{13.8}$$

where $d\sigma(\tau)$ is the surface measure on the sphere $[|x - x_o| = M(T-\tau)]$. Using the constant C in (13.1) and the definition (13.3) of $\mathbf{V}$i, choose M from

$$|\mathbf{V}\cdot\nu_x| \le \sup_{|p|<C} |D_p\mathcal{H}(p)| = \delta M \quad \text{for some } \delta \in (0,1). \tag{13.9}$$

This choice of M in (13.8) gives

$$\begin{aligned} \int_{|x-x_o|<M(T-t)} w^2(x,t)dx + \frac{(1-\delta)M}{\sqrt{1+M^2}} \int_0^t \int_{|x-x_o|=M(T-\tau)} w^2 d\sigma(\tau) d\tau \\ \le \int_{|x-x_o|<MT} w^2(x,0)dx. \end{aligned}$$

Thus

$$\int_{|x-x_o|<M(T-t)} w^2(x,t)dx \le \int_{|x-x_o|<MT} w^2(x,0)dx. \tag{13.10}$$

14 More on Uniqueness and Stability

Multiply (13.2) by $f'(w)$, for some non-negative $f \in C^1(\mathbb{R})$ and then use Lemma 13.1 to get

$$f(w)_t + \operatorname{div}(\mathbf{V} f(w)) \le 0 \qquad \text{in } \mathbb{R}^N \times \mathbb{R}^+.$$

By similar arguments

$$\int_{|x-x_o|<M(T-t)} f(w(x,t))dx \le \int_{|x-x_o|<MT} f(w(x,0))dx. \tag{14.1}$$

By the change of variables $x - x_o = y$

$$\int_{|y|<M(T-t)} f(w(x_o+y,t))d\xi \le \int_{|y|<MT} f(w(x_o+y,0))dy.$$

Integrating this in dx_o over $\mathbb{R}^N$ gives

$$\int_{\mathbb{R}^N} f(w(x,t))dx \le \left(\frac{T}{T-t}\right)^N \int_{\mathbb{R}^N} f(w(x,0))dx$$

provided the integrals are convergent. Fix $0 < t < T$ and let $T \to \infty$ to obtain the stability estimate

$$\int_{\mathbb{R}^N} f(w(x,t))dx \le \int_{\mathbb{R}^N} f(w(x,0))dx. \tag{14.2}$$

14.1 Stability in $L^p(\mathbb{R}^N)$ for All $p \ge 1$

Proposition 14.1 *Let u_1 and u_2 be solutions of (6.4) in the class $\mathcal{C}_o$ introduced in (13.1). If both are in $L^p(\mathbb{R}^N)$ for some $1 \le p \le \infty$, then*

$$\|u_1(\cdot,t) - u_2(\cdot,t)\|_{p,\mathbb{R}^N} \le \|u_1(\cdot,0) - u_2(\cdot,0)\|_{p,\mathbb{R}^N}. \tag{14.3}$$

Proof. If $1 \le p < \infty$, the conclusion follows from (14.2) for $f(w) = |w|^p$. If $p = 1$, take $f(w) = \operatorname{sign}(w)$, modulo an approximation process. If $p = \infty$ write (14.1) with $f(w) = |w|^q$ for $1 < q < \infty$, in the form

$$\begin{aligned}\left(\frac{N}{\omega_N[M(T-t)]^N} \int_{|x-x_o|<M(T-t)} |w(x,t)|^q dx\right)^{1/q} \\ \le \left(\frac{T}{T-t}\right)^{N/q} \left(\frac{N}{\omega_N (MT)^N} \int_{|x-x_o|<MT} |w(x,0)|^q dx\right)^{1/q}\end{aligned}$$

where ω_N is the measure of the unit sphere in $\mathbb{R}^N$. Letting $q \to \infty$ gives

$$\|w(\cdot,t)\|_{\infty,[|x-x_o|<M(T-t)]} \le \|w(\cdot,0)\|_{\infty,[|x-x_o|<MT]}.$$

This implies (14.3) for $p = \infty$ since x_o is arbitrary. ■

14.2 Comparison Principle

Proposition 14.2 *Let u_1 and u_2 be solutions of (6.4) in the class $\mathcal{C}_o$ introduced in (13.1). If $u_{o,1} \le u_{o,2}$, then*

$$u_1(\cdot,t) \le u_2(\cdot,t) \quad \textit{in } \mathbb{R}^N \quad \textit{for all } t>0. \tag{14.4}$$

Proof. In (14.2) choose $f(w) = w_+$ modulo an approximation process. ■

15 Semi-Concave Solutions of the Cauchy Problem

Let u be a solution of the Cauchy problem (6.4) of class $C^2(\mathbb{R}^N \times \mathbb{R}^+)$ with no requirement that the graph of $u(\cdot,t)$ be concave. We require, however, that in those regions where such a graph is convex, the "convexity", roughly speaking, be controlled by some uniform bound of the second derivatives of $u(\cdot,t)$. In a precise way it is assumed that there exists a positive constant γ such that

$$(u_{x_i x_j}) - \gamma \mathbb{I} \le 0 \qquad \text{in } \mathbb{R}^N \times \mathbb{R}^+.$$

Since this matrix inequality is invariant by rotations of the coordinate axes, it is equivalent to

$$u_{\nu\nu} \le \gamma \quad \text{in } \mathbb{R}^N \times \mathbb{R}^+ \quad \text{for all } |\nu| = 1. \tag{15.1}$$

A solution of the Cauchy problem (6.4) satisfying such an inequality for all unit vectors $\nu \in \mathbb{R}^N$ is called *semi-concave.*

15.1 Uniqueness of Semi-Concave Solutions

The example in Section 12.3 shows that initial data, however smooth, might give rise to quasi-concave solutions and solutions for which (15.1) is violated. Introduce the class

$$\mathcal{C}_1 = \begin{cases} u \in C^2(\mathbb{R}^N \times \mathbb{R}^+) \text{ satisfies (15.1) and} \\ |\nabla u| \le C \text{ for some } C>0 \text{ in } \mathbb{R}^N \times \mathbb{R}^+. \end{cases} \tag{15.2}$$

Proposition 15.1 *Let u_1 and u_2 be two solutions of the Cauchy problem (6.4) in the class $\mathcal{C}_1$. Then $u_1 = u_2$.*

Proof. Set $w = u_1 - u_2$ and proceed as in the proof of Proposition 13.1 to arrive at (13.2). Multiplying the latter by $\operatorname{sign} w$ modulo an approximation process gives

$$|w|_t = -\operatorname{div}(\mathbf{V}|w|) + |w| \operatorname{div} \mathbf{V}$$

where $\mathbf{V}$ is defined in (13.3).

Lemma 15.1 *There exists a constant $\bar{\gamma}$ depending only on the constant γ in (15.1) and C in (15.2) such that* $\operatorname{div} \mathbf{V} \le \bar{\gamma}$ *in $\mathbb{R}^N \times \mathbb{R}^+$.*

Proof. With the same notation as in Lemma 13.1

$$\begin{aligned}\mathcal{H}_{p_i p_j}(p) z_{ij} &= \operatorname{trace}(\mathcal{H}_{p_i p_j})(z_{ij}) = \operatorname{trace}(\mathcal{H}_{p_i p_j})((z_{ij}) - \gamma \mathbb{I} + \gamma \mathbb{I}) \\ &\le \operatorname{trace}(\mathcal{H}_{p_i p_j})((z_{ij}) - \gamma \mathbb{I}) + \gamma \operatorname{trace}(\mathcal{H}_{p_i p_j}(p)) \le \gamma \mathcal{H}_{p_i p_i}(p).\end{aligned}$$

Since the Hamiltonian is convex, $\mathcal{H}_{p_i p_i}(p) \ge 0$. Moreover, since the solutions u_1 and u_2 are both uniformly Lipschitz in $\mathbb{R}^N \times \mathbb{R}^+$

$$0 \le \gamma \mathcal{H}_{p_i p_i}(p) \le \gamma \sup_{|p| < C} \mathcal{H}_{p_i p_i}(p) = \bar{\gamma}. \qquad \blacksquare$$

Combining these estimates

$$|w|_t + \operatorname{div}(\mathbf{V}|w|) \le \bar{\gamma}|w| \quad \text{ in } \mathbb{R}^N \times \mathbb{R}^+.$$

Introduce the backward characteristic cone C_M and the truncated backward characteristic cone C_M^t as in (13.6) and (13.7), where the constant M is chosen as in (13.9). Similar calculations yield

$$\int_{|x - x_o| < M(T - t)} |w(x, t)| dx \le \bar{\gamma} \int_0^t \int_{|x - x_o| < M(T - \tau)} |w(x, \tau)| dx d\tau.$$

This implies $w = 0$ by Gronwall's inequality. ■

16 A Weak Notion of Semi-Concavity

The most limiting requirement of the class $\mathcal{C}_1$ is that solutions have to be of class $C^2(\mathbb{R}^N \times \mathbb{R}^+)$. Such a requirement is not natural, since the equation in (6.4) imposes no conditions on the second derivatives of it solutions. In addition, Proposition 10.1 establishes only that variational solutions of (8.3) are Lipschitz continuous, however smooth the initial datum might be. On the other hand, the example of Section 12.3 shows that uniqueness fails if some assumptions are not formulated on the graph of $u(\cdot, t)$ through the second derivatives of solutions. A condition of semi-concavity can be imposed using a discrete form of second derivatives. A solution of the Cauchy problem (6.4) is weakly semi-concave if there exists a positive constant γ such that for every unit vector $\nu \in \mathbb{R}^N$ and all $h \in \mathbb{R}$

$$u(x + h\nu, t) - 2u(x, t) + u(x - h\nu, t) \le \gamma \Big(1 + \frac{1}{t}\Big) h^2. \tag{16.1}$$

Remark 16.1 The t-dependence on the right-hand side allows for nonsemiconcave initial data.

For $\varepsilon > 0$, let k_ε be a mollifying kernel in $\mathbb{R}^N$, and let $u_\varepsilon(\cdot,t)$ be the mollification of $u(\cdot,t)$ with respect to the space variables, i.e.,

$$x \to u_\varepsilon(x,t) = \int_{\mathbb{R}^N} k_\varepsilon(x-y)u(y,t)dy.$$

Lemma 16.1 *Let $u(\cdot,t)$ be weakly semi-concave in the sense of (16.1). Then for every unit vector $\nu \in \mathbb{R}^N$ and all $\varepsilon > 0$,*

$$u_{\varepsilon,\nu\nu} \le \gamma\Big(1+\frac{1}{t}\Big) \qquad \textit{in } \mathbb{R}^N \times \mathbb{R}^+.$$

Proof. Fix $\nu \in \mathbb{R}^N$ and $\varepsilon > 0$ and compute

$$\begin{aligned} u_{\varepsilon,\nu\nu}(x,t) &= \int_{\mathbb{R}^N} k_{\varepsilon,\nu\nu}(x-y)u(y,t)dy \\ &= \lim_{h\to 0}\frac{1}{h^2}\int_{\mathbb{R}^N}\left[k_\varepsilon(x+h\nu-y) - 2k_\varepsilon(x-y,t) + k_\varepsilon(x-h\nu-y)\right]u(y,t)dy \\ &= \lim_{h\to 0}\frac{1}{h^2}\int_{\mathbb{R}^N} k_\varepsilon(x-\eta)\left[u(\eta+h\nu,t) - 2u(\eta,t) + u(\eta-h\nu,t)\right]d\eta \\ &\le \gamma\Big(1+\frac{1}{t}\Big)\int_{\mathbb{R}^N} k_\varepsilon(x-\eta)d\eta = \gamma\Big(1+\frac{1}{t}\Big). \end{aligned}$$

■

Corollary 16.1 *Let $u(\cdot,t)$ be weakly semi-concave in the sense of (16.1). Then for all $\varepsilon > 0$*

$$(u_{\varepsilon,x_ix_j}) - \gamma\Big(1+\frac{1}{t}\Big)\mathbb{I} \le 0 \qquad \textit{in } \mathbb{R}^N \times \mathbb{R}^+.$$

17 Semi-Concavity of Hopf Variational Solutions

The semi-concavity condition (16.1) naturally arises from the variational formula (8.3). Indeed, if u_o is weakly semi-concave, the corresponding variational solution is weakly semi-concave. Moreover, if the Hamiltonian $p \to \mathcal{H}(p)$ is strictly convex, then the corresponding variational solution is weakly semi-concave irrespective of whether the initial datum is weakly semi-concave. The next two sections contain these results. Here we stress that they hold for the variational solutions (8.3) and not necessarily for any solution of the Cauchy problem (6.4).

17.1 Weak Semi-Concavity of Hopf Variational Solutions Induced by the Initial Datum u_o

Proposition 17.1 *Let u_o be weakly semi-concave, that is there exists a positive constant γ_o such that for every unit vector $\nu \in \mathbb{R}^N$ and all $h \in \mathbb{R}$*

$$u_o(x+h\nu)-2u_o(x)+u(x-h\nu)\le\gamma_o h^2 \qquad \textit{in}\ \ \mathbb{R}.$$

Then $x\to u(x,t)$ is weakly semi-concave, uniformly in t, for the same constant γ_o.

Proof. Let $\xi\in\mathbb{R}^N$ be a vector where the minimum in (8.3) is achieved. Then

$$u(x\pm h\nu,t)=\min_{y\in\mathbb{R}^N}\Big[t\mathcal{L}\Big(\frac{x\pm h\nu-y}{t}\Big)+u_o(y)\Big]\le t\mathcal{L}\Big(\frac{x-\xi}{t}\Big)+u_o(\xi\pm h\nu).$$

From this

$$\begin{aligned}u(x+h\nu,t)-&2u(x,t)+u(x-h\nu,t)\\ &\le t\mathcal{L}\Big(\frac{x-\xi}{t}\Big)+u_o(\xi+h\nu)-2t\mathcal{L}\Big(\frac{x-\xi}{t}\Big)\\ &\quad-2u_o(\xi)+t\mathcal{L}\Big(\frac{x-\xi}{t}\Big)+u_o(\xi-h\nu)\\ &=u_o(\xi+h\nu)-2u_o(\xi)+u_o(\xi-h\nu)\le\gamma_o h^2.\end{aligned}$$ ■

17.2 Strictly Convex Hamiltonian

The Hamiltonian $p\to\mathcal{H}(p)$ is *strictly* convex, if there exists a positive constant c_o such that $(\mathcal{H}_{p_ip_j})\ge c_o\mathbb{I}$ in $\mathbb{R}^N$.

Lemma 17.1 *Let $p\to\mathcal{H}(p)$ be strictly convex. then for all $p_1,p_2\in\mathbb{R}^N$*

$$\mathcal{H}\Big(\frac{p_1+p_2}{2}\Big)\le\frac{1}{2}\mathcal{H}(p_1)+\frac{1}{2}\mathcal{H}(p_2)-\frac{c_o}{8}|p_1-p_2|^2.$$

Moreover, if $\mathcal{L}$ is the Lagrangian corresponding to $\mathcal{H}$, then for all $q_1,q_2\in\mathbb{R}^N$

$$\frac{1}{2}\mathcal{L}(q_1)+\frac{1}{2}\mathcal{L}(q_2)\le\mathcal{L}\Big(\frac{q_1+q_2}{2}\Big)+\frac{1}{8c_o}|q_1-q_2|^2.$$

Proof. Fix $p_1\ne p_2$ in $\mathbb{R}^N$ and set

$$\bar p=\tfrac{1}{2}(p_1+p_2),\qquad \ell=|p_2-p_1|,\qquad \nu=(p_2-p_1)/2\ell.$$

Consider the two segments $(p_1,\bar p)$ and $(\bar p,p_2)$ with parametric equations

$$\begin{aligned}(p_1,\bar p)&=\{y(\sigma)=\bar p+\sigma\nu;\ \sigma\in(0,-\tfrac{1}{2}\ell)\}\\ (\bar p,p_2)&=\{y(\sigma)=\bar p+\sigma\nu;\ \sigma\in(0,\tfrac{1}{2}\ell)\}\end{aligned}$$

and compute

$$\begin{aligned}\mathcal{H}(\bar p)&=\mathcal{H}(p_1)+\int_0^{\ell/2}D_\sigma\mathcal{H}(\bar p-\sigma\nu)\cdot\nu d\sigma\\ \mathcal{H}(\bar p)&=\mathcal{H}(p_2)-\int_0^{\ell/2}D_\sigma\mathcal{H}(\bar p+\sigma\nu)\cdot\nu d\sigma.\end{aligned}$$

Adding them up

$$2\mathcal{H}(\bar{p}) = \mathcal{H}(p_1) + \mathcal{H}(p_2) - \int_0^{\ell/2} [D_\sigma \mathcal{H}(\bar{p} + \sigma\nu) - D_\sigma \mathcal{H}(\bar{p} - \sigma\nu)] \cdot \nu d\sigma.$$

By the mean value theorem, there exists some $\sigma' \in (0, \frac{1}{2}\ell)$ such that

$$[D_\sigma \mathcal{H}(\bar{p} + \sigma\nu) - D_\sigma \mathcal{H}(\bar{p} - \sigma\nu)] \cdot \nu = \mathcal{H}_{p_i p_j}(\bar{p} - \sigma'\nu)\nu_i\nu_j 2\sigma \geq c_o 2\sigma.$$

Combining these calculations

$$2\mathcal{H}(\bar{p}) \leq \mathcal{H}(p_1) + \mathcal{H}(p_2) - c_o \int_0^{\ell/2} 2\sigma d\sigma.$$

This proves the first statement. To prove the second, recall that $\mathcal{L}$ is the Legendre transform of $\mathcal{H}$. Therefore

$$\mathcal{L}(q_1) \leq q_1 \cdot p_1 - \mathcal{H}(p_1), \qquad \mathcal{L}(q_2) \leq q_2 \cdot p_2 - \mathcal{H}(p_2)$$

for all $p_1, p_2 \in \mathbb{R}^N$. From this

$$\begin{aligned}\frac{1}{2}\mathcal{L}(q_1) + \frac{1}{2}\mathcal{L}(q_2) &\leq \frac{1}{2}(q_1 \cdot p_1 + q_2 \cdot p_2) - \Big(\frac{1}{2}\mathcal{H}(p_1) + \frac{1}{2}\mathcal{H}(p_2)\Big) \\ &\leq \frac{1}{2}(q_1 \cdot p_1 + q_2 \cdot p_2) - \mathcal{H}\Big(\frac{p_1 + p_2}{2}\Big) - \frac{c_o}{8}|p_1 - p_2|^2.\end{aligned}$$

Transform

$$\frac{1}{2}(q_1 \cdot p_1 + q_2 \cdot p_2) = \Big(\frac{q_1 + q_2}{2}\Big)\Big(\frac{p_1 + p_2}{2}\Big) + \frac{1}{4}(q_1 - q_2) \cdot (p_1 - p_2)$$

and combine with the previous inequality to obtain

$$\begin{aligned}&\frac{1}{2}\mathcal{L}(q_1) + \frac{1}{2}\mathcal{L}(q_2) \leq \Big(\frac{q_1 + q_2}{2}\Big)\Big(\frac{p_1 + p_2}{2}\Big) - \mathcal{H}\Big(\frac{p_1 + p_2}{2}\Big) \\ &- \Big(\frac{c_o}{8}|p_1 - p_2|^2 - \frac{1}{4}(q_1 - q_2) \cdot (p_1 - p_2) + \frac{1}{8c_o}|q_1 - q_2|^2\Big) + \frac{1}{8c_o}|q_1 - q_2|^2 \\ &\leq \max_{p \in \mathbb{R}^N} \Big[\Big(\frac{q_1 + q_2}{2}\Big) \cdot p - \mathcal{H}(p)\Big] - \Big[\sqrt{\frac{c_o}{8}}(p_1 - p_2) - \frac{1}{\sqrt{8c_o}}(q_1 - q_2)\Big]^2 \\ &\quad + \frac{1}{8c_o}|q_1 - q_2|^2 \\ &\leq \mathcal{L}\Big(\frac{q_1 + q_2}{2}\Big) + \frac{1}{8c_o}|q_1 - q_2|^2.\end{aligned}$$

■

Proposition 17.2 *Let $\mathcal{H}$ be strictly convex. Then every variational solution of (8.3) is weakly semi-concave for all $t > 0$, in the sense of (16.1), for a constant γ independent of u_o.*

Proof. Let $\xi \in \mathbb{R}^N$ be a vector where the minimum in (8.3) is achieved. Then

$$\begin{aligned}
u(x+h\nu,t) - 2u(x,t) + u(x-h\nu,t) &= \min_{y\in\mathbb{R}^N}\Big[t\mathcal{L}\Big(\frac{x+h\nu-y}{t}\Big) + u_o(y)\Big] \\
&\quad - 2\Big[t\mathcal{L}\Big(\frac{x-\xi}{t}\Big) + u_o(\xi)\Big] + \min_{y\in\mathbb{R}^N}\Big[t\mathcal{L}\Big(\frac{x-h\nu-y}{t}\Big) + u_o(y)\Big] \\
&\le \Big[t\mathcal{L}\Big(\frac{x+h\nu-\xi}{t}\Big) + u_o(\xi)\Big] - 2\Big[t\mathcal{L}\Big(\frac{x-\xi}{t}\Big) + u_o(\xi)\Big] \\
&\quad + \Big[t\mathcal{L}\Big(\frac{x-h\nu-\xi}{t}\Big) + u_o(\xi)\Big] \\
&= 2t\Big[\frac{1}{2}\mathcal{L}\Big(\frac{x+h\nu-\xi}{t}\Big) + \frac{1}{2}\mathcal{L}\Big(\frac{x-h\nu-\xi}{t}\Big) - \mathcal{L}\Big(\frac{x-\xi}{t}\Big)\Big] \\
&\le 2t\frac{|2h\nu/t|^2}{8c_o} = \frac{h^2}{tc_o}.
\end{aligned}$$

■

18 Uniqueness of Weakly Semi-Concave Variational Hopf Solutions

Introduce the class $\mathcal{C}_2$ of solutions

$$\mathcal{C}_2 = \begin{cases} u \ \text{ is a variational solution of (8.3)} \\ u(\cdot,t) \ \text{ is weakly semi-concave in the sense of (16.1)} \\ |\nabla u| \le C \ \text{ for some } \ C>0 \ \text{ in } \ \mathbb{R}^N\times\mathbb{R}^+. \end{cases} \tag{18.1}$$

Theorem 18.1. *The Cauchy problem has at most one solution within the class $\mathcal{C}_2$.*

Proof. Let u_1 and u_2 be two solutions in $\mathcal{C}_2$ and set $w = u_1 - u_2$. Proceeding as in the proof of Proposition 13.1, we arrive at an analogue of (13.2), which in this context holds a.e. in $\mathbb{R}^N\times\mathbb{R}^+$. From the latter, we derive

$$f(w)_t = -\mathbf{V}\cdot D_x f(w) \qquad \text{a.e. in } \ \mathbb{R}^N\times\mathbb{R}^+$$

for any $f\in C^1(\mathbb{R})$, where $\mathbf{V}$ is defined in (13.3). For $\varepsilon>0$, let $u_{\varepsilon,1}$ and $u_{\varepsilon,2}$ be the mollifications of u_1 and u_2 as in Section 16, and set

$$\mathbf{V}_\varepsilon = \int_0^1 D_p\mathcal{H}(sD_xu_{\varepsilon,2} + (1-s)D_xu_{\varepsilon,1})ds.$$

With this notation

$$f(w)_t = -\operatorname{div}(\mathbf{V}_\varepsilon f(w)) + f(w)\operatorname{div}\mathbf{V}_\varepsilon + (\mathbf{V}_\varepsilon - \mathbf{V})\cdot D_x f(w).$$

Lemma 18.1 *There exists a positive constant $\bar\gamma$, independent of ε, such that*

$$\operatorname{div}\mathbf{V}_\varepsilon \le \bar\gamma\Big(1+\frac{1}{t}\Big) \qquad \text{in } \ \mathbb{R}^N\times\mathbb{R}^+.$$

Proof. Same as in Lemma 15.1 with the proper minor modifications. ■

Putting this in the previous expression of $f(w)_t$, and assuming that $f(\cdot)$ is non-negative, gives

$$f(w)_t \le -\operatorname{div}(\mathbf{V}_\varepsilon f(w)) + \bar\gamma\Big(1+\frac{1}{t}\Big) f(w) + (\mathbf{V}_\varepsilon - \mathbf{V})\cdot D_x f(w). \tag{18.2}$$

Introduce the backward characteristic cone C_M and the truncated backward characteristic cone C_M^t as in (13.6) and (13.7), where the constant M is chosen as in (13.9). For a fixed $0<\sigma<t<T$, introduce also the truncated cone

$$C_M^{t,\sigma} = \big[|x-x_o| < M(T-\tau);\ 0<\sigma<\tau<t<T\big].$$

Now integrate (18.2) over such a cone. Proceeding as in the proof of Proposition 13.1, and taking into account the choice (13.9) of M, yields

$$\begin{aligned}\int_{|x-x_o|<M(T-t)} f(w(x,t))dx \le& \int_{|x-x_o|<M(T-\sigma)} f(w(x,\sigma))dx \\ &+ \bar\gamma \int_\sigma^t \Big(1+\frac{1}{t}\Big)\int_{|x-x_o|<M(T-\tau)} f(w(x,\tau))dxd\tau \\ &+ \int_\sigma^t \int_{|x-x_o|<M(T-\tau)} (\mathbf{V}_\varepsilon - \mathbf{V})\cdot D_x f(w)dxd\tau.\end{aligned} \tag{18.3}$$

By the properties of the class $\mathcal{C}_2$, $|D_x w|$, $|D_x u_1|$, and $|D_x u_2|$ are a.e. bounded in $\mathbb{R}^N \times \mathbb{R}^+$, independent of ε. Therefore

$$\lim_{\varepsilon\to 0}\int_\sigma^t \int_{|x-x_o|<M(T-\tau)} (\mathbf{V}_\varepsilon - \mathbf{V})\cdot D_x w dxd\tau = 0.$$

Observe first that (18.3) continues to hold for non-negative functions $f(\cdot)$, uniformly Lipschitz continuous in $\mathbb{R}$. Choose $f_\delta(w) = (|w|-\delta)_+$, for some fixed $\delta\in(0,1)$. There exists $\sigma>0$ such that $f_\delta(w(\cdot,\tau)) = 0$ for all $\tau\in(0,\sigma]$. Indeed, by virtue of (10.3), for all $\tau\in(0,\sigma]$

$$\begin{aligned}(|w(x,\tau)|-\delta)_+ &\le (|u_1(x,\tau)-u_o(x)| + |u_2(x,\tau)-u_o(x)| - \delta)_+ \\ &\le (2\bar C\sigma - \delta)_+ = 0\end{aligned}$$

provided $\sigma < \delta/2\bar C$. These remarks in (18.3) yield

$$\begin{aligned}&\int_{|x-x_o|<M(T-t)} f_\delta(w(x,t))dx \\ &\qquad \le \bar\gamma \int_\sigma^t \Big(1+\frac{1}{t}\Big)\int_{|x-x_o|<M(T-\tau)} f_\delta(w(x,\tau))dxd\tau \\ &\qquad \le \bar\gamma\Big(1+\frac{1}{\sigma}\Big)\int_\sigma^t \int_{|x-x_o|<M(T-\tau)} f_\delta(w(x,\tau))dxd\tau.\end{aligned}$$

Setting

$$\varphi_\delta(t) = \int_\sigma^t \int_{|x-x_o|<M(T-\tau)} f_\delta(w(x,\tau))dxd\tau$$

the previous inequality reads as

$$\varphi_\delta'(t) \le \bar{\gamma}\Big(1 + \frac{1}{\sigma}\Big)\varphi_\delta(t) \qquad \text{and} \ \ \varphi_\delta(\sigma) = 0.$$

This implies that $\varphi_\delta(\tau) = 0$ for all $\tau \in (0,t)$, and since (x_o, T) is arbitrary, $|w| \le \delta$ in $\mathbb{R}^N \times \mathbb{R}^+$, for all $\delta > 0$. ■

9

LINEAR ELLIPTIC EQUATIONS WITH MEASURABLE COEFFICIENTS

1 Weak Formulations and Weak Derivatives

Let E be a bounded domain in $\mathbb{R}^N$ with boundary ∂E of class C^1. Denote by (a_{ij}) an $N \times N$ symmetric matrix with entries $a_{ij} \in L^\infty(E)$, and satisfying the *ellipticity condition*

$$\lambda|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2 \tag{1.1}$$

for all $\xi \in \mathbb{R}^N$ and all $x \in E$, for some $0 < \lambda \le \Lambda$. The number Λ is the least upper bound of the eigenvalues of (a_{ij}) in E, and λ is their greatest lower bound. A vector-valued function $\mathbf{f} = (f_1, \dots, f_N) : E \to \mathbb{R}^N$ is said to be in $L^p_{\text{loc}}(E)$, for some $p \ge 1$, if all the components $f_j \in L^p_{\text{loc}}(E)$. Given a scalar function $f \in L^1_{\text{loc}}(E)$ and a vector-valued function $\mathbf{f} \in L^1_{\text{loc}}(E)$, consider the formal partial differential equation in divergence form (Section 3.1 of the Preliminaries)

$$-\big(a_{ij}u_{x_i}\big)_{x_j} = \operatorname{div}\mathbf{f} - f \quad \text{ in } E. \tag{1.2}$$

Expanding formally the indicated derivatives gives a PDE of the type of (3.1) of Chapter 1, which, in view of the ellipticity condition (1.1), does not admit real characteristic surfaces (Section 3 of Chapter 1). In this formal sense, (1.2) is a *second-order elliptic* equation.

Multiply (1.2) formally by a function $v \in C_o^\infty(E)$ and formally integrate by parts in E to obtain

$$\int_E \big(a_{ij}u_{x_i}v_{x_j} + f_j v_{x_j} + fv\big)dx = 0. \tag{1.3}$$

This is well defined for all $v \in C_o^\infty(E)$, provided $\nabla u \in L^p_{\text{loc}}(E)$, for some $p \ge 1$. In such a case (1.3) is the *weak formulation* of (1.2), and u is a *weak solution*. Such a weak notion of solution coincides with the classical one whenever the various terms in (1.3) are sufficiently regular. Indeed, assume that $f \in C(E)$ and $a_{ij}, \mathbf{f} \in C^1(E)$; if a function $u \in C^2(E)$ satisfies (1.3) for all $v \in C_o^\infty(E)$, integrating by parts gives

© Springer Nature Switzerland AG 2023

E. DiBenedetto, U. Gianazza, *Partial Differential Equations*,
Cornerstones, https://doi.org/10.1007/978-3-031-46618-2_10

$$\int_E \left[\left(a_{ij}u_{x_i}\right)_{x_j} + \operatorname{div}\mathbf{f} + f)\right] v dx = 0 \qquad \text{for all } \; v \in C_o^\infty(E).$$

Thus u satisfies (1.2) in the classical sense. It remains to clarify the meaning of $\nabla u \in L^p_{\text{loc}}(E)$ for some $p \geq 1$.

1.1 Weak Derivatives

A function $u \in L^p_{\text{loc}}(E)$, for some $p \geq 1$, has a weak partial derivative in $L^p_{\text{loc}}(E)$ with respect to the variable x_j if there exists a function $w_j \in L^p_{\text{loc}}(E)$ such that

$$\int_E u v_{x_j}\, dx = -\int_E w_j v\, dx \qquad \text{for all } \; v \in C_o^\infty(E). \tag{1.4}$$

If $u \in C^1(E)$, then $w_j = u_{x_j}$ in the classical sense. There are functions admitting weak and not classical derivatives. As an example, $u(x) = |x|$ for $x \in (-1, 1)$ does not have a derivative at $x = 0$; however it admits the weak derivative

$$w = \begin{cases} -1 & \text{in } \; (-1, 0) \\ \;\;\,1 & \text{in } \; (0, 1) \end{cases} \qquad \text{as an element of } \; L^1(-1, 1).$$

With a perhaps improper but suggestive symbolism we set $w_j = u_{x_j}$, warning that in general, u_{x_j} need not be the limit of difference quotients along x_j, and it is meant only in the sense of (1.4). The derivatives u_{x_j} in (1.3) are meant in this weak sense, and solutions of (1.1) are sought as functions in the *Sobolev space* ([241])

$$W^{1,p}(E) = \{\text{the set of } u \in L^p(E) \text{ such that } \nabla u \in L^p(E)\}\,. \tag{1.5}$$

A norm in $W^{1,p}(E)$ is

$$\|u\|_{1,p} = \|u\|_p + \|\nabla u\|_p. \tag{1.6}$$

Proposition 1.1 (Meyers and Serrin [182]) *$W^{1,p}(E)$ is a Banach space for the norm (1.6). Moreover, $C^\infty(E)$ is dense in $W^{1,p}(E)$.*

Introduce also the two spaces

$$W_o^{1,p}(E) = \{\text{the closure of } C_o^\infty(E) \text{ in the norm (1.6)}\}\,. \tag{1.7}$$

$$\tilde{W}^{1,p}(E) = \left\{\text{all } u \in W^{1,p}(E) \text{ such that } \int_E u\, dx = 0\right\}. \tag{1.8}$$

Proposition 1.2 *$W_o^{1,p}(E)$ and $\tilde{W}^{1,p}(E)$ equipped with the norm (1.6) are Banach spaces.*

Functions in $W^{1,p}(E)$ are more "regular" than merely elements in $L^p(E)$, on several accounts. First, they are embedded in $L^q(E)$ for some $q > p$. Second, they form a compact subset of $L^p(E)$. Third, they have boundary values (traces) on ∂E, as elements of $L^p(\partial E)$.

2 Embeddings of $W^{1,p}(E)$

Since ∂E is of class C^1, there is a circular spherical cone $\mathcal{C}$ of height h and solid angle ω such that by putting its vertex at any point of ∂E, it can be properly swung, by a rigid rotation, to remain in E. This is the *cone condition* of ∂E. Denote by $\gamma = \gamma(N,p)$ a constant depending on N and p and independent of E and ∂E.

Theorem 2.1 (Sobolev–Nikol'skii [242]). *If $1 < p < N$ then $W^{1,p}(E) \hookrightarrow L^{p^*}(E)$, where $p^* = \frac{Np}{N-p}$, and there exists $\gamma = \gamma(N,p)$, such that*

$$\|u\|_{p^*} \le \frac{\gamma}{\omega}\Big\{\frac{1}{h}\|u\|_p + \|\nabla u\|_p\Big\}, \quad p^* = \frac{Np}{N-p}, \ \ \textit{for all} \ \ u \in W^{1,p}(E). \tag{2.1}$$

If $p = 1$ and $|E| < \infty$, then $W^{1,p}(E) \hookrightarrow L^q(E)$ for all $1 \le q < \frac{N}{N-1}$ and there exists $\gamma = \gamma(N,q)$ such that

$$\|u\|_q \le \frac{\gamma}{\omega}|E|^{\frac{1}{q}-\frac{N-1}{N}}\Big\{\frac{1}{h}\|u\|_1 + \|\nabla u\|_1\Big\} \qquad \textit{for all} \ \ u \in W^{1,1}(E). \tag{2.2}$$

If $p > N$ then $W^{1,p}(E) \hookrightarrow L^\infty(E)$, and there exists $\gamma = \gamma(N,p)$ such that

$$\|u\|_\infty \le \frac{\gamma}{\omega h^{N/p}}\left(\|u\|_p + h\|\nabla u\|_p\right) \qquad \textit{for all} \ \ u \in W^{1,p}(E). \tag{2.3}$$

If $p > N$ and in addition E is convex, then $W^{1,p}(E) \hookrightarrow C^{1-\frac{N}{p}}(\bar{E})$, and there exists $\gamma = \gamma(N,p)$ such that for every pair of points $x, y \in \bar{E}$ with $|x-y| \le h$ ([184])

$$|u(x) - u(y)| \le \frac{\gamma}{\omega}|x-y|^{1-\frac{N}{p}}\|\nabla u\|_p \qquad \textit{for all} \ \ u \in W^{1,p}(E). \tag{2.4}$$

Remark 2.1 The constants $\gamma(N,p)$ can be computed explicitly, and they tend to infinity as $p \to N$. This is expected as $W^{1,N}(E)$ is not embedded in $L^\infty(E)$. Indeed the function

$$[|x| < e^{-1}] - \{0\} \ni x \to \ln|\ln|x|| \in W^{1,N}(|x| < e^{-1})$$

is not essentially bounded about the origin. In this sense these embeddings are sharp ([251]). In (2.2) the value $q = 1^* = \frac{N}{N-1}$ is not permitted, and the corresponding constant $\gamma(N,q) \to \infty$ as $q \to 1^*$. The limiting embedding for $p = N$ takes a special form ([50], Chapter IX, Section 13).

Remark 2.2 The structure of ∂E enters only through the solid angle ω and the height h of the cone condition of ∂E. Therefore (2.1)–(2.4) continue to hold for domains whose boundaries merely satisfy the cone condition.

Remark 2.3 If E is not convex, the estimate (2.4) can be applied locally. Thus if $p > N$, a function $u \in W^{1,p}(E)$ is locally Hölder continuous in E.

A proof of these embeddings is in Section 2c of the Complements.

2.1 Compact Embeddings of $W^{1,p}(E)$

Theorem 2.2 (Rellich–Kondrachov [215, 139]). *Let* $1 \le p < N$. *Then for all* $1 \le q < p^*$, *the embedding* $W^{1,p}(E) \hookrightarrow L^q(E)$ *is compact.*

A proof is in Section 2.2c of the Complements.

Corollary 2.1 *Let* $\{u_n\}$ *be a bounded sequence in* $W^{1,p}(E)$. *If* $1 \le p < N$, *for each fixed* $1 \le q < \frac{Np}{N-p}$ *there exist a subsequence* $\{u_{n'}\} \subset \{u_n\}$, *and* $u \in W^{1,p}(E)$ *such that*

$$\{u_{n'}\} \to u \;\text{ weakly in }\; W^{1,p}(E) \qquad \text{and} \qquad \{u_{n'}\} \to u \;\text{ strongly in }\; L^q(E).$$

3 Multiplicative Embeddings of $W_o^{1,p}(E)$ and $\tilde{W}^{1,p}(E)$

Theorem 3.1 (Gagliardo–Nirenberg [90, 193]). *If* $1 \le p < N$, *then* $W_o^{1,p}(E) \hookrightarrow L^{p^*}(E)$ *for* $p^* = \frac{Np}{N-p}$, *and there exists* $\gamma = \gamma(N,p)$ *such that*

$$\|u\|_{p^*} \le \gamma \|\nabla u\|_p \qquad p^* = \frac{Np}{N-p} \quad \text{for all } u \in W_o^{1,p}(E). \tag{3.1}$$

If $p = N$, *then* $W_o^{1,p}(E) \hookrightarrow L^q(E)$ *for all* $q > p$, *and there exists* $\gamma = \gamma(N,q)$ *such that*

$$\|u\|_q \le \gamma \|\nabla u\|_p^{1-\frac{p}{q}} \|u\|_p^{\frac{p}{q}} \quad \text{for all } u \in W_o^{1,p}(E). \tag{3.2}$$

If $p > N$, *then* $W_o^{1,p}(E) \hookrightarrow L^\infty(E)$, *and there exists* $\gamma = \gamma(N,p)$ *such that*

$$\|u\|_\infty \le \gamma \|\nabla u\|_p^{\frac{N}{p}} \|u\|_p^{1-\frac{N}{p}} \quad \text{for all } u \in W_o^{1,p}(E). \tag{3.3}$$

Remark 3.1 The constants $\gamma(N,p)$ can be computed explicitly independent of ∂E, and they tend to infinity as $p \to N$. Unlike (2.2), the value 1^* is permitted in (3.1).

Functions in $W_o^{1,p}(E)$ are limits of functions in $C_o^\infty(E)$ in the norm of $W_o^{1,p}(E)$, and in this sense they vanish on ∂E. This permits embedding inequalities such as (3.1)–(2.3) with constants γ independent of E and ∂E. Inequalities of this kind would not be possible for functions $u \in W^{1,p}(E)$. For example, a constant nonzero function would not satisfy any of them. This suggest that for them to hold some information is required on some values of u. Let

$$u_E = \frac{1}{|E|} \int_E u \, dx$$

denote the integral average of u over E. The multiplicative embeddings (3.1)–(3.3) continue to hold for functions of zero average. Denote by $\gamma = \gamma(N, E, \partial E)$ a constant depending on N, $|E|$, and the C^1–smoothness of ∂E, but invariant under homothetic transformations of E.

Theorem 3.2 (Golovkin–Poincarè [107]). *If* $1 < p < N$, *then* $W^{1,p}(E) \hookrightarrow L^{p^*}(E)$, *where* $p^* = \frac{Np}{N-p}$, *and there exists* $\gamma = \gamma(N, p, E, \partial E)$ *such that*

$$\|u - u_E\|_{p^*} \le \gamma \|\nabla u\|_p, \qquad p^* = \frac{Np}{N-p} \quad \textit{for all } u \in W^{1,p}(E). \tag{3.4}$$

If $p > N$, *then* $W^{1,p}(E) \hookrightarrow L^\infty(E)$, *and there exists* $\gamma = \gamma(N, p)$ *such that*

$$\|u - u_E\|_\infty \le \gamma \|\nabla u\|_p^{\frac{N}{p}} \|u - u_E\|_p^{1-\frac{N}{p}} \quad \textit{for all } u \in W^{1,p}(E). \tag{3.5}$$

If $p = N$, *then* $W^{1,p}(E) \hookrightarrow L^q(E)$ *for all* $q > p$, *and there exists* $\gamma = \gamma(N, q)$ *such that*

$$\|u - u_E\|_q \le \gamma \|\nabla u\|_p^{1-\frac{p}{q}} \|u - u_E\|_p^{\frac{p}{q}} \quad \textit{for all } u \in W^{1,p}(E). \tag{3.6}$$

Remark 3.2 When E is convex, a simple proof of Theorem 3.2 is due to Poincarè and it is reported in Section 3.2c of the Complements. In such a case the constant $\gamma(N, E, \partial E)$ in (3.4) has the form

$$\gamma(N, E, \partial E) = C \frac{(\operatorname{diam} E)^N}{|E|}$$

for some absolute constants $C > 1$ depending only on N.

3.1 Some Consequences of the Multiplicative Embedding Inequalities

Corollary 3.1 *The norm* $\|\cdot\|_{1,p}$ *in* $W_o^{1,p}(E)$, *introduced in (1.6), is equivalent to* $\|\nabla u\|_p$; *that is, there exists a positive constant* $\gamma_o = \gamma_o(N, p, E)$ *such that*

$$\gamma_o \|u\|_{1,p} \le \|\nabla u\|_p \le \|u\|_{1,p} \quad \textit{for all } u \in W_o^{1,p}(E). \tag{3.7}$$

Corollary 3.2 *The norm* $\|\cdot\|_{1,p}$ *in* $\tilde{W}^{1,p}(E)$, *is equivalent to* $\|\nabla u\|_p$; *that is, there exists a positive constant* $\tilde{\gamma}_o = \tilde{\gamma}_o(N, p, E)$ *such that*

$$\tilde{\gamma}_o \|u\|_{1,p} \le \|\nabla u\|_p \le \|u\|_{1,p} \quad \textit{for all } u \in \tilde{W}^{1,p}(E). \tag{3.8}$$

Corollary 3.3 $W_o^{1,2}(E)$ *and* $\tilde{W}^{1,2}(E)$ *are Hilbert spaces with equivalent inner products*

$$\langle u, v\rangle + \langle \nabla u, \nabla v\rangle \quad \textit{and} \quad \langle \nabla u, \nabla v\rangle \tag{3.9}$$

where $\langle \cdot, \cdot \rangle$ *denotes the standard inner product in* $L^2(E)$.

4 The Homogeneous Dirichlet Problem

Given $\mathbf{f}, f \in L^\infty(E)$, consider the homogeneous Dirichlet problem

$$\begin{aligned} -\big(a_{ij}u_{x_i}\big)_{x_j} &= \operatorname{div}\mathbf{f} - f && \text{in } E \\ u\big|_{\partial E} &= 0 && \text{on } \partial E. \end{aligned} \tag{4.1}$$

The PDE is meant in the weak sense (1.3) by requiring that $u \in W^{1,p}(E)$ for some $p \ge 1$. The homogeneous boundary datum is enforced, in a weak form, by requiring that u be in the space $W_o^{1,p}(E)$ defined in (1.7). Seeking solutions $u \in W_o^{1,p}(E)$ implies that in (1.3), by density, one may take $v = u$. Thus, by taking into account the ellipticity condition (1.1)

$$\lambda \int_E |\nabla u|^2 dx \le \int_E (|\mathbf{f}||\nabla u| + |f||u|)dx.$$

This forces $p = 2$ and identifies $W_o^{1,2}(E)$ as the natural space where solutions of (4.1) should be sought.

Theorem 4.1. *The homogeneous Dirichlet problem (4.1) admits at most one weak solution $u \in W_o^{1,2}(E)$.*

Proof. If $u_1, u_2 \in W_o^{1,2}(E)$ are weak solutions of (4.1)

$$\int_E a_{ij}(u_1 - u_2)_{x_i} v_{x_j} dx = 0 \qquad \text{for all } v \in W_o^{1,2}(E).$$

This and the ellipticity condition (1.1) imply $\|\nabla(u_1 - u_2)\|_2 = 0$. Thus $u_1 = u_2$ a.e. in E, by the embedding of Theorem 3.1. ■

5 Solving the Homogeneous Dirichlet Problem (4.1) by the Riesz Representation Theorem

Regard (1.3) as made out of two pieces

$$a(u, v) = \int_E a_{ij}u_{x_i}v_{x_j} dx \quad \text{and} \quad \ell(v) = -\int_E (f_j v_{x_j} + fv)dx \tag{5.1}$$

for all $v \in W_o^{1,2}(E)$. Finding a solution to (4.1) amounts to finding $u \in W_o^{1,2}(E)$ such that

$$a(u, v) = \ell(v) \quad \text{for all } v \in W_o^{1,2}(E). \tag{5.2}$$

The first term in (5.1) is a bilinear form in $W_o^{1,2}(E)$. By the ellipticity condition (1.1) and Corollary 3.1

$$\lambda\gamma_o^2\|u\|_{1,2}^2 \le \lambda\|\nabla u\|_2^2 \le a(u,u) \le \Lambda\|\nabla u\|_2^2 \le \Lambda\|u\|_{1,2}^2.$$

Therefore $a(\cdot,\cdot)$ is an inner product in $W_o^{1,2}(E)$ equivalent to any one of the inner products in (3.9). The second term $\ell(\cdot)$ in (5.1) is a linear functional in $W_o^{1,2}(E)$, bounded in $\|\cdot\|_{1,2}$, and thus bounded in the norm generated by the inner product $a(\cdot,\cdot)$. Therefore by the Riesz representation theorem it is represented as in (5.2) for a unique $u \in W_o^{1,2}(E)$ ([162]).[1]

6 Solving the Homogeneous Dirichlet Problem (4.1) by Variational Methods

Consider the nonlinear functional

$$W_o^{1,2}(E) \ni u \to J(u) \overset{\text{def}}{=} \int_E \big(\tfrac{1}{2}a_{ij}u_{x_i}u_{x_j} + f_j u_{x_j} + fu\big)dx. \tag{6.1}$$

One verifies that $J(\cdot)$ is strictly convex in $W_o^{1,2}(E)$, that is

$$J\big(tu + (1-t)v\big) < tJ(u) + (1-t)J(v)$$

for every pair (u,v) of nontrivial elements of $W_o^{1,2}(E)$, and all $t \in (0,1)$. Assume momentarily $N > 2$, and let 2^{**} be the Hölder conjugate of 2^*, so that

$$2^* = \frac{2N}{N-2}, \qquad 2^{**} = \frac{2N}{N+2}, \qquad \text{and} \qquad \frac{1}{2^*} + \frac{1}{2^{**}} = 1.$$

The functional $J(u)$ is estimated above using the ellipticity condition (1.1), Hölder's inequality, and the embedding (3.1):

$$\begin{aligned} J(u) &\le \frac{1}{2}\Lambda\|\nabla u\|_2^2 + \|\mathbf{f}\|_2\|\nabla u\|_2 + \|f\|_{2^{**}}\|u\|_{2^*} \\ &\le \frac{1}{2}\Lambda\|\nabla u\|_2^2 + \left(\|\mathbf{f}\|_2 + \gamma\|f\|_{2^{**}}\right)\|\nabla u\|_2 \\ &\le \Lambda\|\nabla u\|_2^2 + \frac{1}{2\Lambda}\left(\|\mathbf{f}\|_2 + \gamma\|f\|_{2^{**}}\right)^2. \end{aligned}$$

Similarly, $J(u)$ is estimated below by

$$J(u) \ge \frac{1}{4}\lambda\|\nabla u\|_2^2 - \frac{1}{\lambda}\left(\|\mathbf{f}\|_2 + \gamma\|f\|_{2^{**}}\right)^2.$$

Therefore

$$-F_o + \tfrac{1}{4}\lambda\|\nabla u\|_2^2 \le J(u) \le \Lambda\|\nabla u\|_2^2 + F_o \tag{6.2}$$

for all $u \in W_o^{1,2}(E)$, where

---

[1] The Riesz representation theorem is in [50], Chapter VI Section 18. This solution method is in [162] and it is referred to as the Lax–Milgram theorem.

$$F_o = \frac{1}{\lambda}\left(\|\mathbf{f}\|_2 + \gamma\|f\|_{2^{**}}\right)^2.$$

Here γ is the constant appearing in the embedding inequality (3.1). These estimates imply that the convex functional $J(\cdot)$ is bounded below, and we denote by J_o its infimum. A minimum can be sought by a minimizing sequence $\{u_n\} \subset W_o^{1,2}(E)$ such that

$$J(u_n) < J(u_{n-1}) < \cdots < J(u_1) \quad \text{and} \quad \lim J(u_n) = J_o. \tag{6.3}$$

By (6.2) and (6.3), the sequence $\{u_n\}$ is bounded in $W_o^{1,2}(E)$. Therefore a subsequence $\{u_{n'}\} \subset \{u_n\}$ can be selected such that $\{u_{n'}\} \to u$ weakly in $W_o^{1,2}(E)$, and $\{J(u_{n'})\} \to J_o$. Since the norm $a(\cdot,\cdot)$ introduced in (5.1) is weakly lower semi-continuous

$$\liminf J(u_{n'}) = J_o \ge J(u).$$

Thus $J(u) = J_o$ and $J(u) \le J(w)$ for all $w \in W_o^{1,2}(E)$. Enforcing this last condition for functions $w = u + \varepsilon v$, for $\varepsilon > 0$ and $v \in C_o^\infty(E)$, gives

$$J(u) \le J(u) + \frac{\varepsilon^2}{2}\int_E a_{ij} v_{x_i} v_{x_j}\,dx + \varepsilon\int_E \big(a_{ij} u_{x_i} v_{x_j} + f_j v_{x_j} + fv\big)\,dx.$$

Divide by ε and let $\varepsilon \to 0$ to get

$$\int_E \big(a_{ij} u_{x_i} v_{x_j} + f_j v_{x_j} + fv\big)\,dx \ge 0 \quad \text{for all } v \in C_o^\infty(E).$$

Changing v into $-v$ shows that this inequality actually holds with equality for all $v \in C_o^\infty(E)$ and establishes that u is a weak solution of the Dirichlet problem (4.1). In view of its uniqueness, as stated in Theorem 4.1 the whole minimizing sequence $\{u_n\}$ converges weakly to the unique minimizer of $J(\cdot)$. Summarizing, the PDE in divergence form (4.1) is associated with a natural functional $J(\cdot)$ in $W_o^{1,2}(E)$, whose minimum is a solutions of the PDE in $W_o^{1,2}(E)$. Conversely, the functional $J(\cdot)$ in $W_o^{1,2}(E)$ generates naturally a PDE in divergence form whose solutions are the solutions of the homogeneous Dirichlet problem (4.1).

6.1 The Case $N = 2$

The arguments are the same except for a different use of the embedding inequality (3.1) in estimating the term containing fu in (6.1). Pick $1 < p < 2$, define p^* as in (3.1), and let p^{**} be its Hölder conjugate, so that

$$p^* = \frac{Np}{N-p}, \qquad p^{**} = \frac{Np}{N(p-1)+p}, \qquad \frac{1}{p^*} + \frac{1}{p^{**}} = 1.$$

One verifies that $p^{**} > 1$ and estimates

$$\int_E fu\,dx \le \|f\|_{p^{**}}\|u\|_{p^*} \le \gamma\|f\|_{p^{**}}\|\nabla u\|_p \le \gamma|E|^{\frac{2-p}{2}}\|f\|_{p^{**}}\|\nabla u\|_2.$$

The proof now proceeds as before except that the term $\gamma\|f\|_{2^{**}}$ is now replaced by $\gamma|E|^{2-p}2\|f\|_{p^{**}}$.

6.2 Gâteaux Derivative and The Euler Equation of $J(\cdot)$

If u is a minimum for $J(\cdot)$ the function of one variable $t \to J(u+tv)$, for an arbitrary but fixed $v \in C_o^\infty(E)$, has a minimum for $t=0$. Therefore

$$0 = \frac{d}{dt} J(u+tv) \bigm|_{t=0} = \int_E \big(a_{ij} u_{x_i} v_{x_j} + f_j v_{x_j} + fv \big) dx. \tag{6.4}$$

This reinforces that solutions of the Dirichlet problem (4.1) are minima of $J(\cdot)$ and vice versa. The procedure leading to (6.4), which is called the Euler equation of $J(\cdot)$, has a broader scope. It can be used to connect stationary points, not necessarily minima, of a functional $J(\cdot)$, for which no convexity information is available, to solutions of its Euler equation.

The derivative of $\varepsilon \to J(u+\varepsilon\varphi)$ at $\varepsilon = 0$ is called the Gâteaux derivative of $J(\cdot)$ at u. Relative variations from $J(u)$ to $J(v)$ are computed along the "line" in $\mathbb{R} \times W_o^{1,2}(E)$ originating at u and "slope" φ. In this sense Gâteaux derivatives are directional derivatives. A more general notion of derivative of $J(\cdot)$ is that of Fréchet derivative, where relative variations from $J(u)$ to $J(v)$ are computed for any v however varying in a $W_o^{1,2}(E)$-neighborhood of u ([9], Chapter I, Section 1,2).

7 Solving the Homogeneous Dirichlet Problem (4.1) by Galerkin Approximations

Let $\{w_n\}$ be a countable, complete, orthonormal system for $W_o^{1,2}(E)$. Such a system exists, since $W_o^{1,2}(E)$ is separable, and it can be constructed, for example, by the Gram–Schmidt procedure, for the natural inner product $a(u,v)$ introduced in (5.1). Thus in particular

$$A_{hk} = \int_E a_{ij} w_{h,x_i} w_{k,x_j} dx = \delta_{hk} \tag{7.1}$$

where δ_{hk} is the Kronecker delta. Every $v \in W_o^{1,2}(E)$ admits the representation as $v = \sum v_k w_k$ for constants v_k. The Galerkin method consists in constructing the solution u of (4.1) as the weak $W_o^{1,2}(E)$ limit, as $n \to \infty$, of the finite-dimensional approximations

$$u_n = \sum_{h=1}^n c_{n,h} w_h$$

where the coefficients $c_{n,h}$ are computed by enforcing an n-dimensional version of the PDE in its weak form (1.3). Precisely, u_n is sought as the solution of

$$\int_E \Big(a_{ij} u_{n,x_i} \sum_{k=1}^n v_k w_{k,x_j} + f_j \sum_{k=1}^n v_k w_{k,x_j} + f \sum_{k=1}^n v_k w_k \Big) dx = 0 \tag{7.2}$$

for all $v \in W_o^{1,2}(E)$. From this

$$\sum_{k=1}^n v_k \Big\{ \sum_{h=1}^n c_{n,h} \int_E a_{ij} w_{h,x_i} w_{k,x_j} dx + \int_E \big(f_j w_{k,x_j} + f w_k \big) dx \Big\} = 0$$

Setting

$$A_{hk} = \int_E a_{ij} w_{h,x_i} w_{k,x_j} dx, \qquad \phi_k = - \int_E \big(f_j w_{k,x_j} + f w_k \big) dx$$

and taking into account that $v \in W_o^{1,2}(E)$ is arbitrary, the coefficients $c_{n,h}$ are computed as the unique solution of the linear algebraic system

$$\sum_{h=1}^n c_{n,h} A_{hk} = \phi_k \qquad \text{for } \; k = 1, \dots, n. \tag{7.3}$$

By (7.1), A_{hk} is the identity matrix, and therefore $c_{n,h} = \phi_h$ for all $n, h \in \mathbb{N}$. With u_n so determined, put $v = u_n$ in (7.2), and assuming momentarily that $N > 2$, estimate

$$\lambda \|\nabla u_n\|_2^2 \le \|\mathbf{f}\|_2 \|\nabla u_n\|_2 + \|f\|_{2^{**}} \|u_n\|_{2^*} \le \big(\|\mathbf{f}\|_2 + \gamma \|f\|_{2^{**}} \big) \|\nabla u_n\|_2$$

where γ is the constant of the embedding inequality (3.1). Therefore $\{u_n\}$ is bounded in $W_o^{1,2}(E)$, and a subsequence $\{u_{n'}\} \subset \{u_n\}$ can be selected, converging weakly to some $u \in W_o^{1,2}(E)$. Letting $n \to \infty$ in (7.2) shows that such a u is a solution of the homogeneous Dirichlet problem (4.1). In view of its uniqueness, the whole sequence $\{u_n\}$ of finite-dimensional approximations converges weakly to u. The same arguments hold true for $N = 2$, by the minor modifications indicated in Section 6.1.

7.1 On the Selection of an Orthonormal System in $W_o^{1,2}(E)$

The selection of the complete system $\{w_n\}$ is arbitrary. For example the Gram–Schmidt procedure could be carried on starting from a countable collection of linearly independent elements of $W_o^{1,2}(E)$, and using anyone of the equivalent inner products in (3.9), or the construction could be completely independent of Gram–Schmidt procedure. The Galerkin method continues to hold, except that the coefficients A_{hk} defined in (7.1) are no longer identified by the Kronecker symbol. This leads to the determination of the coefficients $c_{n,h}$ as solutions of the linear algebraic system (7.3), whose leading $n \times n$ matrix (A_{hk}) is still invertible, for all n, because of the ellipticity condition (1.1). The corresponding unique solutions might depend on the nth approximating truncation, and therefore are labeled by $c_{n,h}$.

While the method is simple and elegant, it hinges on a suitable choice of complete system in $W_o^{1,2}(E)$. Such a choice is suggested by the specific geometry of E and the structure of the matrix (a_{ij}) ([34]).

7.2 Conditions on f and f for the Solvability of the Dirichlet Problem (4.1)

Revisiting the proofs of these solvability methods shows that the only required conditions on $\mathbf{f}$ and f are

$$\mathbf{f} \in L^2(E) \qquad f \in L^q(E) \;\text{ where }\; \begin{cases} q = 2^{**} & \text{if } \; N > 2 \\ \text{any } q > 1 & \text{if } \; N = 2. \end{cases} \tag{7.4}$$

Theorem 7.1. *Let (7.4) hold. Then the homogeneous Dirichlet problem (4.1) admits a unique solution.*

8 Traces on ∂E of Functions in $W^{1,p}(E)$

8.1 The Segment Property

The boundary ∂E has the *segment property* if there exist a locally finite open covering of ∂E with balls $\{B_t(x_j)\}$ centered at $x_j \in \partial E$ with radius t, a corresponding sequence of unit vectors $\mathbf{n}_j$, and a number $t^* \in (0, 1)$ such that

$$x \in \bar{E} \cap B_t(x_j) \quad \Longrightarrow \quad x + t\mathbf{n}_j \in E \qquad \text{for all } \; t \in (0, t^*). \tag{8.1}$$

Such a requirement forces, in some sense, the domain E to lie locally on one side of its boundary. However no smoothness is required on ∂E. As an example for $x \in \mathbb{R}$ let

$$h(x) = \begin{cases} \sqrt{|x|}\sin^2 \dfrac{1}{x} & \text{for } \; |x| > 0 \\ 0 & \text{for } \; x = 0. \end{cases} \qquad \text{and} \qquad E = [y > h]. \tag{8.2}$$

The set E satisfies the segment property. The cone property does not imply the segment property. For example, the unit disc from which a radius is removed satisfies the cone property and does not satisfy the segment property. The segment property does not imply the cone property. For example the set in (8.2), does not satisfy the cone property. The segment property does not imply that ∂E is of class C^1. Conversely, ∂E of class C^1 does not imply the segment property.

A remarkable fact about domains with the segment property is that functions in $W^{1,p}(E)$ can be extended “outside” E to be in $W^{1,p}(E')$, for a larger open set E' containing E. A consequence of such an extension is that functions in $W^{1,p}(E)$ can be approximated in the norm (1.6) by functions smooth up to ∂E. Precisely

Proposition 8.1 *Let E be a bounded open set in $\mathbb{R}^N$ with boundary ∂E of class C^1 and with the segment property. Then $C_o^\infty(\mathbb{R}^N)$ is dense in $W^{1,p}(E)$.*

Proof. Section 8.1c of the Complements.

8.2 Defining Traces

Denote by $\gamma = \gamma(N, p, \partial E)$ a constant that can be quantitatively determined a priori in terms of N, p, and the structure of ∂E only.

Proposition 8.2 *Let ∂E be of class C^1 and satisfy the segment property. If $1 \le p < N$, there exists $\gamma = \gamma(N, p, \partial E)$ such that for all $\varepsilon > 0$*

$$\|u\|_{p^* \frac{N-1}{N};\partial E} \le \varepsilon \|\nabla u\|_p + \gamma\Big(1 + \frac{1}{\varepsilon}\Big)\|u\|_p \quad \textit{for all } \ u \in C_o^\infty(\mathbb{R}^N). \tag{8.3}$$

If $p = N$, then for all $q \ge 1$, there exists $\gamma = \gamma(N, q, \partial E)$ such that for all $\varepsilon > 0$

$$\|u\|_{q;\partial E} \le \varepsilon \|\nabla u\|_p + \gamma\Big(1 + \frac{1}{\varepsilon}\Big)\|u\|_p \quad \textit{for all } \ u \in C_o^\infty(\mathbb{R}^N). \tag{8.4}$$

If $p > N$, there exists $\gamma = \gamma(N, p, \partial E)$ such that for all $\varepsilon > 0$

$$\begin{aligned} \|u\|_{\infty,\partial E} &\le \varepsilon \|\nabla u\|_p + \gamma\Big(1 + \frac{1}{\varepsilon}\Big)\|u\|_p \\ |u(x) - u(y)| &\le \gamma |x - y|^{1 - \frac{N}{p}} \|u\|_{1,p} \ \textit{ for all } \ x, y \in \bar{E}. \end{aligned} \tag{8.5}$$

Proof. Section 8.2c of the Complements.

Remark 8.1 The constants γ in (8.3) and (8.5) tend to infinity as $p \to N$, and the constant γ in (8.4) tends to infinity as $q \to \infty$.

Since $u \in C_o^\infty(\mathbb{R}^N)$, the values of u on ∂E are meant in the classical sense. By Proposition 8.1, given $u \in W^{1,p}(E)$ there exists a sequence $\{u_n\} \subset C_o^\infty(\mathbb{R}^N)$ such that $\{u_n\} \to u$ in $W^{1,p}(E)$. In particular, $\{u_n\}$ is Cauchy in $W^{1,p}(E)$ and from (8.3) and (8.4)

$$\begin{aligned} \|u_n - u_m\|_{p^* \frac{N-1}{N};\partial E} &\le \gamma \|u_n - u_m\|_{1,p} \quad \text{if } \ 1 \le p < N \\ \|u_n - u_m\|_{q;\partial E} &\le \gamma \|u_n - u_m\|_{1,p} \quad \text{for fixed } q \ge 1 \text{ if } \ p \ge N > 1. \end{aligned}$$

By the completeness of the spaces $L^p(\partial E)$, for $p \ge 1$

$$\{u_n\big|_{\partial E}\} \to \operatorname{tr}(u) \ \text{ in } \ \begin{cases} L^{p^* \frac{N-1}{N}}(\partial E) & \text{if } \ 1 \le p < N \\ L^q(\partial E) & \text{for fixed } q \ge 1 \text{ if } \ p \ge N > 1. \end{cases} \tag{8.6}$$

One verifies that $\operatorname{tr}(u)$ is independent of the particular sequence $\{u_n\}$. Therefore given $u \in W^{1,p}(E)$, this limiting process identifies its "boundary values" $\operatorname{tr}(u)$, called the *trace* of u on ∂E, as an element of $L^r(E)$, with r specified by (8.6). With perhaps an improper but suggestive symbolism we write $\operatorname{tr}(u) = u|_{\partial E}$.

8.3 Characterizing the Traces on ∂E of Functions in $W^{1,p}(E)$

The trace of a function in $W^{1,p}(E)$ is somewhat more regular that merely an element in $L^p(\partial E)$ for some $p \ge 1$. For $v \in C^\infty(\partial E)$ and $s \in (0,1)$, set

$$|||v|||^p_{s,p;\partial E} = \int_{\partial E}\int_{\partial E} \frac{|v(x)-v(y)|^p}{|x-y|^{(N-1)+sp}} d\sigma(x)d\sigma(y) < \infty \tag{8.7}$$

where $d\sigma(\cdot)$ is the surface measure on ∂E. Denote by $W^{s,p}(\partial E)$ the collections of functions v in $L^p(\partial E)$ with finite norm

$$\|v\|_{s,p;\partial E} = \|v\|_{p,\partial E} + |||v|||_{s,p;\partial E}. \tag{8.8}$$

The next theorem characterizes the traces of functions in $W^{1,p}(E)$ in terms of the spaces $W^{s,p}(\partial E)$.

Theorem 8.1. *Let ∂E be of class C^1 and satisfy the segment property. If $u \in W^{1,p}(E)$, then tr$(u) \in W^{s,p}(\partial E)$, with $s = 1 - \frac{1}{p}$. Conversely, given $v \in W^{s,p}(\partial E)$, with $s = 1 - \frac{1}{p}$, there exists $u \in W^{1,p}(E)$ such that tr$(u) = v$.*

Proof. Section 8.3c of the Complements.

9 The Inhomogeneous Dirichlet Problem

Assume that ∂E is of class C^1 and satisfies the segment property. Given $\mathbf{f}$ and f satisfying (7.4) and $\varphi \in W^{\frac{1}{2},2}(\partial E)$, consider the Dirichlet problem

$$\begin{aligned} -\big(a_{ij}u_{x_i}\big)_{x_j} &= \operatorname{div}\mathbf{f} - f &&\text{in } E\\ u\big|_{\partial E} &= \varphi &&\text{on } \partial E. \end{aligned} \tag{9.1}$$

By Theorem 8.1 there exists $v \in W^{1,2}(E)$ such that tr$(v) = \varphi$. A solution of (9.1) is sought of the form $u = w + v$, where $w \in W^{1,2}_o(E)$ is the unique weak solution of the auxiliary, homogeneous Dirichlet problem

$$\begin{aligned} -\big(a_{ij}w_{x_i}\big)_{x_j} &= \operatorname{div}\tilde{\mathbf{f}} - f &&\text{in } E\\ w\big|_{\partial E} &= 0 &&\text{on } \partial E \end{aligned} \qquad \text{where } \tilde{f}_j = f_j + a_{ij}v_{x_i}. \tag{9.2}$$

Theorem 9.1. *Assume that ∂E is of class C^1 and satisfies the segment property. For every $\mathbf{f}$ and f satisfying (7.4) and $\varphi \in W^{\frac{1}{2},2}(\partial E)$, the Dirichlet problem (9.1) has a unique weak solution $u \in W^{1,2}(E)$.*

Remark 9.1 The class $W^{1,2}(E)$ where a weak solution is sought characterizes the boundary data φ on ∂E that ensure solvability.

10 The Neumann Problem

Assume that ∂E is of class C^1 and satisfies the segment property. Given $\mathbf{f}$ and f satisfying (7.4), consider the formal Neumann problem

$$\begin{aligned} -\big(a_{ij}u_{x_i}\big)_{x_j} &= \operatorname{div}\mathbf{f} - f &\quad& \text{in } E \\ (a_{ij}u_{x_i} + f_j)\, n_j &= \psi && \text{on } \partial E \end{aligned} \tag{10.1}$$

where $\mathbf{n} = (n_1, \dots, n_N)$ is the outward unit normal to ∂E and $\psi \in L^p(\partial E)$ for some $p \ge 1$. If $a_{ij} = \delta_{ij}$, $\mathbf{f} = f = 0$, and ψ are sufficiently regular, this is precisely the Neumann problem (1.3) of Chapter 2. Since $a_{ij} \in L^\infty(E)$ and $\mathbf{f} \in L^2(E)$, neither the PDE nor the boundary condition in (10.1) are well defined, and they have to be interpreted in some weak form. Multiply formally the first of (10.1) by $v \in C_o^\infty(\mathbb{R}^N)$ and integrate by parts over E, as if both the PDE and the boundary condition were satisfied in the classical sense. This gives formally

$$\int_E \big(a_{ij}u_{x_i}v_{x_j} + f_j v_{x_j} + f v\big)dx = \int_{\partial E} \psi v \, d\sigma. \tag{10.2}$$

If $v \in C_o^\infty(\mathbb{R}^N)$ is constant in a neighborhood of E, this implies the necessary condition of solvability[2]

$$\int_E f\, dx = \int_{\partial E} \psi \, d\sigma. \tag{10.3}$$

It turns out that this condition linking the data f and ψ is also sufficient for the solvability of (10.1), provided a precise class for ψ is identified. By Proposition 8.1, if (10.2) holds for all $v \in C_o^\infty(\mathbb{R}^N)$, it must hold for all $v \in W^{1,2}(E)$, provided a solution u is sought in $W^{1,2}(E)$. In such a case, the right-hand side is well defined if ψ is in the conjugate space of integrability of the traces of functions in $W^{1,2}(E)$. Therefore the natural class for the Neumann datum is

$$\psi \in L^q(\partial E), \qquad \text{where} \quad \begin{cases} q = \dfrac{2(N-1)}{N} & \text{if } N > 2 \\[2ex] \text{any } q > 1 & \text{if } N = 2. \end{cases} \tag{10.4}$$

Theorem 10.1. *Let ∂E be of class C^1 and satisfy the segment property. Let $\mathbf{f}$ and f satisfy (7.4) and ψ satisfy (10.4) and be linked by the compatibility condition (10.3). Then the Neumann problem (10.1) admits a solution in the weak form (10.2) for all $v \in W^{1,2}(E)$. The solution is unique up to a constant.*

Proof. Consider the nonlinear functional in $W^{1,2}(E)$

---

[2] This is a version of the compatibility condition (1.4) of Chapter 2; see also Theorem 6.1 of Chapter 3, and Section 6 of Chapter 4.

$$J(u) \overset{\text{def}}{=} \int_E \big(\tfrac{1}{2} a_{ij} u_{x_i} u_{x_j} + f_j u_{x_j} + fu\big) dx - \int_{\partial E} \psi \operatorname{tr}(u) d\sigma. \tag{10.5}$$

By the compatibility condition (10.3), $J(u) = J(u - u_E)$, where u_E is the integral average of u over E. Therefore $J(\cdot)$ can be regarded as defined in the space $\tilde{W}^{1,2}(E)$ introduced in (1.8). One verifies that $J(\cdot)$ is strictly convex in $\tilde{W}^{1,2}(E)$. Assume momentarily that $N > 2$, let 2^{**} be the Hölder conjugate of 2^*, and estimate

$$\begin{aligned} \Big| \int_E f_j u_{x_j} dx \Big| &\le \|\mathbf{f}\|_2 \|\nabla u\|_2 \\ \Big| \int_E f u dx \Big| &\le \|f\|_{2^{**}} \|u\|_{2^*} \le \gamma \|f\|_{2^{**}} \|\nabla u\|_2 \end{aligned}$$

where γ is the constant in the embedding inequality (3.4). Similarly, using the trace inequality (8.3)

$$\begin{aligned} \Big| \int_{\partial E} \psi \operatorname{tr}(u) d\sigma \Big| &\le \|\psi\|_{q;\partial E} \|\operatorname{tr}(u)\|_{2^* \frac{N-1}{N};\partial E} \\ &\le \gamma \|\psi\|_{q;\partial E} \left(\|u\|_2 + \|\nabla u\|_2 \right) \le 2\gamma^2 \|\psi\|_{q;\partial E} \|\nabla u\|_2 \end{aligned}$$

where γ is the largest of the constants in (3.4) and (8.3). Therefore

$$-F_1 + \tfrac{1}{4}\lambda \|\nabla u\|_2^2 \le J(u) \le \Lambda \|\nabla u\|_2^2 + F_1 \tag{10.6}$$

for all $u \in \tilde{W}^{1,2}(E)$, where

$$F_1 = \frac{1}{\lambda} \left(\|\mathbf{f}\|_2 + \gamma \|f\|_{2^{**}} + 2\gamma^2 \|\psi\|_{q;\partial E} \right)^2 .$$

By Corollary 3.2, the $\|u\|_{1,2}$ norm of $\tilde{W}^{1,2}(E)$ is equivalent to $\|\nabla u\|_2$. With these estimates in hand the proof can now be concluded by a minimization process in $\tilde{W}^{1,2}(E)$, similar to that of Section 6. The minimum $u \in \tilde{W}^{1,2}(E)$ satisfies (10.2), for all $v \in W^{1,2}(E)$ and the latter can be characterized as the Euler equation of $J(\cdot)$.

Essentially the same arguments continue to hold for $N = 2$, modulo minor variants that can be modeled after those in Section 6.1.

10.1 A Variant of (10.1)

The compatibility condition (10.3) has the role of estimating $J(\cdot)$ above and below as in (10.6), via the multiplicative embeddings of Theorem 3.2. Consider next the Neumann problem

$$\begin{aligned} -\left(a_{ij} u_{x_i}\right)_{x_j} + \mu u &= \operatorname{div} \mathbf{f} - f \qquad && \text{in } E \\ \left(a_{ij} u_{x_i} + f_j\right) n_j &= \psi && \text{on } \partial E \end{aligned} \tag{10.7}$$

where $\mu > 0$, $\mathbf{f}, f \in L^2(E)$, and ψ satisfies (10.4). The problem is meant in its weak form

$$\int_E \big(a_{ij}u_{x_i}v_{x_j} + f_j v_{x_j} + \mu uv + fv\big)dx = \int_{\partial E} \psi v \, d\sigma \tag{10.8}$$

for all $v \in W^{1,2}(E)$. No compatibility conditions are needed on the data f and ψ for a solution to exist, and in addition, the solution is unique.

Theorem 10.2. *Let ∂E be of class C^1 and satisfy the segment property. Let $\mathbf{f}, f \in L^2(E)$, and let ψ satisfy (10.4). Then the Neumann problem (10.7) with $\mu > 0$ admits a unique solution in the weak form (10.8).*

Proof. If u_1 and u_2 are two solutions in $W^{1,2}(E)$, their difference w satisfies

$$\int_E \big(a_{ij}w_{x_i}w_{x_j} + \mu w^2\big)dx = 0.$$

The nonlinear functional in $W^{1,2}(E)$

$$J(u) \overset{\text{def}}{=} \int_E \big(\tfrac{1}{2}a_{ij}u_{x_i}u_{x_j} + \tfrac{1}{2}\mu u^2 + f_j u_{x_j} + fu\big)dx - \int_{\partial E} \psi \mathrm{tr}(u) d\sigma \tag{10.9}$$

is strictly convex. Then a solution can be constructed by the variational method of Section 6, modulus establishing an estimate analogous to (6.2) or (10.6), with $\|\nabla u\|_2$ replaced by the norm $\|u\|_{1,2}$ of $W^{1,2}(E)$. Estimate

$$\Big|\int_E f_j u_{x_j} dx\Big| \le \|\mathbf{f}\|_2 \|\nabla u\|_2, \qquad \Big|\int_E fu dx\Big| \le \|f\|_2 \|u\|_2$$
$$\Big|\int_{\partial E} \psi \mathrm{tr}(u) d\sigma\Big| \le \gamma \|\psi\|_{q;\partial E} \left(\|u\|_2 + \|\nabla u\|_2\right)$$

where γ is the constant of the trace inequality (8.3). Then the functional $J(\cdot)$ is estimated above and below by

$$-F_\mu + \tfrac{1}{4}\min\{\lambda; \mu\}\|u\|_{1,2}^2 \le J(u) \le \tfrac{3}{2}\max\{\Lambda; \mu\}\|u\|_{1,2}^2 + F_\mu \tag{10.10}$$

for all $u \in W^{1,2}(E)$, where

$$F_\mu = \frac{1}{\min\{\lambda; \mu\}} \left(\|\mathbf{f}\|_2 + \|f\|_2 + \gamma\|\psi\|_{q;\partial E}\right)^2 . \qquad \blacksquare$$

11 The Eigenvalue Problem

Consider the problem of finding a nontrivial pair (μ, u) with $\mu \in \mathbb{R}$ and $u \in W_o^{1,2}(E)$, a solution of

$$\begin{aligned} -\big(a_{ij}u_{x_i}\big)_{x_j} &= \mu u \qquad \text{in } E \\ u &= 0 \qquad \text{on } \partial E. \end{aligned} \tag{11.1}$$

This is meant in the weak sense

$$\int_E \big(a_{ij}u_{x_i}v_{x_j} - \mu uv\big)dx = 0 \qquad \text{for all } v \in W_o^{1,2}(E). \tag{11.2}$$

If (μ, u) is a solution pair, μ is an eigenvalue and u is an eigenfunction of (11.1). In principle, the pair (μ, u) is sought for $\mu \in \mathbb{C}$, and for u in the complex-valued Hilbert space $W_o^{1,2}(E)$, with complex inner product as in Section 1 of Chapter 4. However by considerations analogous to those of Proposition 7.1 of that Chapter, eigenvalues of (11.1) are real, and eigenfunctions can be taken to be real-valued. Moreover, any two distinct eigenfunctions corresponding to two distinct eigenvalues are orthogonal in $L^2(E)$.

Proposition 11.1 *Eigenvalues of (11.1) are positive. Moreover, to each eigenvalue μ there correspond at most finitely many eigenfunctions, linearly independent, and orthonormal in $L^2(E)$.*

Proof. If $\mu \le 0$, the functional

$$W_o^{1,2}(E) \ni u \to \int_E (a_{ij}u_{x_i}u_{x_j} - \mu u^2)\,dx \tag{11.3}$$

is strictly convex and bounded below by $\lambda\|\nabla u\|_2^2$. Therefore it has a unique minimum, which is the unique solution of its Euler equation (11.1). Since $u = 0$ is a solution, it is the only one. Let $\{u_n\}$ be a sequence of eigenfunctions linearly independent in $L^2(E)$, corresponding to μ. Without loss of generality we may assume they are orthonormal. Then from (11.2), $\|\nabla u_n\|_2 \le \sqrt{\mu/\lambda}$ for all n, and $\{u_n\}$ is equi-bounded in $W_o^{1,2}(E)$. If $\{u_n\}$ is infinite, a subsequence can be selected, and relabeled with n, such that $\{u_n\} \to u$ weakly in $W_o^{1,2}(E)$ and strongly in $L^2(E)$. However, $\{u_n\}$ cannot be a Cauchy sequence in $L^2(E)$, since $\|u_n - u_m\|_2 = \sqrt{2}$ for all n, m. ■

Let $\{u_{\mu,1}, \dots, u_{\mu,n_\mu}\}$ be the linearly independent eigenfunctions corresponding to the eigenvalue μ. The number n_μ is the multiplicity of μ. If $n_\mu = 1$, then μ is said to be simple.

12 Constructing The Eigenvalues of (11.1)

Minimize the strictly convex functional $a(\cdot,\cdot)$ on the unit sphere S_1 of $L^2(E)$, that is

$$\min_{\substack{u\in W_o^{1,2}(E)\\ \|u\|_2=1}} a(u,u) = \min_{\substack{u\in W_o^{1,2}(E)\\ \|u\|_2=1}} \int_E a_{ij}u_{x_i}u_{x_j}dx$$

and let $\mu_1 \ge 0$ be its minimum value. A minimizing sequence $\{u_n\} \subset S_1$ is bounded in $W_o^{1,2}(E)$, and a subsequence $\{u_{n'}\} \subset \{u_n\}$ can be selected such that $\{u_{n'}\} \to w_1$ weakly in $W_o^{1,2}(E)$ and strongly in $L^2(E)$. Therefore $w_1 \in S_1$, it is nontrivial, $\mu_1 > 0$, and

$$\mu_1 = \lim a(u_{n'}, u_{n'}) \ge \lim a(w_1, w_1) \ge \mu_1. \tag{12.1}$$

Thus

$$\mu_1 \le \frac{a(w_1 + v, w_1 + v)}{\|w_1 + v\|_2^2} \qquad \text{for all } \ v \in W_o^{1,2}(E). \tag{12.2}$$

It follows that the functional

$$\begin{aligned} W_o^{1,2}(E) \ni v \to I_{\mu_1}(v) = \int_E &\big(a_{ij} w_{1,x_i} v_{x_j} - \mu_1 w_1 v\big) dx \\ &+ \frac{1}{2} \int_E \big(a_{ij} v_{x_i} v_{x_j} - \mu_1 v^2\big) dx \end{aligned}$$

is non-negative, its minimum is zero, and the minimum is achieved for $v = 0$. Thus

$$\frac{d}{dt} I_{\mu_1}(tv)\Big|_{t=0} = 0 \qquad \text{for all } \ v \in W_o^{1,2}(E).$$

The latter is precisely (11.2) for the pair (μ_1, w_1). While this process identifies μ_1 uniquely, the minimizer $w_1 \in S_1$ depends on the choice of subsequence $\{u_{n'}\} \subset \{u_n\}$. Thus a priori, to μ_1 there might correspond several eigenfunctions in $W_o^{1,2}(E) \cap S_1$.

If μ_n and the set of its linearly independent eigenfunctions have been found, set

$$\begin{aligned} \mathcal{E}_n &= \{\text{span of the eigenfunctions of } \mu_n\} \\ W_n^{1,2}(E) &= W_o^{1,2}(E) \cap [\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_n]^\perp \end{aligned}$$

consider the minimization problem

$$\min_{\substack{u \in W_n^{1,2}(E) \\ \|u\|_2 = 1}} a(u, u) = \min_{\substack{u \in W_n^{1,2}(E) \\ \|u\|_2 = 1}} \int_E a_{ij} u_{x_i} u_{x_j} dx$$

and let $\mu_{n+1} > \mu_n$ be its minimum value. A minimizing sequence $\{u_n\} \subset S_1$ is bounded in $W_n^{1,2}(E)$, and a subsequence $\{u_{n'}\} \subset \{u_n\}$ can be selected such that $\{u_{n'}\} \to w_{n+1}$ weakly in $W_n^{1,2}(E)$ and strongly in $L^2(E)$. Therefore $w_{n+1} \in S_1$ is nontrivial, and

$$\mu_{n+1} = \lim a(u_{n'}, u_{n'}) \ge \lim a(w_{n+1}, w_{n+1}) \ge \mu_{n+1}.$$

Thus

$$\mu_{n+1} \le \frac{a(w_{n+1} + v, w_{n+1} + v)}{\|w_{n+1} + v\|_2^2} \qquad \text{for all } \ v \in W_n^{1,2}(E).$$

It follows that the functional

$$W_n^{1,2}(E) \ni v \to I_{\mu_{n+1}}(v) = \int_E \big(a_{ij} w_{n+1,x_i} v_{x_j} - \mu_{n+1} w_{n+1} v\big) dx \\ + \frac{1}{2} \int_E \big(a_{ij} v_{x_i} v_{x_j} - \mu_{n+1} v^2\big) dx$$

is non-negative, its minimum is zero, and the minimum is achieved for $v = 0$. Thus

$$\frac{d}{dt} I_{\mu_{n+1}}(tv)\,\Big|_{t=0} = 0 \qquad \text{for all } \; v \in W_n^{1,2}(E).$$

This implies

$$\int_E \big(a_{ij} w_{n+1,x_i} v_{x_j} - \mu_{n+1} w_{n+1} v\big) dx = 0 \tag{12.3}$$

for all $v \in W_n^{1,2}(E)$. The latter coincides with (11.2), except that the test functions v are taken out of $W_n^{1,2}(E)$ instead of the entire $W_o^{1,2}(E)$. Any $v \in W_o^{1,2}(E)$ can be written as $v = v^\perp + v_o$, where $v^\perp \in W_n^{1,2}(E)$ and v_o has the form

$$v_o = \sum_{j=1}^{k_n} v_j w_j$$

where v_j are constants, and w_j are eigenfunctions of (11.1) corresponding to eigenvalues μ_j, for $j \le n$. By construction

$$\int_E \big(a_{ij} w_{n+1,x_i} v_{o,x_j} - \mu_{n+1} w_{n+1} v_o\big) dx = 0.$$

Hence (11.2) holds for all $v \in W_o^{1,2}(E)$, and (μ_{n+1}, w_{n+1}) is a nontrivial solution pair of (11.1). While this process identifies μ_{n+1} uniquely, the minimizer $w_{n+1} \in S_1$ depends on the choice of subsequence $\{u_{n'}\} \subset \{u_n\}$. Thus a priori, to μ_{n+1} there might correspond several eigenfunctions.

13 The Sequence of Eigenvalues and Eigenfunctions

This process generates a sequence of eigenvalues $\mu_n < \mu_{n+1}$ each with its own multiplicity. The linearly independent eigenfunctions $\{w_{\mu_n,1}, \dots, w_{\mu_n,n_{\mu_n}}\}$ corresponding to μ_n can be chosen to be orthonormal. The eigenfunctions are relabeled with n to form an orthonormal sequence $\{w_n\}$, and each is associated with its own eigenvalue, which in this reordering remains the same as the index of the corresponding eigenfunctions ranges over its own multiplicity. We then write

$$\begin{matrix} \mu_1 \le \mu_2 \le \cdots \le \mu_n \le \cdots \\ w_1 \quad\; w_2 \quad\;\; \cdots \quad\;\; w_n \quad\; \cdots \end{matrix} \tag{13.1}$$

Proposition 13.1 *Let $\{\mu_n\}$ and $\{w_n\}$ be as in (13.1). Then $\{\mu_n\} \to \infty$ as $n \to \infty$. The orthonormal system $\{w_n\}$ is complete in $L^2(E)$. The system $\{\sqrt{\mu_n} w_n\}$ is orthonormal and complete in $W_o^{1,2}(E)$ with respect to the inner product $a(\cdot, \cdot)$.*

Proof. If $\{\mu_n\} \to \mu_\infty < \infty$, the sequence $\{w_n\}$ will be bounded in $W_o^{1,2}(E)$, and by compactness, a subsequence $\{w_{n'}\} \subset \{w_n\}$ can be selected such that $\{w_{n'}\} \to w$ strongly in $L^2(E)$. Since $\{w_n\}$ is orthonormal in $L^2(E)$

$$2 = \lim_{n',m' \to \infty} \|w_{n'} - w_{m'}\|^2 \to 0.$$

Let $f \in L^2(E)$ be nonzero and orthogonal to the $L^2(E)$-closure of $\{w_n\}$. Let $u_f \in W_o^{1,2}(E)$ be the unique solution of the homogeneous Dirichlet problem (4.1) with $\mathbf{f} = 0$, for such a given f. Since $f \neq 0$, the solution $u_f \neq 0$ can be renormalized so that $\|u_f\|_2 = 1$. Then, for all $n \in \mathbb{N}$

$$\mu_n = \inf_{\substack{u \in W_n^{1,2}(E) \\ \|u\|_2 = 1}} a(u, u) \leq a(u_f, u_f) \leq \|f\|_2.$$

It is apparent that $\{\sqrt{\mu_n} w_n\}$ is an orthogonal system in $W_o^{1,2}(E)$ with respect to the inner product $a(\cdot, \cdot)$. To establish its completeness in $W_o^{1,2}(E)$ it suffices to verify that $a(w_n, u) = 0$ for all w_n implies $u = 0$. This in turn follows from the completeness of $\{w_n\}$ in $L^2(E)$. ■

Proposition 13.2 *μ_1 is simple and $w_1 > 0$ in E.*

Proof. Let (μ_1, w) be a solution pair for (11.1) for the first eigenvalue μ_1. Since $w \in W_o^{1,2}(E)$, also $w^\pm \in W_o^{1,2}(E)$, and either of these can be taken as a test function in the corresponding weak form (11.2) for the pair (μ_1, w). This gives

$$a(w^\pm, w^\pm) = \mu_1 \|w^\pm\|_2^2.$$

Therefore in view of the minimum problem (12.2), the two functions $w^\pm$ are both non-negative solutions of

$$\begin{aligned} -\big(a_{ij} w_{x_i}^\pm\big)_{x_j} &= \mu_1 w^\pm && \text{weakly in } E \\ w^\pm &= 0 && \text{on } \partial E. \end{aligned} \tag{13.2}$$

Lemma 13.1 *The functions $w^\pm$ are Hölder continuous in E, and if $w^+(x_o) > 0$ $(w^-(x_o) > 0)$, for some $x_o \in E$, then $w^+ > 0$ $(w^- > 0)$ in E.*

Assuming the lemma for the moment, either $w^+ \equiv 0$ or $w^- \equiv 0$ in E. Therefore, since $w = w^+ - w^-$, the eigenfunction w can be chosen to be strictly positive in E. If v and w are two linearly independent eigenfunctions corresponding to μ_1, they can be selected to be both positive in E and thus cannot be orthogonal. Thus $v = \gamma w$ for some $\gamma \in \mathbb{R}$, and μ_1 is simple. ■

The proof of Lemma 13.1 will follow from the Harnack Inequality of Section 8 of Chapter 11.

14 A Priori $L^\infty(E)$ Estimates for Solutions of the Dirichlet Problem (9.1)

A weak sub(super)-solution of the Dirichlet problem (9.1) is a function $u \in W^{1,2}(E)$, whose trace on ∂E satisfies $\operatorname{tr}(u) \le (\ge)\varphi$ and such that

$$\int_E \big(a_{ij} u_{x_i} v_{x_j} + f_j v_{x_j} + f v\big) dx \le (\ge) 0 \tag{14.1}$$

for all non-negative $v \in W_o^{1,2}(E)$. A function $u \in W^{1,2}(E)$ is a weak solution of the Dirichlet problem (9.1), if and only if is both a weak sub- and super-solution of that problem.

Proposition 14.1 *Let $u \in W^{1,2}(E)$ be a weak sub-solution of (9.1) for $N \ge 2$. Assume*

$$\varphi_+ \in L^\infty(\partial E), \quad \mathbf{f} \in L^{N+\varepsilon}(E), \quad f_+ \in L^{\frac{N+\varepsilon}{2}}(E) \tag*{(14.2)$_+$}$$

for some $\varepsilon > 0$. Then $u_+ \in L^\infty(E)$ and there exists a constant C_ε that can be determined a priori only in terms of λ, Λ, N, ε, and the constant γ in the Sobolev embedding (3.1)–(3.2), such that

$$\operatorname*{ess\,sup}_E u_+ \le \max\big\{ \operatorname*{ess\,sup}_{\partial E} \varphi_+ ; C_\varepsilon \big[\|\mathbf{f}\|_{N+\varepsilon}; |E|^\delta \|f_+\|_{\frac{N+\varepsilon}{2}}\big] |E|^\delta \big\} \tag*{(14.3)$_+$}$$

where

$$\delta = \frac{\varepsilon}{N(N+\varepsilon)}. \tag{14.4}$$

A similar statement holds for supersolutions. Precisely

Proposition 14.2 *Let $u \in W^{1,2}(E)$ be a weak super-solution of (9.1) for $N \ge 2$. Assume*

$$\varphi_- \in L^\infty(\partial E), \quad \mathbf{f} \in L^{N+\varepsilon}(E), \quad f_- \in L^{\frac{N+\varepsilon}{2}}(E) \tag*{(14.2)$_-$}$$

for some $\varepsilon > 0$. Then $u_- \in L^\infty(E)$ and

$$\operatorname*{ess\,sup}_E u_- \le \max\big\{ \operatorname*{ess\,sup}_{\partial E} \varphi_- ; C_\varepsilon \big[\|\mathbf{f}\|_{N+\varepsilon}; |E|^\delta \|f_-\|_{\frac{N+\varepsilon}{2}}\big] |E|^\delta \big\} \tag*{(14.3)$_-$}$$

for the same constants C_ε and δ.

Remark 14.1 The constant C_ε in $(14.3)_\pm$ is "stable" as $\varepsilon \to \infty$, in the sense that if $\mathbf{f}$ and $f_\pm$ are in $L^\infty(E)$, then $u_\pm \in L^\infty(E)$ and there exists a constant C_∞ depending on the indicated quantities except ε, such that

$$\operatorname*{ess\,sup}_E u_\pm \le \max\Big\{ \operatorname*{ess\,sup}_{\partial E} \varphi_\pm ; C_\infty \big[\|\mathbf{f}\|_\infty; |E|^{\frac{1}{N}} \|f_\pm\|_\infty\big] |E|^{\frac{1}{N}} \Big\}. \tag{14.5}$$

Remark 14.2 The constant C_ε tends to infinity as $\varepsilon \to 0$. Indeed, the propositions are false for $\varepsilon = 0$, as shown by the following example. For $N > 2$, the two equations

$$\begin{aligned} \Delta u &= f \qquad \text{where } f = \frac{N-2}{|x|^2 \ln|x|} - \frac{1}{|x|^2 \ln^2|x|} \\ \Delta u &= f_{j,x_j} \qquad \text{where } f_j = \frac{x_j}{|x|^2 \ln|x|} \end{aligned}$$

are both solved, in a neighborhood E of the origin by $u(x) = \ln\big|\ln|x|\big|$. One verifies that

$$\begin{aligned} &f \in L^{\frac{N}{2}}(E) \qquad \text{and } f \notin L^{\frac{N+\varepsilon}{2}}(E) \quad \text{for any } \varepsilon > 0 \\ &\mathbf{f} \in L^N(E) \qquad \text{and } \mathbf{f} \notin L^{N+\varepsilon}(E) \quad \text{for any } \varepsilon > 0. \end{aligned}$$

Remark 14.3 The propositions can be regarded as a weak form of the maximum principle (Section 4.1 of Chapter 2). Indeed, if u is a weak sub(super)-solution of the Dirichlet problem (9.1), with $\mathbf{f} = f = 0$, then $u_+ \le \operatorname{tr}(u)_+$ $(u_- \le \operatorname{tr}(u)_-)$.

15 Proof of Propositions 14.1–14.2

It suffices to establish Proposition 14.1. Let $u \in W^{1,2}(E)$ be a weak sub-solution of the Dirichlet problem (9.1), in the sense of (14.1) for all non-negative $v \in W_o^{1,2}(E)$. Let $k \ge \|\varphi_+\|_{\infty,\partial E}$ to be chosen, and set

$$k_n = k\Big(2 - \frac{1}{2^{n-1}}\Big), \qquad A_n = [u > k_n], \qquad n = 1, 2, \dots . \tag{15.1}$$

Then $(u-k_n)_+ \in W_o^{1,2}(E)$ for all $n \in \mathbb{N}$, and it can be taken as a test function in the weak formulation (14.1) to yield

$$\int_E \big[\,[a_{ij}u_{x_i} + f_j]\,(u-k_n)_{+x_j} + f_+(u-k_n)_+\big]dx \le 0.$$

From this, estimate

$$\begin{aligned} \lambda\|\nabla(u-k_n)_+\|_2^2 &\le \|\mathbf{f}\chi_{A_n}\|_2\|\nabla(u-k_n)_+\|_2 + \int_E f_+(u-k_n)_+dx \\ &\le \frac{\lambda}{4}\|\nabla(u-k_n)_+\|_2^2 + \frac{1}{\lambda}\|\mathbf{f}\chi_{A_n}\|_2^2 + \int_E f_+(u-k_n)_+dx \\ &\le \frac{\lambda}{4}\|\nabla(u-k_n)_+\|_2^2 + \frac{1}{\lambda}\|\mathbf{f}\|_{N+\varepsilon}^2|A_n|^{1-\frac{2}{N+\varepsilon}} \\ &\quad + \int_E f_+(u-k_n)_+dx. \end{aligned}$$

The last term is estimated by Hölder's inequality as

$$\int_E f_+(u-k_n)_+dx \le \|f_+\|_{\frac{p^*}{p^*-1}}\|(u-k_n)_+\|_{p^*}$$

where

$$\frac{p^*}{p^*-1}=\frac{N+\varepsilon}{2} \quad\text{and}\quad p^*=\frac{Np}{N-p}. \tag{15.2}$$

For these choices one verifies that $1<p<2$ for all $N\ge 2$. Therefore by (3.1) of the embedding of Theorem 3.1

$$\begin{aligned}\int_E f_+(u-k_n)_+dx &\le \gamma\|f_+\|_{\frac{N+\varepsilon}{2}}\|\nabla(u-k_n)_+\|_p\\ &\le \gamma\|\nabla(u-k_n)_+\|_2\|f_+\|_{\frac{N+\varepsilon}{2}}|A_n|^{\frac{1}{p}(1-\frac{p}{2})}\\ &\le \frac{\lambda}{4}\|\nabla(u-k_n)_+\|_2^2+\frac{\gamma^2}{\lambda}\|f_+\|^2_{\frac{N+\varepsilon}{2}}|A_n|^{\frac{2}{p}-1}.\end{aligned}$$

Combining these estimates gives

$$\|\nabla(u-k_n)_+\|_2^2 \le C_o^2|A_n|^{1-\frac{2}{N}+2\delta} \tag{15.3}$$

where

$$C_o^2=\frac{2\gamma^2}{\lambda^2}\max\left\{\|\mathbf{f}\|^2_{N+\varepsilon};|E|^{2\delta}\|f_+\|^2_{\frac{N+\varepsilon}{2}}\right\}$$

and δ is defined in (14.4).

15.1 An Auxiliary Lemma on Fast Geometric Convergence

Lemma 15.1 *Let $\{Y_n\}$ for $n=1,2,\dots,$ be a sequence of positive numbers linked by the recursive inequalities*

$$Y_{n+1}\le b^nKY_n^{1+\sigma} \tag{15.4}$$

for some $b>1$, $K>0$, and $\sigma>0$. If

$$Y_1\le b^{-1/\sigma^2}K^{-1/\sigma} \tag{15.5}$$

Then $\{Y_n\}\to 0$ as $n\to\infty$.

Proof. By direct verification by applying (15.5) recursively. ■

15.2 Proof of Proposition 14.1 for $N>2$

By the embedding inequality (3.1) for $W_o^{1,2}(E)$ and (15.3)

$$\begin{aligned}\frac{k^2}{4^n}|A_{n+1}| &\le \int_E (u-k_n)_+^2\chi_{[u>k_{n+1}]}dx \le \|(u-k_n)_+\|_2^2\\ &\le \|(u-k_n)_+\|^2_{2^*}|A_n|^{\frac{2}{N}} \le \gamma^2\|\nabla(u-k_n)_+\|_2^2|A_n|^{\frac{2}{N}}\\ &\le \gamma^2C_o^2|A_n|^{1+2\delta}.\end{aligned} \tag{15.6}$$

From this

$$|A_{n+1}| \le \frac{4^n\gamma^2 C_o^2}{k^2}|A_n|^{1+2\delta} \qquad \text{for all } \ n\in\mathbb{N} \tag{15.7}$$

where γ is the constant of the embedding inequality (3.1). If $\{|A_n|\}\to 0$ as $n\to\infty$, then $u\le 2k$ a.e. in E. By Lemma 15.1, this occurs if

$$|A_1| \le |E| \le 2^{-1/2\delta^2} C_o^{-1/\delta} k^{1/\delta}.$$

This in turn is satisfied if k is chosen from $k = 2^{1/2\delta}C_o|E|^\delta$. ∎

15.3 Proof of Proposition 14.1 for $N = 2$

The main difference is in the application of the embedding inequality in (15.6), leading to the recursive inequalities (15.7). Let $q>2$ to be chosen, and modify (15.6) by applying the embedding inequality (3.2) of Theorem 3.1, as follows. First

$$\begin{aligned}\|(u-k_n)_+\|_2 &\le \|(u-k_n)_+\|_q |A_n|^{\frac{1}{2}(1-\frac{2}{q})}\\ &\le \gamma(q)\|\nabla(u-k_n)_+\|_2^{1-\frac{2}{q}}\|(u-k_n)_+\|_2^{\frac{2}{q}}|A_n|^{\frac{1}{2}(1-\frac{2}{q})}\\ &\le \frac{1}{2}\|(u-k_n)_+\|_2 + \gamma(q)\|\nabla(u-k_n)_+\|_2|A_n|^{\frac{1}{2}}.\end{aligned}$$

Therefore

$$\begin{aligned}\frac{k^2}{4^n}|A_{n+1}| &\le \int_E (u-k_n)_+^2\chi_{[u>k_{n+1}]}dx\\ &\le \|(u-k_n)_+\|_2^2 \le 2\gamma(q)C_o^2|A_n|^{1+2\delta}.\end{aligned}$$ ∎

16 A Priori $L^\infty(E)$ Estimates for Solutions of the Neumann Problem (10.1)

A weak sub(super)-solution of the Neumann problem (10.1) is a function $u\in W^{1,2}(E)$ satisfying

$$\int_E \big(a_{ij}u_{x_i}v_{x_j} + f_j v_{x_j} + fv\big)dx \le (\ge) \int_{\partial E}\psi v\, d\sigma \tag{16.1}$$

for all non-negative test functions $v\in W^{1,2}(E)$. A function $u\in W^{1,2}(E)$ is a weak solution of the Neumann problem (10.1), if and only if is both a weak sub- and super-solution of that problem.

Proposition 16.1 *Let ∂E be of class C^1 and satisfying the segment property. Let $u\in W^{1,2}(E)$ be a weak sub-solution of (10.1) for $N\ge 2$, and assume that*

$$\psi_+\in L^{N-1+\sigma}(\partial E), \quad \mathbf{f}\in L^{N+\varepsilon}(E), \quad f_+\in L^{\frac{N+\varepsilon}{2}}(E) \tag*{(16.2)$_+$}$$

for some $\sigma > 0$ and $\varepsilon > 0$. Then $u_+ \in L^\infty(E)$, and there exists a positive constant C_ε that can be determined quantitatively a priori only in terms of the set of parameters $\{N, \lambda, \Lambda, \varepsilon, \sigma\}$, the constant γ in the embeddings of Theorem 2.1, the constant γ of the trace inequality of Proposition 8.2, and the structure of ∂E through the parameters h and ω of its cone condition such that

$$\operatorname*{ess\,sup}_E u_+ \le C_\varepsilon \max\big\{\|u_+\|_2; \|\psi_+\|_{q;\partial E}; \|\mathbf{f}\|_{N+\varepsilon}; |E|^\delta \|f_+\|_{\frac{N+\varepsilon}{2}}\big\} \tag{16.3$_+$}$$

where

$$q = N - 1 + \sigma, \quad \sigma = \varepsilon\frac{N-1}{N}, \quad \text{and} \quad \delta = \frac{\varepsilon}{N(N+\varepsilon)}. \tag{16.4}$$

Proposition 16.2 *Let ∂E be of class C^1 and satisfying the segment property. Let $u \in W^{1,2}(E)$ be a weak super-solution of (10.1) for $N \ge 2$, and assume that*

$$\psi_- \in L^{N-1+\sigma}(\partial E), \quad \mathbf{f} \in L^{N+\varepsilon}(E), \quad f_- \in L^{\frac{N+\varepsilon}{2}}(E) \tag{16.2$_-$}$$

for some $\sigma > 0$ and $\varepsilon > 0$. Then $u_- \in L^\infty(E)$

$$\operatorname*{ess\,sup}_E u_- \le C_\varepsilon \max\big\{\|u_-\|_2; \|\psi_-\|_{q;\partial E}; \|\mathbf{f}\|_{N+\varepsilon}; |E|^\delta \|f_-\|_{\frac{N+\varepsilon}{2}}\big\} \tag{16.3$_-$}$$

where the parameters q, σ, δ and C_ε are the same as in (16.3)$_+$ and (16.4).

Remark 16.1 The dependence on some norm of u, for example $\|u_\pm\|_2$, is expected, since the solutions of (10.1) are unique up to constants.

Remark 16.2 The constant C_ε in (16.3)$_\pm$ is “stable” as $\varepsilon \to \infty$, in the sense that if

$$\psi_\pm \in L^\infty(\partial E), \quad \mathbf{f} \in L^\infty(E), \quad f_\pm \in L^\infty(E) \tag{16.5}$$

then $u_\pm \in L^\infty(E)$ and there exists a constant C_∞ depending on the indicated quantities except ε and σ such that

$$\operatorname*{ess\,sup}_E u_\pm \le C_\infty \max\big\{\|u_\pm\|_2; \operatorname*{ess\,sup}_{\partial E} \psi_\pm; \|\mathbf{f}\|_\infty; |E|^{\frac{1}{N}} \|f_\pm\|_\infty\big\}. \tag{16.6}$$

Remark 16.3 The constant C_ε in (16.3)$_\pm$ tends to infinity as $\varepsilon \to 0$. The order of integrability of $\mathbf{f}$ and f required in (16.2)$_\pm$ is optimal for $u_\pm$ to be in $L^\infty(E)$. This can be established by the same local solutions in Remark 14.2. Also, the order of integrability of $\psi_\pm$ is optimal for $u_\pm$ to be in $L^\infty(E)$, even if $\mathbf{f} = f = 0$. Indeed, the propositions are false for $\sigma = 0$ and $\mathbf{f} = f = 0$, as shown by the following counterexample. Consider the family of functions parametrized by $\eta > 0$ (Section 8 of Chapter 2)

$$\mathbb{R} \times \mathbb{R}^+ = E \ni (x, y) \to F_\eta(x, y) = \frac{1}{2\pi} \ln \sqrt{x^2 + (y+\eta)^2}.$$

One verifies that F_η are harmonic in E, and on the boundary $y = 0$ of E

$$F_{\eta,y}\Big|_{y=0} = \frac{\eta}{2\pi(x^2+\eta^2)}.$$

One also verifies that

$$\int_{\mathbb{R}} F_{\eta,y}(x,0)dx = \frac{1}{2} \qquad \text{for all } \ \eta > 0.$$

Therefore if an estimate of the type of $(16.3)_\pm$ were to exist for $\sigma = 0$, and with C independent of σ, we would have, for all (x, y) in a neighborhood E_o of the origin

$$|F_\eta(x,y)| \le C(1 + \|F_{\eta,y}(\cdot,0)\|_{1,\partial E}) = \tfrac{3}{2}C \qquad \text{for all } \ \eta > 0.$$

Letting $\eta \to 0$ gives a contradiction. While the counterexample is set in $\mathbb{R}\times\mathbb{R}^+$, it generates a contradiction in a subset of E_o about the origin of $\mathbb{R}^2$.

Remark 16.4 The constants C_ε in $(16.3)_\pm$ and C_∞ in (16.6) depend on the embedding constants of Theorem 2.1. As such, they depend on the structure of ∂E through the parameters h and ω of its cone condition. Because of this dependence, C_ε and C_∞ tend to ∞ as either $h \to 0$ or $\omega \to 0$.

Remark 16.5 The propositions are a priori estimates, which assume that a sub(super)-solution exists. Sufficient conditions for the existence of a solution require that f and ψ must be linked by the compatibility condition (10.3). Such a requirement, however, plays no role in the a priori $L^\infty(E)$ estimates.

17 Proof of Propositions 16.1–16.2

It suffices to establish Proposition 16.1. Let $u \in W^{1,2}(E)$ be a sub-solution of the Neumann problem (10.1), in the sense of (16.1), and for $k > 0$ to be chosen, define k_n and A_n as in (15.1). In the weak formulation (16.1) take $v = (u - k_{n+1})_+ \in W^{1,2}(E)$, to obtain

$$\int_E [(a_{ij}u_{x_i} + f_j)(u-k_{n+1})_{+x_j} + f_+(u-k_{n+1})_+]dx \le \int_{\partial E} \psi_+(u-k_{n+1})_+ d\sigma.$$

Estimate the various terms by making use of the embedding (2.1), as follows

$$\begin{aligned} \Big| \int_E f_j(u-k_{n+1})_{+x_j}dx \Big| &\le \|\nabla(u-k_{n+1})_+\|_2 \|\mathbf{f}\chi_{A_{n+1}}\|_2 \\ &\le \frac{\lambda}{4}\|\nabla(u-k_{n+1})_+\|_2^2 + \frac{1}{\lambda}\|\mathbf{f}\|_{N+\varepsilon}^2 |A_{n+1}|^{1-\frac{2}{N+\varepsilon}}. \end{aligned}$$

Next, for the same choices of p_* as in (15.2), by the embedding (2.1) of Theorem 2.1

$$
\begin{aligned}
\Big|\int_E f_+(u-k_{n+1})_+dx\Big| &\le \|f_+\|_{\frac{N+\varepsilon}{2}}\|(u-k_{n+1})_+\|_{p^*}\\
&\le \frac{\gamma}{\omega}\|f_+\|_{\frac{N+\varepsilon}{2}}\Big(\frac{1}{h}\|(u-k_{n+1})_+\|_p+\|\nabla(u-k_{n+1})_+\|_p\Big)\\
&\le \frac{\lambda}{4}\|\nabla(u-k_{n+1})_+\|_2^2+\|(u-k_{n+1})_+\|_2^2\\
&\quad+\frac{\gamma^2}{\omega^2}\big(\frac{1}{\lambda}+\frac{1}{4h^2}\big)|E|^\delta\|f_+\|^2_{\frac{N+\varepsilon}{2}}|A_{n+1}|^{1-\frac{2}{N}+2\delta}
\end{aligned}
$$

where γ is the constant of the embedding inequality (2.1), and δ is defined in (14.4). Setting

$$
F^2=\|\mathbf{f}\|^2_{N+\varepsilon}+|E|^\delta\|f_+\|^2_{\frac{N+\varepsilon}{2}},\qquad C_1=\frac{\gamma^2}{\omega^2}\big(\frac{1}{\lambda}+\frac{1}{4h^2}\big)
$$

the previous remarks imply

$$
\begin{aligned}
\frac{\lambda}{2}\|\nabla(u-k_{n+1})_+\|_2^2\le \|(u-k_{n+1})_+\|_2^2+C_1F^2|A_{n+1}|^{1-\frac{2}{N}+2\delta}\\
+\Big|\int_{\partial E}\psi_+(u-k_{n+1})_+d\sigma\Big|.
\end{aligned}
$$

The last integral is estimated by means of the trace inequality (8.3). Let

$$
q=N-1+\sigma=(N-1)\Big(\frac{N+\varepsilon}{N}\Big)
$$

be the order of integrability of ψ on ∂E and determine p^* and p from

$$
1-\frac{1}{q}=\frac{1}{p^*}\frac{N}{N-1},\qquad p^*=\frac{Np}{N-p},\qquad \frac{1}{q}=\frac{p-1}{p}\frac{N}{N-1}.
$$

One verifies that for these choices, $1<p<2<N$, and the trace inequality (8.3) can be applied. Therefore

$$
\begin{aligned}
&\Big|\int_{\partial E}\psi_+(u-k_{n+1})_+d\sigma\Big|\le\|\psi\|_{q;\partial E}\|(u-k_{n+1})_+\|_{p^*\frac{N-1}{N};\partial E}\\
&\le\|\psi_+\|_{q;\partial E}\big[\|\nabla(u-k_{n+1})_+\|_p+2\gamma\|(u-k_{n+1})_+\|_p\big]\\
&\le\|\psi_+\|_{q;\partial E}\big[\|\nabla(u-k_{n+1})_+\|_2+2\gamma\|(u-k_{n+1})_+\|_2\big]|A_{n+1}|^{\frac{1}{p}-\frac{1}{2}}\\
&\le\frac{\lambda}{4}\|\nabla(u-k_{n+1})_+\|_2^2+\|(u-k_{n+1})_+\|_2^2\\
&\quad+\Big(\gamma^2+\frac{1}{\lambda}\Big)\|\psi_+\|^2_{q;\partial E}|A_{n+1}|^{\frac{2}{p}-1}.
\end{aligned}
$$

Denote by C_ℓ, $\ell=1,2,\dots$ generic positive constants that can be determined quantitatively a priori, only in terms of the set of parameters $\{N,\lambda,\Lambda\}$, the constant γ in the embeddings of Theorem 2.1, the constant γ of the trace inequality of Proposition 8.2, and the structure of ∂E through the parameters h and ω of the cone condition. Then combining the previous estimates yields the existence of constants C_2 and C_3 such that

$$\|\nabla(u-k_{n+1})_+\|_2^2 \le C_2\|(u-k_{n+1})_+\|_2^2 + C_3F_*^2|A_{n+1}|^{1-\frac{2}{N}+2\delta} \tag{17.1}$$

where we have set

$$F_*^2 = \max\big\{\|\psi\|_{q;\partial E}^2\,;\ \|\mathbf{f}\|_{N+\varepsilon}^2\,;\ |E|^{2\delta}\|f_+\|_{\frac{N+\varepsilon}{2}}^2\big\}.$$

17.1 Proof of Proposition 16.1 for $N>2$

By the embedding inequality (2.1) of Theorem 2.1 for $W^{1,2}(E)$ and (17.1)

$$\begin{aligned}\|(u-k_{n+1})_+\|_2^2 &\le \|(u-k_{n+1})_+\|_{2^*}^2|A_{n+1}|^{\frac{2}{N}}\\ &\le \frac{2\gamma^2}{\omega^2}\big(\|\nabla(u-k_{n+1})_+\|_2^2+\frac{1}{h^2}\|(u-k_{n+1})_+\|_2^2\big)|A_{n+1}|^{\frac{2}{N}}\\ &\le C_4\|(u-k_n)_+\|_2^2|A_{n+1}|^{\frac{2}{N}}+C_5F_*^2|A_{n+1}|^{1+2\delta}.\end{aligned} \tag{17.2}$$

For all $n\in\mathbb{N}$

$$Y_n \overset{\text{def}}{=} \int_E (u-k_n)_+^2dx \ge \int_{A_{n+1}}(u-k_{n+1})_+^2dx \ge \frac{k^2}{4^n}|A_{n+1}|.$$

Therefore the previous inequality yields

$$Y_{n+1} \le \frac{4^nC_4}{k^{4\delta}}Y_n^{1+2\delta}\left(\frac{1}{k^2}\int_E u^2dx\right)^{\frac{2}{N+\varepsilon}} + \frac{F_*^2}{k^2}\frac{4^{2n}C_5}{k^{4\delta}}Y_n^{1+2\delta}.$$

Take $k\ge\max\big\{\|u\|_2;F_*\big\}$, so that

$$\frac{1}{k^2}\int_E u^2dx\le 1 \qquad\text{and}\qquad \frac{F_*^2}{k^2}\le 1.$$

This choice leads to the recursive inequalities

$$Y_{n+1}\le \frac{4^{2n}C_6}{k^{4\delta}}Y_n^{1+2\delta} \tag{17.3}$$

for a constant C_6 that can be determined a priori only in terms of $\{N,\lambda,\Lambda\}$, the constants γ in the embedding inequalities of Theorem 2.1, the trace inequalities of Proposition 8.2, the smoothness of ∂E through the parameters ω and h of its cone condition, and is otherwise independent of $\mathbf{f}$, f, and ψ. By the fast geometric convergence Lemma 15.1, $\{Y_n\}\to 0$ as $n\to\infty$, provided

$$Y_1\le 2^{-1/\delta^2}C_6^{-1/2\delta}k^2.$$

We conclude that by choosing

$$k = 2^{1/2\delta^2}C_6^{1/4\delta}\max\{\|u\|_2\,;\,F_*\}$$

then $Y_\infty=\|(u-2k)_+\|_2=0$, and therefore $u\le 2k$ in E. ∎

17.2 Proof of Proposition 16.1 for $N = 2$

The only differences occur in the application of the embedding inequalities of Theorem 2.1, in the inequalities (17.2), leading to the recursive inequalities (17.3). Inequality (17.2) is modified by fixing $1 < p < 2$ and applying the embedding inequality (2.1) of Theorem 2.1 for $1 < p < N$. This gives

$$\begin{aligned}\|(u-k_{n+1})_+\|_2^2 &\le \|(u-k_{n+1})_+\|_{p^*}^2|A_{n+1}|^{2\frac{p-1}{p}}\\ &\le \gamma(p,h,\omega)\big(\|\nabla(u-k_{n+1})_+\|_p^2+\|(u-k_{n+1})_+\|_p^2\big)|A_{n+1}|^{2\frac{p-1}{p}}\\ &\le \gamma(p,h,\omega)\big(\|\nabla(u-k_{n+1})_+\|_2^2+\|(u-k_{n+1})_+\|_2^2\big)|A_{n+1}|\\ &\le \gamma\|(u-k_n)_+\|_2^2|A_{n+1}|+C_5F_*^2|A_{n+1}|^{1+2\delta}. \qquad \blacksquare\end{aligned}$$

18 Miscellaneous Remarks on Further Regularity

A function $u \in W^{1,2}_{\mathrm{loc}}(E)$ is a *local* weak solution of (1.2), irrespective of possible boundary data, if it satisfies (1.3) for all $v \in W^{1,2}_o(E_o)$ for all open sets E_o such that $\bar{E}_o \subset E$. On the data $\mathbf{f}$ and f assume

$$\mathbf{f} \in L^{N+\varepsilon}(E), \qquad f_\pm \in L^{\frac{N+\varepsilon}{2}}(E), \qquad \text{for some } \varepsilon > 0. \tag{18.1}$$

The set of parameters $\{N, \lambda, \Lambda, \varepsilon, \|\mathbf{f}\|_{N+\varepsilon}, \|f\|_{\frac{N+\varepsilon}{2}}\}$ are the data, and we say that a constant $C, \gamma, \dots$ depends on the data if it can be quantitatively determined a priori in terms of only these quantities. Continue to assume that the boundary ∂E is of class C^1 and with the segment property. For a compact set $\mathcal{K} \subset \mathbb{R}^N$ and $\eta \in (0,1)$ continue to denote by $|\!|\!| \cdot |\!|\!|_{\eta;\mathcal{K}}$ the Hölder norms introduced in (8.3) of Chapter 2.

Theorem 18.1. *Let $u \in W^{1,2}_{\mathrm{loc}}(E)$ be a local weak solution of (1.2) and let (18.1) hold. Then u is locally bounded and locally Hölder continuous in E, and for every compact set $K \subset E$, there exist positive constants γ_K, and C_K depending upon the data and* $\mathrm{dist}\{K;\partial E\}$*, and $\alpha \in (0,1)$ depending only on the data and independent of* $\mathrm{dist}\{K;\partial E\}$*, such that*

$$|\!|\!| u |\!|\!|_{\alpha;K} \le \gamma_K(\mathrm{data}, \mathrm{dist}\{K;\partial E\}). \tag{18.2}$$

Theorem 18.2. *Let $u \in W^{1,2}(E)$ be a solution of the Dirichlet problem (9.1), with $\mathbf{f}$ and f satisfying (18.1) and $\varphi \in C^\epsilon(\partial E)$ for some $\epsilon \in (0,1)$. Then u is Hölder continuous in $\bar{E}$ and there exist constants $\gamma > 1$ and $\alpha \in (0,1)$, depending upon the data, the C^1 structure of ∂E, and the Hölder norm $|\!|\!|\varphi|\!|\!|_{\epsilon;\partial E}$, such that*

$$|\!|\!| u |\!|\!|_{\alpha,\bar{E}} \le \gamma(\mathrm{data}, \varphi, \partial E). \tag{18.3}$$

Theorem 18.3. *Let $u \in W^{1,2}(E)$ be a solution of the Neumann problem (10.1), with $\mathbf{f}$ and f satisfying (16.1) and $\psi \in L^{N-1+\sigma}(\partial E)$ for some $\sigma \in$*

$(0,1)$. *Then u is Hölder continuous in $\bar{E}$, and there exist constants γ and $\alpha \in (0,1)$, depending on the data, the C^1 structure of ∂E, and $\|\psi\|_{N-1+\sigma;\partial E}$, such that*

$$|\!|\!|u|\!|\!|_{\alpha,\bar{E}} \le \gamma(\text{data}, \psi, \partial E). \tag{18.4}$$

The precise structure of these estimates in terms of the Dirichlet data φ or the Neumann ψ, as well as the dependence on the structure of ∂E is specified in more general theorems for functions in the DeGiorgi classes (Theorem 7.1 and Theorem 8.1 of the next Chapter). These are the key, seminal facts in the theory of regularity of solutions of elliptic equations. They can be used, by boot-strap arguments, to establish further regularity on the solutions, whenever further regularity is assumed on the data.

Problems and Complements

1c Weak Formulations and Weak Derivatives

1.1c The Chain Rule in $W^{1,p}(E)$

Proposition 1.1c *Let $u \in W^{1,p}(E)$ for some $p \ge 1$, and let $f \in C^1(\mathbb{R})$ satisfy* $\sup|f'| \le M$, *for some positive constant M. Then $f(u) \in W^{1,p}(E)$ and $\nabla f(u) = f'(u)\nabla u$.*

Proposition 1.2c *Let $u \in W^{1,p}(E)$ for some $p \ge 1$. Then $u^\pm \in W^{1,p}(E)$ and*

$$\nabla u^\pm = \begin{cases} \operatorname{sign}(u)\nabla u & \textit{a.e. in } [u^\pm > 0] \\ 0 & \textit{a.e. in } [u = 0]. \end{cases}$$

Proof (Hint). To prove the statement for u^+, for $\varepsilon > 0$, apply the previous proposition with

$$f_\varepsilon(u) = \begin{cases} \sqrt{u^2+\varepsilon^2} - \varepsilon & \text{for } u > 0 \\ 0 & \text{for } u \le 0. \end{cases}$$

Then let $\varepsilon \to 0$. ■

Corollary 1.1c *Let $u \in W^{1,p}(E)$ for some $p \ge 1$. Then $|u-k| \in W^{1,p}(E)$, for all $k \in \mathbb{R}$, and $\nabla u = 0$ a.e. on any level set of u.*

Corollary 1.2c *Let $f, g \in W^{1,p}(E)$ for some $p \geq 1$. Then $f \wedge g$ and $f \vee g$ are in $W^{1,p}(E)$ and*

$$\nabla f \wedge g = \begin{cases} \nabla f & a.e.\ in\ [f > g] \\ \nabla g & a.e.\ in\ [f < g] \\ 0 & a.e.\ in\ [f = g]. \end{cases}$$

A similar formula holds for $f \vee g$.

1.2. Prove Propostion 1.2 and the first part of Propostion 1.1.

In the same way as done in (1.5), one can introduce Sobolev spaces of higher order. Indeed, for $k \in \mathbb{N}$ and $p \geq 1$, one defines

$$W^{k,p}(E) = \{u \in L^p(E) : D^\alpha u \in L^p(E) \text{ for } |\alpha| \leq k\}.$$

For $1 \leq p < \infty$, a norm for $W^{k,p}(E)$ is given by

$$\|u\|_{k,p} = \left(\sum_{|\alpha| \leq k} \|D^\alpha u\|_p^p \right)^{\frac{1}{p}},$$

whereas for $p = \infty$ we have

$$\|u\|_{k,\infty} = \sup_{|\alpha| \leq k} \|D^\alpha u\|_\infty.$$

The spaces $W^{k,p}(E)$ are Banach spaces when endowed with their norms. Moreover, we have

$$W_o^{k,p}(E) = \{\text{the closure of } C_o^\infty(E) \text{ in the norm of } W^{k,p}(E)\}.$$

2c Embeddings of $W^{1,p}(E)$

It suffices to prove the various assertions for $u \in C^\infty(E)$. Fix $x \in E$ and let $\mathcal{C}_x \subset \bar{E}$ be a cone congruent to the cone $\mathcal{C}$ of the cone property. Let $\mathbf{n}$ be the unit vector exterior to $\mathcal{C}_x$, ranging over its same solid angle, and compute

$$|u(x)| = \Big| \int_0^h \frac{\partial}{\partial \rho} \Big(1 - \frac{\rho}{h}\Big) u(\rho \mathbf{n}) d\rho \Big| \leq \int_0^h |\nabla u(\rho \mathbf{n})| d\rho + \frac{1}{h} \int_0^h |u(\rho \mathbf{n})| d\rho.$$

Integrating over the solid angle of $\mathcal{C}_x$ gives

$$\begin{aligned} \omega |u(x)| &\leq \int_{\mathcal{C}_x} \frac{|\nabla u(y)|}{|x-y|^{N-1}} dy + \frac{1}{h} \int_{\mathcal{C}_x} \frac{|u(y)|}{|x-y|^{N-1}} dy \qquad (2.1c) \\ &\leq \int_E \frac{|\nabla u(y)|}{|x-y|^{N-1}} dy + \frac{1}{h} \int_E \frac{|u(y)|}{|x-y|^{N-1}} dy. \end{aligned}$$

The right-hand side is the sum of two Riesz potentials of the form (10.1) of Chapter 2. The embeddings (2.1)–(2.3) are now established from this and the estimates of Riesz potentials (10.2) of Proposition 10.1 of Chapter 2. Complete the estimates and compute the constants γ explicitly.

2.1c Proof of (2.4)

Let $\mathcal{C}_{x,\rho}$ be the cone of vertex at x, radius $0 < \rho \le h$, coaxial with $\mathcal{C}_x$ and with the same solid angle ω. Denote by $(u)_{x,\rho}$ the integral average of u over $\mathcal{C}_{x,\rho}$.

Lemma 2.1c *For every pair $x, y \in E$ such that $|x-y| = \rho \le h$*

$$|u(y) - (u)_{x,\rho}| \le \frac{\gamma(N,p)}{\omega} \rho^{1-\frac{N}{p}} \|\nabla u\|_p.$$

Proof. For all $\xi \in \mathcal{C}_{x,\rho}$

$$|u(y) - u(\xi)| = \Big| \int_0^1 \frac{\partial}{\partial t} u\big(y + t(\xi - y)\big) dt \Big|.$$

Integrate in $d\xi$ over $\mathcal{C}_{x,\rho}$, and then in the resulting integral perform the change of variables $y + t(\xi - y) = \eta$. The Jacobian is t^{-N}, and the new domain of integration is transformed into those η for which $|y - \eta| = t|\xi - y|$ as ξ ranges over $\mathcal{C}_{x,\rho}$. Such a transformed domain is contained in the ball $B_{2\rho t}(y)$. These operations give

$$\begin{aligned}
\frac{\omega}{N} \rho^N |u(y) - (u)_{x,\rho}| &\le \int_0^1 \Big(\int_{\mathcal{C}_{x,\rho}} |\xi - y|\, |\nabla u(y + t(\xi - y))| d\xi \Big) dt \\
&\le \int_0^1 t^{-(N+1)} \int_{E \cap B_{2\rho t}(y)} |\eta - y| |\nabla u(\eta)| d\eta\, dt \\
&\le \gamma(N,p) \int_0^1 t^{-(N+1)} (2\rho t)^{N(1-\frac{1}{p})+1} \|\nabla u\|_p t. \qquad \blacksquare
\end{aligned}$$

To conclude the proof of (2.4), fix $x, y \in E$, let $z = \frac{1}{2}(x+y)$, $\rho = \frac{1}{2}|x-y|$, and estimate

$$|u(x) - u(y)| \le |u(x) - (u)_{z,\rho}| + |u(y) - (u)_{z,\rho}| \le \frac{\gamma(N,p)}{\omega} |x-y|^{1-\frac{N}{p}} \|\nabla u\|_p. \ \blacksquare$$

2.2c Compact Embeddings of $W^{1,p}(E)$

The proof consists in verifying that a bounded subset of $W^{1,p}(E)$ satisfies the conditions for a subset of $L^q(E)$ to be compact ([50], Chapter V). For $\delta > 0$ let

$$E_\delta = \big\{ x \in E \big| \operatorname{dist}\{x; \partial E\} > \delta \big\}.$$

For $q \in [1, p^*)$ and $u \in W^{1,p}(E)$

$$\|u\|_{q, E - E_\delta} \le \|u\|_{p^*} |E - E_\delta|^{\frac{1}{q} - \frac{1}{p^*}}.$$

Next, for $h \in \mathbb{R}^N$ of length $|h| < \delta$ compute

$$\begin{aligned}\int_{E_\delta}|u(x+h)-u(x)|dx &\le \int_{E_\delta}\int_0^1\Big|\frac{d}{dt}u(x+th)\Big|dtdx\\ &\le |h|\int_0^1\int_{E_\delta}|\nabla u(x+th)|dx\,dt \le |h||E|^{\frac{p-1}{p}}\|\nabla u\|_p.\end{aligned}$$

Therefore for all $\sigma\in(0,\frac{1}{q})$

$$\begin{aligned}\int_{E_\delta}|T_hu-u|^q dx &= \int_{E_\delta}|T_hu-u|^{q\sigma+q(1-\sigma)}dx\\ &\le \Big(\int_{E_\delta}|T_hu-u|dx\Big)^{q\sigma}\Big(\int_{E_\delta}|T_hu-u|^{\frac{q(1-\sigma)}{1-q\sigma}}dx\Big)^{1-q\sigma}.\end{aligned}$$

Choose σ so that

$$\frac{q(1-\sigma)}{1-q\sigma}=p^*,\qquad \text{that is,}\qquad \sigma q=\frac{p^*-q}{p^*-1}.$$

Such a choice is possible if $1<q<p^*$. Applying the embedding Theorem 2.1 gives

$$\int_{E_\delta}|T_hu-u|^q dx\le \gamma^{(1-\sigma)q}\Big(\int_{E_\delta}|T_hu-u|dx\Big)^{\frac{p^*-q}{p^*-1}}\|u\|_{1,p}^{(1-\sigma)q}$$

for a constant γ depending only on N,p and the geometry of the cone property of E. Combining these estimates

$$\|T_hu-u\|_{q,E_\delta}\le\gamma_1|h|^\sigma\|u\|_{1,p}.\qquad\blacksquare$$

3c Multiplicative Embeddings of $W_o^{1,p}(E)$ and $\tilde{W}^{1,p}(E)$

3.1c Proof of Theorem 3.1 for $1\le p<N$

Lemma 3.1c *Let $u\in C_o^\infty(E)$ and $N>1$. Then*

$$\|u\|_{\frac{N}{N-1}}\le\prod_{j=1}^N\|u_{x_j}\|_1^{1/N}.$$

Proof. If $N=2$

$$\begin{aligned}\iint_E u^2(x_1,x_2)dx_1dx_2 &= \iint_E u(x_1,x_2)u(x_1,x_2)dx_1dx_2\\ &\le \iint_E \max_{x_2}u(x_1,x_2)\max_{x_1}u(x_1,x_2)dx_1dx_2\\ &= \int_{\mathbb{R}}\max_{x_2}u(x_1,x_2)dx_1\int_{\mathbb{R}}\max_{x_1}u(x_1,x_2)dx_2\\ &\le \iint_E|u_{x_1}|dx\iint_E|u_{x_2}|dx.\end{aligned}$$

Thus the lemma holds for $N = 2$. Assuming that it does hold for N, set

$$\overline{x} = (x_1, \dots, x_N) \quad \text{and} \quad x = (\bar{x}, x_{N+1}).$$

By repeated application of Hölder's inequality and the induction

$$\begin{aligned}
\|u\|_{\frac{N+1}{N}}^{\frac{N+1}{N}} &= \int_{\mathbb{R}} \int_{\mathbb{R}^N} |u(\overline{x}, x_{N+1})|^{\frac{N+1}{N}} d\overline{x} dx_{N+1} \\
&= \int_{\mathbb{R}} dx_{N+1} \int_{\mathbb{R}^N} |u(\overline{x}, x_{N+1})| |u(\overline{x}, x_{N+1})|^{\frac{1}{N}} d\overline{x} \\
&\le \int_{\mathbb{R}} dx_{N+1} \Big(\int_{\mathbb{R}^N} |u(\overline{x}, x_{N+1})| d\overline{x} \Big)^{\frac{1}{N}} \Big(\int_{\mathbb{R}^N} |u(\overline{x}, x_{N+1})|^{\frac{N}{N-1}} d\overline{x} \Big)^{\frac{N-1}{N}} \\
&\le \Big(\int_E |u_{x_{N+1}}| dx \Big)^{\frac{1}{N}} \int_{\mathbb{R}} \prod_{j=1}^{N} \Big(\int_{\mathbb{R}^N} |u_{x_j}(\bar{x}, x_{N+1})| d\bar{x} \Big)^{\frac{1}{N}} dx_{N+1} \\
&\le \Big(\int_E |u_{x_{N+1}}| dx \Big)^{\frac{1}{N}} \Big(\prod_{j=1}^{N} \int_E |u_{x_j}| dx \Big)^{\frac{1}{N}} = \Big(\prod_{j=1}^{N+1} \int_E |u_{x_j}| dx \Big)^{\frac{1}{N}} \qquad \blacksquare
\end{aligned}$$

Next, for $1 \le p < N$ write

$$\|u\|_{\frac{Np}{N-p}} = \Big(\int_E w^{\frac{N}{N-1}} dx \Big)^{\frac{N-1}{N} \frac{N-p}{p(N-1)}} \qquad \text{where } \ w = |u|^{\frac{p(N-1)}{N-p}}$$

and apply Lemma 3.1c to the function w. This gives

$$\begin{aligned}
\|u\|_{\frac{Np}{N-p}} &\le \Big[\prod_{j=1}^{N} \Big(\int_E |w_{x_j}| dx \Big)^{\frac{1}{N}} \Big]^{\frac{N-p}{p(N-1)}} \\
&= \gamma(N,p) \prod_{j=1}^{N} \Big(\int_E |u|^{\frac{p(N-1)}{N-p} - 1} |u_{x_j}| dx \Big)^{\frac{N-p}{Np(N-1)}}
\end{aligned}$$

where

$$\gamma = \Big(\frac{p(N-1)}{N-p} \Big)^{\frac{N-p}{p(N-1)}}.$$

Now for all $j = 1, \dots, N$, by Hölder's inequality

$$\int_E |u|^{\frac{p(N-1)}{N-p} - 1} |u_{x_j}| dx \le \Big(\int_E |u_{x_i}|^p dx \Big)^{\frac{1}{p}} \Big(\int_E |u|^{\frac{Np}{N-p}} dx \Big)^{\frac{p-1}{p}}.$$

Therefore

$$\begin{aligned}
\prod_{j=1}^{N} \Big(\int_E |u|^{\frac{p(N-1)}{N-p} - 1} |u_{x_j}| dx \Big)^{\frac{N-p}{Np(N-1)}} &= \prod_{j=1}^{N} \|u_{x_i}\|_p^{\frac{N-p}{Np(N-1)}} \|u\|_{\frac{Np}{N-p}}^{\frac{p-1}{p(N-1)}} \\
&\le \|\nabla u\|_p^{\frac{N-p}{p(N-1)}} \|u\|_{\frac{Np}{N-p}}^{\frac{p-1}{p} \frac{N}{N-1}}. \qquad \blacksquare
\end{aligned}$$

3.2c Proof of Theorem 3.1 for $p \ge N > 1$

Let $F(x;y)$ be the fundamental solution of the Laplacean. Then for $u \in C_o^\infty(E)$, by the Stokes formula (2.3)–(2.4) of Chapter 2

$$\begin{aligned} u(x) &= -\int_{\mathbb{R}^N} F(x;y)\Delta u(y)dy = \int_{\mathbb{R}^N} \nabla u(y)\cdot\nabla_y F(x;y)dy \\ &= \int_{|x-y|<\rho} \nabla u(y)\cdot\nabla_y F(x;y)dy + \int_{|x-y|>\rho} \nabla u(y)\cdot\nabla_y F(x;y)dy. \end{aligned}$$

The last integral can be computed by an integration by parts, and equals

$$\int_{|x-y|>\rho} \nabla u(y)\cdot\nabla_y F(x;y)dy = \frac{1}{\omega_N \rho^{N-1}}\int_{|x-y|=\rho} u(y)d\sigma$$

since $F(x;\cdot)$ is harmonic in $\mathbb{R}^N - \{x\}$. Here $d\sigma$ denotes the surface measure on the sphere $|x-y| = \rho$. Put this in the previous expression of $u(x)$, multiply by $N\omega_N\rho^{N-1}$, and integrate in $d\rho$ over $(0,R)$, where R is a positive number to be chosen later. This gives

$$\begin{aligned} \omega_N R^N |u(x)| \le N\int_0^R &\left(\int_{|x-y|<\rho}\frac{|\nabla u(y)|}{|x-y|^{N-1}}dy\right)\rho^{N-1}d\rho \\ &+ N\int_0^R\left(\int_{|x-y|=\rho}|u(y)|d\sigma\right)d\rho. \end{aligned}$$

From this, for all $x \in E$

$$\begin{aligned} \omega_N|u(x)| &\le \int_{B_R(x)}\frac{|\nabla u(y)|}{|x-y|^{N-1}}dy + \frac{N}{R^N}\int_{B_R(x)}|u(y)|dy \\ &= I_1(x,R) + NI_2(x,R). \end{aligned}$$

3.2.1c Estimate of $I_1(x,R)$

Choose two positive numbers $a, b < N$ such that

$$\frac{a}{q} + b\left(1-\frac{1}{p}\right) = N-1.$$

Since $p \ge N$, this choice is possible for the indicated range of q. Now write

$$\frac{|\nabla u|}{|x-y|^{N-1}} = |\nabla u|^{p(\frac{1}{p}-\frac{1}{q})}\frac{|\nabla u|^{\frac{p}{q}}}{|x-y|^{\frac{a}{q}}}\frac{1}{|x-y|^{b(1-\frac{1}{p})}}$$

and apply Hölder's inequality with the conjugate exponents

$$\left(\frac{1}{p}-\frac{1}{q}\right) + \frac{1}{q} + \left(1-\frac{1}{p}\right) = 1.$$

This gives

$$I_1(x,R) \le \|\nabla u\|_p^{1-\frac{p}{q}} \Big(\int_{B_R(x)} \frac{|\nabla u(y)|^p}{|x-y|^a} dy \Big)^{\frac{1}{q}} \Big(\int_{B_R(x)} \frac{1}{|x-y|^b} dy \Big)^{1-\frac{1}{p}}.$$

Taking the qth power and integrating over E gives

$$\|I_1(R)\|_q \le \frac{\omega_N^{1-\frac{1}{p}+\frac{1}{q}} R^{N(\frac{1}{N}-\frac{1}{p}+\frac{1}{q})}}{(N-a)^{\frac{1}{q}}(N-b)^{1-\frac{1}{p}}} \|\nabla u\|_p.$$

3.2.2c Estimate of $I_2(x,R)$

$$\begin{aligned} I_2(x,R) &\le R^{-N} \Big(\int_{|x-y|<R} |u(y)|^p dy \Big)^{\frac{1}{p}} \Big(\int_{|x-y|<R} 1 dy \Big)^{1-\frac{1}{p}} \\ &\le \Big(\frac{\omega_N}{N} \Big)^{1-\frac{1}{p}} R^{-\frac{N}{p}} \|u\|_p^{1-\frac{p}{q}} \Big(\int_{|\xi|<R} |u(x+\xi)|^p d\xi \Big)^{\frac{1}{q}}. \end{aligned}$$

Take the qth power and integrate in dx over $\mathbb{R}^N$ to obtain

$$\|I_2\|_q \le \Big(\frac{\omega_N}{N} \Big)^{1+\frac{1}{q}-\frac{1}{p}} R^{-N(\frac{1}{p}-\frac{1}{q})} \|u\|_p.$$

3.2.3c Proof of Theorem 3.1 for $p \ge N > 1$ (Concluded)

Combining these estimates yields

$$\|u\|_q \le \gamma\big(R^{1-\delta} \|\nabla u\|_p + R^{-\delta} \|u\|_p \big), \qquad \delta = N\Big(\frac{1}{p} - \frac{1}{q} \Big)$$

for a constant $\gamma(N,p,q,a,b)$. Minimizing the right-hand side with respect to the parameter R proves the estimate.

3.3c Proof of Theorem 3.2 for $1 \le p < N$ and E Convex

Having fixed $x, y \in E$, let $R(x,y)$ be the distance from x to ∂E along $y - x$ and write

$$|u(x) - u(y)| \le \int_0^{R(x,y)} \Big| \frac{\partial}{\partial \rho} u(x + \rho \mathbf{n}) \Big| d\rho, \qquad \mathbf{n} = \frac{y-x}{|y-x|}.$$

Integrate in dy over E to obtain

$$|E||u(x) - u_E| \le \int_E \Big(\int_0^{R(x,y)} |\nabla u(x + \rho \mathbf{n})| d\rho \Big) dy.$$

The integral in dy is calculated by introducing polar coordinates with pole at x. Therefore if $\mathbf{n}$ is the angular variable spanning the sphere $|\mathbf{n}|=1$, the right-hand side is majorized by

$$\begin{aligned}(\operatorname{diam} E)^{N-1}\int_0^{\operatorname{diam} E}\int_{|\mathbf{n}|=1}\int_0^{R(x,y)}&\rho^{N-1}\frac{|\nabla u(x+\rho\mathbf{n})|}{|x-y|^{N-1}}d\rho d\mathbf{n}dr\\ &\le(\operatorname{diam} E)^N\int_E\frac{|\nabla u|}{|x-y|^{N-1}}dy.\end{aligned}$$

Therefore

$$|u(x)-u_E|\le\frac{(\operatorname{diam} E)^N}{|E|}\int_E\frac{|\nabla u(y)|}{|x-y|^{N-1}}dy.$$

The proof is now concluded using the estimates of the Riesz potentials in Section 10 of Chapter 2. The remaining cases for $p\ge N$ are left as an exercise following similar arguments in the analogous multiplicative embeddings of $W_o^{1,p}(E)$.

5c Solving the Homogeneous Dirichlet Problem (4.1) by the Riesz Representation Theorem

Consider formally the linear operator with variable coefficients

$$\mathcal{L}(u)=-\big(a_{ij}u_{x_i}+a_ju\big)_{x_j}+b_iu_{x_i}+cu \tag{5.1c}$$

and the associated, formal bilinear form

$$a(u,v)=\int_E\big(a_{ij}u_{x_i}v_{x_j}+a_juv_{x_j}+b_iu_{x_i}v+cuv\big)\,dx. \tag{5.2c}$$

Assuming that (a_{ij}) satisfies the ellipticity condition (1.1), all the various terms are well defined for $u,v\in W_o^{1,2}(E)$, provided $c\in L^\infty(E)$ and

$$\left.\begin{array}{l}\mathbf{a}=(a_1,\dots,a_N)\\ \mathbf{b}=(b_1,\dots,b_N)\end{array}\right\}\in\left\{\begin{array}{ll}L^N(E) & \text{if } N>2\\ L^q(E) & \text{for some } q>2 \text{ if } N=2.\end{array}\right. \tag{5.3c}$$

The homogeneous Dirichlet problem (4.1) takes the form

$$\mathcal{L}(u)=\operatorname{div}\mathbf{f}-f\ \text{ in } E,\qquad\text{and } u\big|_{\partial E}=0\ \text{ on } \partial E. \tag{5.4c}$$

The latter is meant in the weak form of seeking $u\in W_o^{1,2}(E)$ such that

$$a(u,v)=-\int_E\big(\mathbf{f}\cdot\nabla v+fv\big)dx \tag{5.5c}$$

for all $v\in W_o^{1,2}(E)$. The unique solvability of this problem can be established almost verbatim by any one of the methods of Sections 5–7, provided the bilinear form $a(\cdot,\cdot)$ introduced in (5.2c) generates an inner product in $W_o^{1,2}(E)$, equivalent to any one of the inner products in (3.9), precisely, if there are constants $0 < \lambda_o < \Lambda_o$ such that

$$\lambda_o \|\nabla u\|_2^2 \le a(u,u) \le \Lambda_o \|\nabla u\|_2^2 \qquad \text{for all } u \in W_o^{1,2}(E).$$

This can be ensured by a number of conditions on **a**, **b**, and c, and on the size of E. Let $c = c^+ - c^-$ be partitioned into its positive and negative parts. Prove that if $N > 2$, the following condition is sufficient for the unique solvability of (5.4c):

$$\gamma \left(\|\mathbf{a}\|_N + \|\mathbf{b}\|_N + \gamma \|c^-\|_\infty \right) \le (1-\varepsilon)\lambda \tag{5.6c}$$

for some $\varepsilon \in (0,1)$, where γ is the constant appearing in the embedding inequality (3.1). The latter occurs, for example, if $c \ge 0$, $\mathbf{b} \in L^q(E)$ for some $q > N$, and $|E|$ is sufficiently small. Prove that another sufficient condition is

$$c \ge c_o > 0 \quad \text{and} \quad \frac{1}{4(1-\varepsilon)\lambda} \left(\|\mathbf{a}\|_\infty + \|\mathbf{b}\|_\infty \right) \le c_o. \tag{5.7c}$$

6c Solving the Homogeneous Dirichlet Problem (4.1) by Variational Methods

The homogeneous Dirichlet problem (5.4c), can also be solved by variational methods. The corresponding functional is

$$2J(u) = \int_E \{ [a_{ij} u_{x_i} + b_i u_{x_i} + (a_j + b_j)u + 2f_j] u_{x_j} + (\mathbf{b} \cdot \nabla u + cu + 2f)u \} dx.$$

The same minimization procedure can be carried out, provided **b** and c satisfy either (5.6c) or (5.7c).

6.1c More General Variational Problems

More generally one might consider minimizing functionals of the type

$$W_o^{1,p}(E) \ni u \to J(u) = \int_E F(x,u,\nabla u)\,dx, \qquad p > 1 \tag{6.1c}$$

where the function

$$E \times \mathbb{R} \times \mathbb{R}^N \ni (x,z,\mathbf{q}) \to F(x,z,\mathbf{q})$$

is measurable in x for a.e. $(z,\mathbf{q}) \in \mathbb{R}^{N+1}$, differentiable in z and $\mathbf{q}$ for a.e. $x \in E$, and satisfies the structure condition

$$\lambda |\mathbf{q}|^p - f(x) \le F(x,z,\mathbf{q}) \le \Lambda |\mathbf{q}|^p + f(x) \tag{6.2c}$$

for a given non-negative $f \in L^1(E)$. On F impose also the convexity (ellipticity) condition, that is, $F(x,z,\cdot) \in C^2(\mathbb{R}^N)$ for a.e. $(x,z) \in E \times \mathbb{R}$, and

$$F_{q_i q_j} \xi_i \xi_j \ge \lambda |\xi|^p \quad \text{for all } \xi \in \mathbb{R}^N \text{ for a.e. } (x,z) \in E \times \mathbb{R}. \tag{6.3c}$$

A Prototype Example

Let (a_{ij}) denote a symmetric $N \times N$ matrix with entries $a_{ij} \in L^\infty(E)$ and satisfying the ellipticity condition (1.1), and consider the functional

$$W_o^{1,p}(E) \ni u \to pJ(u) = \int_E (|\nabla u|^{p-2} a_{ij} u_{x_i} u_{x_j} + pfu)dx \tag{6.4c}$$

for a given $f \in L^q(E)$, where $q \geq 1$ satisfies

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{N} + 1 \quad \text{if } 1 < p < N, \quad \text{and } q \geq 1 \text{ if } p \geq N. \tag{6.5c}$$

Let $\mathbf{v} = (v_1, \dots, v_N)$ be a vector-valued function defined in E. Verify that the map

$$[L^p(E)]^N \ni \mathbf{v} = \int_E |\mathbf{v}|^{p-2} a_{ij} v_i v_j dx$$

defines a norm in $[L^p(E)]^N$ equivalent to $\|\mathbf{v}\|_p$. Since the norm is weakly lower semi-continuous, for every sequence $\{\mathbf{v}_n\} \subset [L^p(E)]^N$ weakly convergent to some $\mathbf{v} \in [L^p(E)]^N$

$$\liminf \int_E |\mathbf{v}_n|^{p-2} a_{ij} v_{i,n} v_{j,n} dx \geq \int_E |\mathbf{v}|^{p-2} a_{ij} v_i v_j dx.$$

The convexity condition (6.3c), called also the Legendre condition, ensures that a similar notion of semi-continuity holds for the functional $J(\cdot)$ in (6.1c) (see[185]).

Lower Semi-Continuity

A functional J from a topological space X into $\mathbb{R}$ is lower semi-continuous if $[J > a]$ is open in X for all $a \in \mathbb{R}$. Prove the following:

Proposition 6.1c *Let X be a topological space satisfying the first axiom of countability. A functional $J : X \to \mathbb{R}$ is lower semi-continuous if and only if for every sequence $\{u_n\} \subset X$ convergent to some $u \in X$*

$$\liminf J(u_n) \geq J(u).$$

6.3. The epigraph of J is the set

$$\mathcal{E}_J = \{(x,a) \in X \times \mathbb{R} \bigm| J(x) \leq a\}.$$

Assume that X satisfies the first axiom of countability and prove that J is lower semi-continuous if and only if its epigraph is closed.

6.4. Prove that $J : X \to \mathbb{R}$ is convex if and only if its epigraph is convex.

6.5. Prove the following:

Proposition 6.2c *Let $J : W_o^{1,p}(E) \to \mathbb{R}$ be the functional in (6.1c) where F satisfies (6.2c)–(6.3c). Then J is weakly lower semi-continuous.*

Hint: Assume first that F is independent of x and z and depends only on $\mathbf{q}$. Then J may be regarded as a convex functional from $\tilde{J} : [L^p(E)]^N \to \mathbb{R}$. Prove that its epigraph is (strongly and hence weakly) closed in $[L^p(E)]^N$.

6.6. Prove the following:

Proposition 6.3c *Let $J : W_o^{1,p}(E) \to \mathbb{R}$ be the functional in (6.1c) where F satisfies (6.2c)–(6.3c). Then J has a minimum in $W_o^{1,p}(E)$.*

Hint: Parallel the procedure of Section 6.

6.7. The minimum claimed by Proposition 6.3c need not be unique. Provide a counterexample. Formulate sufficient assumptions on F to ensure uniqueness of the minimum.

6.8c Gâteaux Derivatives, Euler Equations and Quasi-Linear Elliptic Equations

Let X be a Hausdorff space. A functional $J : X \to \mathbb{R}$ is Gâteaux differentiable at $w \in X$ in the direction of some $v \in X$ if there exists an element $J'(w; v) \in \mathbb{R}$ such that

$$\lim_{t \to 0} \frac{J(w + tv) - J(w)}{t} = J'(w; v).$$

The equation

$$J'(w; v) = 0 \qquad \text{for all } \ v \in X$$

is called the Euler equation of J. In particular, (6.4) is the Euler equation of the functional in (6.1). The Euler equation of the functional in (6.4c) is

$$-\left(|\nabla u|^{p-2} a_{ij} u_{x_i}\right)_{x_j} = f. \tag{6.6c}$$

In the special case $(a_{ij}) = \mathbb{I}$ this is the p-Laplacean equation

$$-\operatorname{div} |\nabla u|^{p-2} \nabla u = f. \tag{6.7c}$$

The Euler equation of the functional in (6.1c) is

$$-\operatorname{div} \mathbf{A}(x, u, \nabla u) + B(x, u, \nabla u) = 0, \qquad u \in W_o^{1,p}(E) \tag{6.8c}$$

where $\mathbf{A} = \nabla_{\mathbf{q}} F$ and $B = F_z$. The equation is elliptic in the sense that $(a_{ij}) = (F_{q_i q_j})$ satisfies (6.3c). Thus the functional in (6.1c) generates the PDE in (6.8c) as its Euler equation, and minima of J are solutions of (6.8c).

6.8.1c Quasi-Linear Elliptic Equations

Consider now (6.8c) independently of its variational origin, where

$$E \times \mathbb{R} \times \mathbb{R}^N \ni (x, z, \mathbf{q}) \to \begin{cases} \mathbf{A}(x, z, \mathbf{q}) \in \mathbb{R}^N \\ B(x, z, \mathbf{q}) \in \mathbb{R} \end{cases} \tag{6.9c}$$

are continuous functions of their arguments and subject to the structure conditions

$$\begin{cases} \mathbf{A}(x, z, \mathbf{q}) \cdot \mathbf{q} \geq \lambda |\mathbf{q}|^p - C^p \\ |\mathbf{A}(x, z, \mathbf{q})| \leq \Lambda |\mathbf{q}|^{p-1} + C^{p-1} \\ |B(x, z, \mathbf{q})| \leq C|\mathbf{q}|^{p-1} + C^p \end{cases} \tag{6.10c}$$

for all $(x, z, \mathbf{q}) \in E \times \mathbb{R} \times \mathbb{R}^N$, for given positive constants $\lambda \leq \Lambda$ and non-negative constant C. A local solution of (6.8c)–(6.10c), irrespective of possible prescribed boundary data, is a function $u \in W^{1,p}_{\text{loc}}(E)$ satisfying

$$\int_E \big[\mathbf{A}(x, u, \nabla u)\nabla v + B(x, u, \nabla u)v\big]\,dx = 0 \ \text{ for all } \ v \in W^{1,p}_o(E_o) \tag{6.11c}$$

for every open set E_o such that $\bar{E}_o \subset E$. In general, there is not a function F satisfying (6.2c)–(6.3c) and a corresponding functional as in (6.1c) for which (6.8c) is its Euler equation. It turns out, however, that local solutions of (6.8c)–(6.10c), whenever they exist, possess the same local behavior, regardless of their possible variational origin (Chapter 10).

6.8.2c Quasi-Minima

Let $J : W^{1,p}_{\text{loc}}(E) \to \mathbb{R}$ be given by (6.1c), where F satisfies (6.2c) but not necessarily (6.3c). A function $u \in W^{1,p}_{\text{loc}}(E)$ is a Q-minimum for J if there is a number $Q \geq 1$ such that

$$J(u) \leq QJ(u + v) \qquad \text{for all } \ v \in W^{1,p}_o(E_o)$$

for every open set E_o such that $\bar{E}_o \subset E$. The notion is of local nature. Minima are Q-minima, but the converse is false. Every functional of the type (6.1c)–(6.3c) generates a quasi-linear elliptic PDE of the type of (6.8c)–(6.10c). The converse is in general false. However every *local* solution $u \in W^{1,p}_{\text{loc}}(E)$ of (6.8c)–(6.10c) is a Q-minimum, in the sense that there exists some F satisfying (6.2c), but not necessarily (6.3c), such that u is a Q-minimum for the function J in (6.1c) for such a F ([98]).

8c Traces on ∂E of Functions in $W^{1,p}(E)$

8.1c Extending Functions in $W^{1,p}(E)$

Establish Proposition 8.1 by the following steps. Let $\mathbb{R}^N_+$ be the upper-half space $x_N > 0$ and denote its coordinates by $x = (\bar{x}, x_N)$, where

$\bar{x} = (x_1, \dots, x_{N-1})$. Assume first that $E = \mathbb{R}^N_+$ so that ∂E is the hyperplane $x_N = 0$. Given $u \in W^{1,p}(\mathbb{R}^N_+)$, set ([172])

$$\tilde{u}(\bar{x}, x_N) = \begin{cases} \quad u(\bar{x}, x_N) & \text{if } x_N > 0 \\ -3u(\bar{x}, -x_N) + 4u(\bar{x}, -\frac{1}{2}x_N) & \text{if } x_N < 0. \end{cases}$$

Prove that $\tilde{u} \in W^{1,p}(\mathbb{R}^N)$, and that $C^\infty_o(\mathbb{R}^N)$ is dense in $W^{1,p}(\mathbb{R}^N_+)$.

If ∂E is of class C^1 and has the segment property, it admits a finite covering with balls $B_t(x_j)$ for some $t > 0$, and $x_j \in \partial E$ for $j = 1, \dots, m$. Let then

$$U = \{B_o, B_{2t}(x_1), \dots, B_{2t}(x_m)\}, \qquad B_o = E - \bigcup_{j=1}^m \bar{B}_t(x_j)$$

be an open covering of E, and let Φ be a partition of unity subordinate to U. Set

$$\psi_j = \{\text{the sum of the } \varphi \in \Phi \text{ supported in } B_{2t}(x_j)\}$$

so that

$$u = \sum_{j=1}^m u_j \quad \text{where} \quad u_j = \begin{cases} u\psi_j & \text{in } E \\ 0 & \text{otherwise.} \end{cases}$$

By construction, $u_j \in W^{1,p}(B_{2t}(x_j))$ with bounds depending on t. By choosing t sufficiently small, the portion $\partial E \cap B_{2t}(x_j)$ can be mapped, in a local system of coordinates, into a portion of the hyperplane $x_N = 0$. Denote by U_j an open ball containing the image of $B_{2t}(x_j)$ and set $U^+_j = U_j \cap [x_N > 0]$. The transformed functions $\bar{u}_j$ belong to $W^{1,p}(U^+_j)$. Perform the extension as indicated earlier, return to the original coordinates and piece together the various integrals each relative to the balls $B_{2t}(x_j)$ of the covering U. This technique is refereed to as local “flattening of the boundary”.

8.2c The Trace Inequality

Proposition 8.1c *Let $u \in C^\infty_o(\mathbb{R}^N)$. If $1 \le p < N$, there exists a constant $\gamma = \gamma(N,p)$ such that*

$$\|u(\cdot, 0)\|_{p^* \frac{N-1}{N}, \mathbb{R}^{N-1}} \le \gamma \|\nabla u\|_{p, \mathbb{R}^N_+}. \tag{8.1c}$$

If $p > N$, there exist constants $\gamma = \gamma(N,p)$ such that

$$\|u(\cdot, 0)\|_{\infty, \mathbb{R}^{N-1}} \le \gamma \|u\|^{1-\frac{N}{p}}_{p, \mathbb{R}^N_+} \|\nabla u\|^{\frac{N}{p}}_{p, \mathbb{R}^N_+} \tag{8.2c}$$

$$|u(\bar{x}, 0) - u(\bar{y}, 0)| \le \gamma |\bar{x} - \bar{y}|^{1-\frac{N}{p}} \|\nabla u\|_{p, \mathbb{R}^N_+} \tag{8.3c}$$

for all $\bar{x}, \bar{y} \in \mathbb{R}^{N-1}$.

Proof. For all $\bar{x} \in \mathbb{R}^{N-1}$ and all $r \ge 1$

$$|u(\bar{x},0)|^r \le r \int_0^\infty |u(\bar{x},x_N)|^{r-1} |u_{x_N}(\bar{x},x_N)| dx_N.$$

Integrate both sides in $d\bar{x}$ over $\mathbb{R}^{N-1}$ and apply Hölder's inequality to the resulting integral on the right-hand side to obtain

$$\|u(\cdot,0)\|^r_{r,\mathbb{R}^{N-1}} \le r \|\nabla u\|_{p,\mathbb{R}^N_+} \|u\|^{r-1}_{q,\mathbb{R}^N_+}, \qquad \text{where } q = \frac{p}{p-1}(r-1). \tag{8.4c}$$

Apply this with $r = p^* \frac{N-1}{N}$, and use the embedding (3.1) of Theorem 3.1 to get

$$\|u(\cdot,0)\|_{p^* \frac{N-1}{N},\mathbb{R}^{N-1}} \le \gamma \|u\|^{1-\frac{1}{r}}_{p^*,\mathbb{R}^N_+} \|\nabla u\|^{\frac{1}{r}}_{p,\mathbb{R}^N_+} \le \gamma \|\nabla u\|_{p,\mathbb{R}^N_+}.$$

The domain $\mathbb{R}^N_+$ satisfies the cone condition with cone $\mathcal{C}$ of solid angle $\frac{1}{2}\omega_N$ and height $h \in (0,\infty)$. Then (8.3c) follows from (2.4) of Theorem 2.1, whereas (8.2c) follows from (2.3) of the same theorem, by minimizing over $h \in (0,\infty)$. ■

Prove Proposition 8.2 by a local flattening of ∂E.

8.3c Characterizing the Traces on ∂E of Functions in $W^{1,p}(E)$

Set $\mathbb{R}^{N+1}_+ = \mathbb{R}^N \times \mathbb{R}_+$ and denote the coordinates in $\mathbb{R}^{N+1}_+$ by (x,t) where $x \in \mathbb{R}^N$ and $t \ge 0$. Also set

$$\nabla_N = \Big(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_N}\Big), \qquad \nabla = \Big(\nabla_N, \frac{\partial}{\partial t}\Big).$$

Proposition 8.2c *Let $u \in C^\infty_o(\mathbb{R}^{N+1}_+)$. Then*

$$|\!|\!| u(\cdot,0) |\!|\!|_{1-\frac{1}{p},p;\mathbb{R}^N} \le \gamma \|u_t\|^{\frac{1}{p}}_{p,\mathbb{R}^{N+1}_+} \|\nabla_N u\|^{1-\frac{1}{p}}_{p,\mathbb{R}^{N+1}_+} \tag{8.5c}$$

where $\gamma = \gamma(p)$ depends only on p and $\gamma(p) \to \infty$ as $p \to 1$.

Proof. For every pair $x, y \in \mathbb{R}^N$, set $2\xi = x - y$ and consider the point $z \in \mathbb{R}^{N+1}_+$ of coordinates $z = (\frac{1}{2}(x+y), \lambda|\xi|)$, where λ is a positive parameter to be chosen. Then

$$\begin{aligned} |u(x,0) - u(y,0)| &\le |u(z) - u(x,0)| + |u(z) - u(y,0)| \\ &\le |\xi| \int_0^1 |\nabla_N u(x - \rho\xi, \lambda\rho|\xi|)| d\rho + |\xi| \int_0^1 |\nabla_N u(y + \rho\xi, \lambda\rho|\xi|)| d\rho \\ &\quad + \lambda|\xi| \int_0^1 |u_t(x - \rho\xi, \lambda\rho|\xi|)| d\rho + \lambda|\xi| \int_0^1 |u_t(y + \rho\xi, \lambda\rho|\xi|)| d\rho. \end{aligned}$$

From this

$$\begin{aligned}\frac{|u(x,0)-u(y,0)|^p}{|x-y|^{N+(p-1)}} \le{}& \frac{1}{2^p}\bigg(\int_0^1 \frac{|\nabla_N u(x-\rho\xi,\lambda\rho|\xi|)|}{|x-y|^{\frac{N-1}{p}}}d\rho\bigg)^p \\ &+ \frac{1}{2^p}\bigg(\int_0^1 \frac{|\nabla_N u(y+\rho\xi,\lambda\rho|\xi|)|}{|x-y|^{\frac{N-1}{p}}}d\rho\bigg)^p \\ &+ \frac{1}{2^p}\lambda^p\bigg(\int_0^1 \frac{|u_t(x-\rho\xi,\lambda\rho|\xi|)|}{|x-y|^{\frac{N-1}{p}}}d\rho\bigg)^p \\ &+ \frac{1}{2^p}\lambda^p\bigg(\int_0^1 \frac{|u_t(y+\rho\xi,\lambda\rho|\xi|)|}{|x-y|^{\frac{N-1}{p}}}d\rho\bigg)^p.\end{aligned}$$

Next integrate both sides over $\mathbb{R}^N\times\mathbb{R}^N$. In the resulting inequality take the $\frac{1}{p}$ power and estimate the various integrals on the right-hand side by the continuous version of Minkowski's inequality. This gives

$$\begin{aligned}|\!|\!|u(\cdot,0)|\!|\!|_{1-\frac{1}{p},\mathbb{R}^N} \le{}& \int_0^1\bigg(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\nabla_N u(x-\rho\xi,\lambda\rho|\xi|)|^p}{|x-y|^{N-1}}dxdy\bigg)^{\frac{1}{p}}d\rho \\ &+\lambda\int_0^1\bigg(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_t(x-\rho\xi,\lambda\rho|\xi|)|^p}{|x-y|^{N-1}}dxdy\bigg)^{\frac{1}{p}}d\rho.\end{aligned}$$

Compute the first integral by integrating first in dy and perform such integration in polar coordinates with pole at x. Denoting by $\mathbf{n}$ the unit vector spanning the unit sphere in $\mathbb{R}^N$ and recalling that $2|\xi|=|x-y|$, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\nabla_N u(x-\rho\xi,\lambda\rho|\xi|)|^p}{|x-y|^{N-1}}dxdy \\ &\qquad = 2\int_{|\mathbf{n}|=1}d\mathbf{n}\int_0^\infty d|\xi|\int_{\mathbb{R}^N}|\nabla_N u(x+\rho\mathbf{n}|\xi|,\lambda\rho|\xi|)|^p dx \\ &\qquad = 2\frac{\omega_N}{\lambda\rho}\int_{\mathbb{R}^{N+1}_+}|\nabla_N u|^p dx.\end{aligned}$$

Compute the second integral in a similar fashion and combine them into

$$\begin{aligned}|\!|\!|u(\cdot,0)|\!|\!|_{1-\frac{1}{p},p;\mathbb{R}^N} \le{}& 2^{1/p}\lambda^{-\frac{1}{p}}\|\nabla_N u\|_{p,\mathbb{R}^{N+1}_+}\int_0^1\rho^{-\frac{1}{p}}d\rho \\ &+ 2^{1/p}\lambda^{1-\frac{1}{p}}\|u_t\|_{p,\mathbb{R}^{N+1}_+}\int_0^1\rho^{-\frac{1}{p}}d\rho \\ ={}& 2^{1/p}\frac{p}{p-1}\Big(\lambda^{-\frac{1}{p}}\|\nabla_N u\|_{p,\mathbb{R}^{N+1}_+}+\lambda^{1-\frac{1}{p}}\|u_t\|_{p,\mathbb{R}^{N+1}_+}\Big).\end{aligned}$$

The proof is completed by minimizing with respect to λ. ■

Prove Theorem 8.1 by the following steps:

8.4. Proposition 8.2c shows that a function in $W^{1,p}(\mathbb{R}^{N+1}_+)$ has a trace on $\mathbb{R}^N = [x_{N+1} = 0]$ in $W^{1-\frac{1}{p},p}(\mathbb{R}^N)$. Prove the direct part of the theorem for general ∂E of class C^1 and with the segment property by a local flattening technique.

8.5. Every $v \in W^{1-\frac{1}{p},p}(\mathbb{R}^N)$ admits an extension $u \in W^{1,p}(\mathbb{R}^{N+1}_+)$ such that $v = \operatorname{tr}(u)$. To construct such an extension, assume first that v is continuous and bounded in $\mathbb{R}^N$. Let $H_v(x,t)$ be its harmonic extension in $\mathbb{R}^{N+1}_+$ constructed in Section 8 of Chapter 2, and in particular in (8.3), and set

$$u(x, x_{N+1}) = H_v(x, x_{N+1}) e^{-x_{N+1}}.$$

Verify that $u \in W^{1,p}(\mathbb{R}^{N+1}_+)$ and that $\operatorname{tr}(u) = v$. Modify the construction to remove the assumption that v is bounded and continuous in $\mathbb{R}^N$.

8.6. Prove that the Poisson kernel $K(\cdot;\cdot)$ in $\mathbb{R}^N \times \mathbb{R}^+$, constructed in (8.2) of Section 8 of Chapter 2, is not in $W^{1,p}(\mathbb{R}^N \times \mathbb{R}^+)$ for any $p \geq 1$. Argue indirectly by examining its trace on $x_{N+1} = 0$.

8.7. Prove a similar fact for the kernel in the Poisson representation of harmonic functions in a ball B_R (formula (3.9) of Section 3 of Chapter 2).

9c The Inhomogeneous Dirichlet Problem

9.1c The Lebesgue Spike

The segment property on ∂E is required to ensure an extension of φ into E by a function $v \in W^{1,2}(E)$. Whence such an extension is achieved, the structure of ∂E does not play any role. Indeed, the problem is recast into one with homogeneous Dirichlet data on ∂E whose solvability by either methods of Sections 5–7 use only the embeddings of $W^{1,2}_o(E)$ of Theorem 3.1, whose constants are independent of ∂E. Verify that the domain of Section 7.2 of Chapter 2 does not satisfy the segment property. Nevertheless the Dirichlet problem (7.3), while not admitting a classical solution, has a unique weak solution given by (7.2). Specify in what sense such a function is a weak solution.

9.2c Variational Integrals and Quasi-Linear Equations

Consider the quasi-linear Dirichlet problem

$$\begin{aligned} -\operatorname{div} \mathbf{A}(x,u,\nabla u) + B(x,u,\nabla u) &= 0 && \text{in } E \\ u\big|_{\partial E} = \varphi &\in W^{1-\frac{1}{p},p}(\partial E) && \text{on } \partial E \end{aligned} \tag{9.1c}$$

where the functions $\mathbf{A}$ and B satisfy the structure condition (6.10c). Assume moreover, that (9.1c) has a variational structure, that is, there exists a function F, as in Section6.1c, and satisfying (6.2c)–(6.3c), such that $\mathbf{A} = \nabla_{\mathbf{q}} F$

and $B = F_z$. A weak solution is a function $u \in W^{1,p}(E)$ such that $\mathrm{tr}(u) = \varphi$ and satisfying (6.11c). Introduce the set

$$K_\varphi = \big\{ u \in W^{1,p}(E) \ \text{ such that } \ \mathrm{tr}(u) = \varphi \big\} \tag{9.2c}$$

and the functional

$$K_\varphi \ni u \to J(u) = \int_E F(x, u, \nabla u) dx \qquad p > 1. \tag{9.3c}$$

Prove the following:

9.3. K_φ is convex and weakly (and hence strongly) closed. **Hint:** Use the trace inequalities (8.3)–(8.5).

9.4. Subsets of K_φ, bounded in $W^{1,p}(E)$ are weakly sequentially compact.

9.5. The functional J in (9.3c) has a minimum in K_φ. Such a minimum is a solution of (9.1c) and the latter is the Euler equation of J. **Hint:** Use Proposition 6.2c.

9.6. Solve the inhomogeneous Dirichlet problem for the more general linear operator in (5.1c).

9.7. Explain why the method of extending the boundary datum φ and recasting the problem as a homogeneous Dirichlet problem might not be applicable for quasi-linear equations of the form (9.1c). **Hint:** Examine the functionals in (6.4c) and their Euler equations (6.6c)–(6.7c).

10c The Neumann Problem

Consider the quasi-linear Neumann problem

$$\begin{aligned} -\operatorname{div} \mathbf{A}(x, u, \nabla u) + B(x, u, \nabla u) &= 0 \qquad \text{in } \ E \\ \mathbf{A}(x, u, \nabla u) \cdot \mathbf{n} &= \psi \qquad \text{on } \ \partial E \end{aligned} \tag{10.1c}$$

where $\mathbf{n}$ is the outward unit normal to ∂E and ψ satisfies (10.4). The functions $\mathbf{A}$ and B satisfy the structure condition (6.10c) and have a variational structure in the sense of Section 9.2c. Introduce the functional

$$W^{1,p}(E) \ni u \to J(u) = \int_E F(x, u, \nabla u) dx - \int_{\partial E} \psi \mathrm{tr}(u) d\sigma. \tag{10.2c}$$

In dependence of various assumptions on F, identify the correct weakly closed subspace of $W^{1,p}(E)$, where the minimization of J should be set, and find such a minimum, to coincide with a solution of (10.1c).

As a starting point, formulate sufficient conditions on the various parts of the operators in (5.1c), (6.6c), and (6.7c) that would ensure solvability of the corresponding Neumann problem. Discuss uniqueness.

11c The Eigenvalue Problem

11.1. Formulate the eigenvalue problem for homogeneous Dirichlet data as in (11.1) for the more general operator (5.1c). Formulate conditions on the coefficients for an analogue of Proposition 11.1 to hold.

11.2. Formulate the eigenvalue problem for homogeneous Neumann data. State and prove a proposition analogous to Proposition 11.1. Extend it to the more general operator (5.1c).

12c Constructing the Eigenvalues

12.1. Set up the proper variational functionals to construct the eigenvalues for homogeneous Dirichlet data for the more general operator (5.1c). Formulate conditions on the coefficients for such a variational problem to be well posed.

12.2. Set up the proper variational functionals to construct the eigenvalues for homogeneous Neumann data. Extend these variational integrals and formulate sufficient conditions to include the more general operator (5.1c).

13c The Sequence of Eigenvalues and Eigenfunctions

13.1. It might seem that the arguments of Proposition 13.2 would apply to all eigenvalues and eigenfunctions. Explain where the argument fails for the eigenvalues following the first.

13.2. Formulate facts analogous to Proposition 13.1 for the sequence of eigenvalues and eigenfunctions for homogeneous Dirichlet data for the more general operator (5.1c).

13.3. Formulate facts analogous to Proposition 13.1 for the sequence of eigenvalues and eigenfunctions for homogeneous Neumann data.

14c A Priori $L^\infty(E)$ Estimates for Solutions of the Dirichlet Problem (9.1)

The proof of Propositions 14.1–14.2 shows that the $L^\infty(E)$-estimate stems only from the recursive inequalities (15.3), and a sup-bound would hold for any function satisfying them. For these inequalities to hold the linearity of the PDE in (9.1) is immaterial. As an example, consider the quasi-linear Dirichlet problem (9.1c) where $B = 0$ and $\mathbf{A}$ is subject to the structure condition (6.10c). In particular the problem is not required to have a variational structure.

14.1. Prove that weak solutions of such a quasi-linear Dirichlet problem satisfy recursive inequalities analogous to (15.3). Prove that they are essentially bounded with an upper of the form (14.5), with $\mathbf{f} = 0$ and $f = C^p$, where C is the constant in the structure conditions (6.10c).

14.2. Prove that the boundedness of u continues to hold, if $\mathbf{A}$ and B satisfy the more general conditions

$$\begin{aligned} \mathbf{A}(x,z,\mathbf{q})\cdot\mathbf{q} &\ge \lambda|\mathbf{q}|^p - f(x) \\ |B(x,z,\mathbf{q})| &\le f(x) \end{aligned} \qquad \text{for some } \ f \in L^{\frac{N+\varepsilon}{p}}(E).$$

Prove that an upper bound for $\|u\|_\infty$ has the same form as $(14.3)_\pm$ with $\mathbf{f} = 0$ and the same value of δ.

15c A Priori $L^\infty(E)$ Estimates for Solutions of the Neumann Problem (10.1)

The estimates $(16.3)_\pm$ and (16.6) are a sole consequence of the recursive inequalities (17.3) and therefore continue to hold for weak solutions of equations from which they can be derived.

15.1. Prove that they can be derived for weak solutions of the quasi-linear Neumann problem (10.1c), where $\mathbf{A}$ and B satisfy the structure conditions (6.10c) and are not required to be variational. Prove that the estimate takes the form

$$\|u\|_\infty \le C_\sigma \max\big\{\|u\|_2\,;\ \|\psi\|_{N-1+\sigma}\,;\ |E|^{\frac{1}{N}}\big\}.$$

15.2. Prove that $L^\infty(E)$ estimates continue to hold if the constant C in the structure conditions (6.10c) is replaced by a non-negative function $f \in L^{\frac{N+\varepsilon}{p}}$ for some $\varepsilon > 0$. In such a case the estimate takes exactly the form (16.6) with $\mathbf{f} = 0$.

15.3. Establish $L^\infty(E)$ estimates for weak solutions to the Neumann problem for the operator $\mathcal{L}(\cdot)$ in (5.1c).

15.4. The estimates deteriorate if either the opening or the height of the circular spherical cone of the cone condition of ∂E tend to zero (Remark 16.4). Generate examples of such occurrences for the Laplacean in dimension $N = 2$.

15.1c Back to the Quasi-Linear Dirichlet Problem (9.1c)

The main difference between the estimates $(14.3)_\pm$ and $(16.3)_\pm$ is that the right-hand side contains the norm $\|u_\pm\|_2$ of the solution. Having the proof of Proposition 14.1 as a guideline, establish $L^\infty(E)$ bounds for solutions of the quasi-linear Dirichlet (9.1c) where $\mathbf{A}$ and B satisfy the full quasi-linear structure (6.10c), where, in addition, C may be replaced by a non-negative function $f \in L^{\frac{N+\varepsilon}{p}}$ for some $\varepsilon > 0$. Prove that the resulting estimate has the form

$$\|u\|_\infty \le \max\{\|\varphi\|_{\infty,\partial E}; C_\varepsilon[\|u\|_2; |E|^{p\delta}\|f\|_{\frac{N+\varepsilon}{p}}]\}.$$

10

DEGIORGI CLASSES

1 Quasi-Linear Equations and DeGiorgi Classes

A quasi-linear elliptic equation in an open set $E \subset \mathbb{R}^N$ is an expression of the form

$$-\operatorname{div} \mathbf{A}(x, u, \nabla u) + B(x, u, \nabla u) = 0 \tag{1.1}$$

where for $u \in W^{1,p}_{\mathrm{loc}}(E)$, the functions

$$E \ni x \to \begin{cases} \mathbf{A}\big(x, u(x), \nabla u(x)\big) \in \mathbb{R}^N \\ B\big(x, u(x), \nabla u(x)\big) \in \mathbb{R} \end{cases}$$

are measurable and satisfy the structure conditions

$$\begin{aligned} \mathbf{A}\big(x, u, \nabla u\big) \cdot \nabla u &\ge \lambda |\nabla u|^p - f^p \\ |\mathbf{A}\big(x, u, \nabla u\big)| &\le \Lambda |\nabla u|^{p-1} + f^{p-1} \\ |B\big(x, u, \nabla u\big)| &\le \Lambda_o |\nabla u|^{p-1} + f_o \end{aligned} \tag{1.2}$$

for given constants $0 < \lambda \le \Lambda$ and $\Lambda_o > 0$, and given non-negative functions

$$f \in L^{N+\varepsilon}(E), \qquad f_o \in L^{\frac{N+\varepsilon}{p}}(E), \qquad \text{for some } \ \varepsilon > 0. \tag{1.3}$$

The Dirichlet and Neumann problems for these equations were introduced in Sections 9.2c and 9c of the Complements of Chapter 9, their solvability was established for a class of functions $\mathbf{A}$ and B, and $L^\infty(E)$ bounds were derived for suitable data. Here we are interested in the local behavior of these solutions irrespective of possible prescribed boundary data. A function $u \in W^{1,p}_{\mathrm{loc}}(E)$ is a local weak sub(super)-solution of (1.1), if

$$\int_E [\mathbf{A}(x, u, \nabla u)\nabla v + B(x, u, \nabla u)v]\, dx \le (\ge) 0 \tag{1.4}$$

for all non-negative test functions $v \in W^{1,p}_o(E_o)$, for every open set E_o such that $\bar{E}_o \subset E$. A local weak solution to (1.1) is a function $u \in W^{1,p}_{\mathrm{loc}}(E)$ satisfying (1.4) with the equality sign, for all $v \in W^{1,p}_o(E_o)$. No further requirements

© Springer Nature Switzerland AG 2023

E. DiBenedetto, U. Gianazza, *Partial Differential Equations*,
Cornerstones, https://doi.org/10.1007/978-3-031-46618-2_11

are placed on $\mathbf{A}$ and B other than the structure conditions (1.2). Specific examples of these PDEs are those introduced in the previous chapter. In particular they include the class of linear equations (1.2), those in (5.1c)–(5.4c), and the nonlinear p-Laplacian-type equations in (6.6c)–(6.7c) of the Complements of Chapter 9. In all these examples the coefficients of the principal part are only measurable. Nevertheless local weak solutions of (1.1) are locally Hölder continuous in E. If $p > N$, this follows from the embedding inequality (2.4) of Theorem 2.1 of Chapter 9. If $1 < p \le N$, this follows from their membership in more general classes of functions called DeGiorgi classes, which are introduced next. Let $B_\rho(y) \subset E$ denote a ball of center y and radius ρ; if y is the origin, write $B_\rho(0) = B_\rho$. For $\sigma \in (0,1)$, consider the concentric ball $B_{\sigma\rho}(y)$ and denote by ζ a non-negative, piecewise smooth cutoff function that equals 1 on $B_{\sigma\rho}(y)$, vanishes outside $B_\rho(y)$ and such that $|\nabla\zeta| \le [(1-\sigma)\rho]^{-1}$. Let u be a local sub(super)-solution of (1.1). For $k \in \mathbb{R}$, the localized truncations $\pm\zeta^p(u-k)_\pm$ belong to $W_o^{1,p}(E)$ and can be taken as test functions v in (1.4). Using the structure conditions (1.2) yields

$$
\begin{aligned}
\lambda \int_{B_\rho(y)} & |\nabla(u-k)_\pm|^p \zeta^p dx \\
&\le \int_{B_\rho(y)} |\nabla(u-k)_\pm|^{p-1}\zeta^{p-1}(p\Lambda|\nabla\zeta| + \Lambda_o\zeta)(u-k)_\pm dx \\
&\quad + \int_{B_\rho(y)} \big\{ f^p\zeta^p\chi_{[(u-k)_\pm>0]} + pf^{p-1}\zeta^{p-1}(u-k)_\pm|\nabla\zeta|) \big\} dx \\
&\quad + \int_{B_\rho(y)} f_o(u-k)_\pm\zeta^p dx \\
&\le \frac{\lambda}{2}\int_{B_\rho(y)} |\nabla(u-k)_\pm|^p\zeta^p dx + \frac{\gamma(\Lambda,p)}{(1-\sigma)^p\rho^p}\int_{B_\rho(y)} (u-k)_\pm^p dx \\
&\quad + \int_{B_\rho(y)} f^p\chi_{[(u-k)_\pm>0]} dx + \int_{B_\rho(y)} f_o(u-k)_\pm\zeta^p dx
\end{aligned}
$$

where ρ has been taken so small that $\rho \le \max\{1;\Lambda_o\}^{-1}$. Next estimate

$$
\int_{B_\rho(y)} f^p\chi_{[(u-k)_\pm>0]} dx \le \|f\|_{N+\varepsilon}^p |A_{k,\rho}^\pm|^{1-\frac{p}{N}+p\delta}
$$

where we have assumed $1 < p \le N$, and

$$
A_{k,\rho}^\pm = [(u-k)_\pm > 0] \cap B_\rho(y) \quad \text{and} \quad \delta = \frac{\varepsilon}{N(N+\varepsilon)}. \tag{1.5}
$$

The term involving f_o is estimated by Hölder's inequality with conjugate exponents

$$
\frac{N+\varepsilon}{p} = \frac{q^*}{q^*-1}, \qquad q^* = \frac{Nq}{N-q}.
$$

Continuing to assume $1 < p \le N$, one checks that $1 < q < p < N$ for all $N \ge 2$ and the Sobolev embedding of Theorem 3.1 of Chapter 9, can be applied since $(u-k)_\pm\zeta \in W_o^{1,q}(B_\rho(y))$. Therefore

$$\begin{aligned}\int_{B_\rho(y)} f_o(u-k)_\pm\zeta^p dx &\le \|f_o\|_{\frac{N+\varepsilon}{p}}\|(u-k)_\pm\zeta\|_{q^*}\\ &\le \gamma(N,p)\|f_o\|_{\frac{N+\varepsilon}{p}}\|\nabla[(u-k)_\pm\zeta]\|_q\\ &\le \gamma(N,p)\|\nabla[(u-k)_\pm\zeta]\|_p\|f_o\|_{\frac{N+\varepsilon}{p}}|A^\pm_{k,\rho}|^{\frac{1}{q}-\frac{1}{p}}\\ &\le \frac{\lambda}{4}\int_E|\nabla(u-k)_\pm|^p\zeta^p dx + \int_E (u-k)^p_\pm|\nabla\zeta|^p dx\\ &\quad + \gamma(N,p,\lambda)\|f_o\|^{\frac{p}{p-1}}_{\frac{N+\varepsilon}{p}}|A^\pm_{k,\rho}|^{1-\frac{p}{N}+\frac{p}{p-1}p\delta}.\end{aligned}$$

Continue to assume that $\rho \le \max\{1;\Lambda_o\}^{-1}$ and combine these estimates to conclude that there exists a constant $\gamma = \gamma(N,p,\lambda,\Lambda)$ dependent only on the indicated quantities and independent of ρ, y, k, and σ such that for $1 < p \le N$

$$\begin{aligned}\|\nabla(u-k)_\pm\|^p_{p,B_{\sigma\rho}(y)} \le \frac{\gamma}{(1-\sigma)^p\rho^p}&\|(u-k)_\pm\|^p_{p,B_\rho(y)}\\ &+ \gamma_*^p|A^\pm_{k,\rho}|^{1-\frac{p}{N}+p\delta}\end{aligned} \tag{1.6}$$

where δ is given by (1.5) and

$$\gamma_*^p = \gamma(N,p)\big(\|f\|^p_{N+\varepsilon} + \|f_o\|^{\frac{p}{p-1}}_{\frac{N+\varepsilon}{p}}\big). \tag{1.7}$$

1.1 DeGiorgi Classes

Let E be an open subset of $\mathbb{R}^N$, let $p \in (1,N]$, and let γ, γ_*, and δ be given positive constants. The DeGiorgi class $\mathrm{DG}^+(E,p,\gamma,\gamma_*,\delta)$ is the collection of all functions $u \in W^{1,p}_{\mathrm{loc}}(E)$ such that $(u-k)_+$ satisfy (1.6) for all $k \in \mathbb{R}$, and for all pair of balls $B_{\sigma\rho}(y) \subset B_\rho(y) \subset E$. Local weak sub-solutions of (1.1) belong to DG^+, for the constants γ, γ_* and δ identified in (1.5)–(1.7). The DeGiorgi class $\mathrm{DG}^-(E,p,\gamma,\gamma_*,\delta)$ are defined similarly, with $(u-k)_+$ replaced by $(u-k)_-$. Local weak super-solutions of (1.1) belong to DG^-. The DeGiorgi classes $\mathrm{DG}(E,p,\gamma,\gamma_*,\delta)$ are the intersection of $\mathrm{DG}^+ \cap \mathrm{DG}^-$, or equivalently the collection of all functions $u \in W^{1,p}_{\mathrm{loc}}(E)$ satisfying (1.6) for all pair of balls $B_{\sigma\rho}(y) \subset B_\rho(y) \subset E$ and all $k \in \mathbb{R}$. We refer to these classes as homogeneous if $\gamma_* = 0$. In such a case the choice of the parameter δ is immaterial. The set of parameters $\{N,p,\gamma\}$ are the homogeneous data of the DG classes, whereas γ_* and δ are the inhomogeneous parameters. This terminology stems from the structure of (1.6) versus the structure of the quasi-linear elliptic equations in (1.1), and is evidenced by (1.7).

Functions in DG have remarkable properties, irrespective of their connection with the quasi-linear equations (1.1). In particular, they are locally bounded, and locally Hölder continuous in E. Even more striking is that non-negative functions in DG satisfy the Harnack inequality of Section 5.1 of Chapter 2, which is typical of non-negative harmonic functions.

2 Local Boundedness of Functions in the DeGiorgi Classes

We say that constants $C, \gamma, \dots$ depend only on the data, and are independent of γ_* and δ, if they can be quantitatively determined a priori only in terms of the homogeneous parameters $\{N, p, \gamma\}$. The dependence on the inhomogeneous parameters $\{\gamma_*, \delta\}$ will be traced, as a way to identify those additional properties afforded by inhomogeneous structures.

Theorem 2.1 (DeGiorgi [47]). *Let $u \in DG^{\pm}$ and $\tau \in (0,1)$. There exists a constant C depending only on the data such that for every pair of concentric balls $B_{\tau\rho}(y) \subset B_\rho(y) \subset E$*

$$\operatorname*{ess\,sup}_{B_{\tau\rho}(y)} u_\pm \le \max\left\{\gamma_* \rho^{N\delta}\,;\ \frac{C}{(1-\tau)^{\frac{1}{\delta}}}\left(\fint_{B_\rho} u_\pm^p\,dx\right)^{\frac{1}{p}}\right\}. \tag{2.1}$$

For homogeneous $DG^{\pm}$ classes, $\gamma_* = 0$ and δ can be taken $\delta = \frac{1}{N}$.

Proof. Having fixed the pair of balls $B_{\tau\rho}(y) \subset B_\rho(y) \subset E$ assume $y = 0$ and consider the sequences of nested concentric balls $\{B_n\}$ and $\{\tilde B_n\}$, and the sequences of increasing levels $\{k_n\}$

$$\begin{aligned} B_n &= B_{\rho_n}(0) \quad \text{where } \rho_n = \tau\rho + \frac{1-\tau}{2^{n-1}}\rho \\ \tilde B_n &= B_{\tilde\rho_n}(0) \quad \text{where } \tilde\rho_n = \frac{\rho_n + \rho_{n+1}}{2} = \tau\rho + \frac{3}{2}\frac{1-\tau}{2^n}\rho \\ &\qquad\qquad k_n = k - \frac{1}{2^{n-1}}k \end{aligned} \tag{2.2}$$

where $k > 0$ is to be chosen. Introduce also non-negative piecewise smooth cutoff functions

$$\zeta_n(x) = \begin{cases} 1 & \text{for } x \in B_{n+1} \\ \dfrac{\tilde\rho_n - |x|}{\tilde\rho_n - \rho_{n+1}} = \dfrac{2^{n+1}}{(1-\tau)\rho}(\tilde\rho_n - |x|) & \text{for } \rho_{n+1} \le |x| \le \tilde\rho_n \\ 0 & \text{for } |x| \ge \tilde\rho_n \end{cases} \tag{2.3}$$

for which

$$|\nabla\zeta_n| \le \frac{2^{n+1}}{(1-\tau)\rho}.$$

Write down the inequalities (1.6) for $(u - k_{n+1})_+$, for the levels k_{n+1} over the pair of balls $\tilde B_n \subset B_n$ for which $(1-\sigma) = 2^{-(n+1)}(1-\tau)$, to get

$$\begin{aligned}\|\nabla(u-k_{n+1})_+\|^p_{p,\tilde{B}_n} \le{}& \frac{2^{(n+1)p}\gamma}{(1-\tau)^p\rho^p}\|(u-k_{n+1})_+\|^p_{p,B_n}\\ &+\gamma_*^p|A^+_{k_{n+1},\rho_n}|^{1-\frac{p}{N}+p\delta}.\end{aligned}$$

In the arguments below, γ is a positive constant depending only on the data and that might be different in different contexts.

2.1 Proof of Theorem 2.1 for $1 < p < N$

Apply the embedding inequality (3.1) of Theorem 3.1 of Chapter 9 to the functions $(u-k_{n+1})_+\zeta_n$ over the balls $\tilde{B}_n$ to get

$$\begin{aligned}\|(u-k_{n+1})_+\|^p_{p,B_{n+1}} &\le \|(u-k_{n+1})_+\zeta_n\|^p_{p,\tilde{B}_n}\\ &\le \|(u-k_{n+1})_+\zeta_n\|^{\frac{p}{p^*}}_{p^*,\tilde{B}_n}|A^+_{k_{n+1},\tilde{\rho}_n}|^{\frac{p}{N}}\\ &\le \gamma\|\nabla[(u-k_{n+1})_+\zeta_n]\|^p_{p,\tilde{B}_n}|A^+_{k_{n+1},\tilde{\rho}_n}|^{\frac{p}{N}}\\ &\le \gamma\bigg(\frac{2^{pn}}{(1-\tau)^p\rho^p}\|(u-k_{n+1})_+\|^p_{p,B_n}\\ &\qquad+\gamma_*^p|A^+_{k_{n+1},\rho_n}|^{1-\frac{p}{N}+p\delta}\bigg)|A^+_{k_{n+1},\rho_n}|^{\frac{p}{N}}.\end{aligned}\tag{2.4}$$

Next

$$\begin{aligned}\|(u-k_n)_+\|^p_{p,B_n} &= \int_{B_n}(u-k_n)^p_+dx \ge \int_{B_n\cap[u>k_{n+1}]}(u-k_n)^p_+dx\\ &\ge \int_{B_n\cap[u>k_{n+1}]}(k_{n+1}-k_n)^p dx \ge \frac{k^p}{2^{np}}|A^+_{k_{n+1},\rho_n}|.\end{aligned}\tag{2.5}$$

Therefore

$$|A^+_{k_{n+1},\rho_n}| \le \frac{2^{np}}{k^p}\|(u-k_n)_+\|^p_{p,B_n}.\tag{2.6}$$

Combining these estimates yields

$$\begin{aligned}\|(u-k_{n+1})_+\|^p_{p,B_{n+1}} \le{}& \gamma\frac{2^{np\frac{N+p}{N}}}{(1-\tau)^p\rho^p}\frac{1}{k^{p\frac{p}{N}}}\|(u-k_n)_+\|^{p(1+\frac{p}{N})}_{p,B_n}\\ &+\gamma\gamma_*^p\frac{2^{np(1+p\delta)}}{k^{p(1+p\delta)}}\|(u-k_n)_+\|^{p(1+p\delta)}_{p,B_n}.\end{aligned}\tag{2.7}$$

Set

$$Y_n = \frac{1}{k^p}\fint_{B_n}(u-k_n)^p_+dx = \frac{\|(u-k_n)_+\|^p_{p,B_n}}{k^p\,|B_n|},\qquad b = 2^{\frac{N+p}{N}}$$

and rewrite the previous recursive inequalities as

$$Y_{n+1} \le \frac{\gamma b^{pn}}{(1-\tau)^p}\left(Y_n^{1+\frac{p}{N}} + \gamma_*^p\frac{\rho^{Np\delta}}{k^p}Y_n^{1+p\delta}\right).\tag{2.8}$$

Stipulate to take k so large that

$$k \ge \gamma_* \rho^{N\delta}, \qquad k > \left(\fint_{B_\rho} u_+^p \, dx \right)^{\frac{1}{p}}. \tag{2.9}$$

Then $Y_n \le 1$ for all n and $Y_n^{\frac{p}{N}} \le Y_n^{p\delta}$. With these remarks and stipulations, the previous recursive inequalities take the form

$$Y_{n+1} \le \frac{\gamma b^{pn}}{(1-\tau)^p} Y_n^{1+p\delta} \quad \text{for all } \; n = 1, 2, \dots \tag{2.10}$$

From the fast geometric convergence Lemma 15.1 of Chapter 9, it follows that $\{Y_n\} \to 0$ as $n \to \infty$, provided

$$Y_1 = \frac{1}{k^p} \fint_{B_\rho} u_+^p \, dx \le b^{-\frac{1}{p\delta^2}} \gamma^{-\frac{1}{p\delta}} (1-\tau)^{\frac{1}{\delta}}.$$

Therefore, taking also into account (2.9), choosing

$$k = \max \left\{ \gamma_* \rho^{N\delta} \, ; \, \frac{b^{\frac{1}{(p\delta)^2}} \gamma^{\frac{1}{p^2\delta}}}{(1-\tau)^{\frac{1}{p\delta}}} \left(\fint_{B_\rho} u_+^p \, dx \right)^{\frac{1}{p}} \right\}$$

one derives

$$Y_\infty = \frac{1}{k^p} \fint_{B_{\tau\rho}} (u-k)_+^p \, dx = 0 \quad \Longrightarrow \quad \operatorname*{ess\,sup}_{B_{\tau\rho}} u_+ \le k.$$

If $\gamma_* = 0$, then (2.8) are already in the form (2.10) with $\delta = \frac{1}{N}$. ■

2.2 Proof of Theorem 2.1 for $p = N$

The main difference occurs in the application of the embedding inequality (3.2) of Theorem 3.1 of Chapter 9 to the functions $(u-k_{n+1})_+\zeta_n$ over the balls $\tilde{B}_n$ to derive inequalities analogous to (2.4). Let $q > N$ to be chosen and estimate

$$\begin{aligned}
&\|(u-k_{n+1})_+\|_{p,B_{n+1}}^p \le \|(u-k_{n+1})_+\zeta_n\|_{p,\tilde{B}_n}^p \\
&\le \|(u-k_{n+1})_+\zeta_n\|_{q,\tilde{B}_n}^{\frac{p}{q}} |A_{k_{n+1},\tilde{\rho}_n}^+|^{1-\frac{p}{q}} \\
&\le \gamma(N,q) \left(\|\nabla[(u-k_{n+1})_+\zeta_n]\|_{p,\tilde{B}_n}^{1-\frac{p}{q}} \|(u-k_{n+1})_+\zeta\|_{p,\tilde{B}_n}^{\frac{p}{q}} \right)^p |A_{k_{n+1},\tilde{\rho}_n}^+|^{1-\frac{p}{q}} \\
&\le \gamma(N,q) \left(\frac{2^{pn}}{(1-\tau)^p \rho^p} \|(u-k_{n+1})_+\|_{p,B_n}^p + \gamma_* |A_{k_{n+1},\rho_n}^+|^{p\delta} \right) |A_{k_{n+1},\rho_n}^+|^{1-\frac{p}{q}}.
\end{aligned}$$

Choose $q = 2/\delta$, estimate $|A_{k_{n+1},\rho_n}^+|$ as in (2.5)–(2.6), and arrive at the analogues of (2.7), which now take the form

$$\begin{aligned}\|(u-k_{n+1})_+\|^p_{p,B_{n+1}} \le \gamma \frac{2^{np(2-\frac{p}{2}\delta)}}{(1-\tau)^p\rho^p}\frac{1}{k^{p(2-\frac{p}{2}\delta)}}\|(u-k_n)_+\|^{p(2-\frac{p}{2}\delta)}_{p,B_n}\\ + \gamma\gamma_*\frac{2^{np(1+\frac{p}{2}\delta)}}{k^{p(1+\frac{p}{2}\delta)}}\|(u-k_n)_+\|^{p(1+\frac{p}{2}\delta)}_{p,B_n}.\end{aligned}$$

Set

$$Y_n = \frac{1}{k^p}\fint_{B_n}(u-k_n)^p_+ dx \qquad \text{and} \qquad b = 2^{2-\frac{p}{2}\delta}$$

and rewrite the previous recursive inequalities as

$$Y_{n+1} \le \frac{\gamma b^{pn}}{(1-\tau)^p}\left(Y_n^{1+\frac{p}{2}\delta+(1-\frac{p}{2}\delta)} + \gamma_*^p\frac{\rho^{N\frac{p}{2}\delta}}{k^p}Y_n^{1+\frac{p}{2}\delta}\right).$$

Stipulate to take k as in (2.9) with δ replaced by $\frac{1}{2}\delta$, and recast these recursive inequalities in the form (2.10) with δ replaced by $\frac{1}{2}\delta$. ■

3 Hölder Continuity of Functions in the DG Classes

For a function $u \in \mathrm{DG}(E, p, \gamma, \gamma_*, \delta)$ and $B_{2\rho}(y) \subset E$ set

$$\mu^+ = \operatorname*{ess\,sup}_{B_{2\rho}(y)} u, \quad \mu^- = \operatorname*{ess\,inf}_{B_{2\rho}(y)} u, \quad \omega(2\rho) = \mu^+ - \mu^- = \operatorname*{ess\,osc}_{B_{2\rho}(y)} u. \tag{3.1}$$

These quantities are well defined since $u \in L^\infty_{\mathrm{loc}}(E)$.

Theorem 3.1 (DeGiorgi [47]). *Let $u \in DG(E, p, \gamma, \gamma_*, \delta)$. There exist constants $C > 1$ and $\alpha \in (0,1)$ depending only upon the data and independent of u, such that for every pair of balls $B_\rho(y) \subset B_R(y) \subset E$*

$$\omega(\rho) \le C \max\Big\{\omega(R)\left(\frac{\rho}{R}\right)^\alpha ; \gamma_*\rho^{N\delta}\Big\}. \tag{3.2}$$

The Hölder continuity is local to E, with Hölder exponent $\alpha_o = \min\{\alpha; N\delta\}$. An upper bound for the Hölder constant is

$$\{\text{Hölder constant}\} \le C\max\{2MR^{-\alpha}; \gamma_*\}, \qquad \text{where} \qquad M = \|u\|_\infty.$$

This implies that the local Hölder estimates deteriorate near ∂E. Indeed, fix $x, y \in E$ and let

$$R = \min\{\mathrm{dist}\{x; \partial E\}\,;\, \mathrm{dist}\{y; \partial E\}\}.$$

If $|x-y| < R$, then (3.2) implies

$$|u(x) - u(y)| \le C\max\{\omega(R)R^{-\alpha_o}; \gamma_*\}|x-y|^{\alpha_o}.$$

If $|x-y| \ge R$, then

$$|u(x) - u(y)| \le 2MR^{-\alpha_o}|x-y|^{\alpha_o}.$$

Corollary 3.1 *Let u be a local weak solution of (1.1)–(1.4). Then for every compact subset $K \subset E$, and for every pair $x, y \in K$*

$$|u(x) - u(y)| \le C \max\left\{\frac{2M_K}{\operatorname{dist}\{K; \partial E\}^\alpha}; \gamma_*\right\} |x-y|^{\alpha_o}$$

where $M_K = \operatorname{ess\,sup}_K |u|$.

3.1 On the Proof of Theorem 3.1

Although the parameters δ and p are fixed, in view of the value of δ in (1.5), which naturally arises from quasi-linear equations, we will assume $\delta \le \frac{1}{N}$. The value $\delta = \frac{1}{N}$ would occur if $\varepsilon \to \infty$ in the integrability requirements (1.3). For homogeneous DG classes $\gamma_* = 0$, while immaterial, we take $\delta = 1/N$. The proof will be carried on for $1 < p < N$. The case $p = N$ only differs in the application of the embedding Theorem 3.1 of Chapter 9, and the minor modifications needed to cover this case can be modeled after almost identical arguments in Section 2.2 above. In what follows we assume that $u \in \mathrm{DG}$ is given, the ball $B_{2\rho}(y) \subset E$ is fixed, $\mu^\pm$ and $\omega(2\rho)$ are defined as in (3.1), and denote by ω any number larger than $\omega(2\rho)$.

4 Estimating the Values of u by the Measure of the Set Where u Is Either Near μ^+ or Near μ^-

Proposition 4.1 *For every $a \in (0,1)$, there exists $\nu \in (0,1)$ depending only on the data and a, but independent of ω, such that if for some $\varepsilon \in (0,1)$*

$$\big|[u > \mu^+ - \varepsilon\omega] \cap B_\rho(y)\big| \le \nu |B_\rho| \tag{4.1$_+$}$$

then either $\varepsilon\omega \le \gamma_* \rho^{N\delta}$ or

$$u \le \mu^+ - a\varepsilon\omega \qquad \text{a.e. in } B_{\frac{1}{2}\rho}(y). \tag{4.2$_+$}$$

Similarly, if

$$\big|[u < \mu^- + \varepsilon\omega] \cap B_\rho(y)\big| \le \nu |B_\rho| \tag{4.1$_-$}$$

then either $\varepsilon\omega \le \gamma_* \rho^{N\delta}$ or

$$u \ge \mu^- + a\varepsilon\omega \qquad \text{a.e. in } B_{\frac{1}{2}\rho}(y). \tag{4.2$_-$}$$

Proof. We prove only $(4.1)_+$–$(4.2)_+$, the arguments for $(4.1)_-$–$(4.2)_-$ being analogous. Set $y = 0$ and consider the sequence of balls $\{B_n\}$ and $\{\tilde{B}_n\}$ introduced in (2.2) for $\tau = \frac{1}{2}$ and the cutoff functions ζ_n introduced in (2.3). For $n \in \mathbb{N}$, introduce also the increasing levels $\{k_n\}$, the nested sets $\{A_n\}$, and their relative measure $\{Y_n\}$ by

$$k_n = \mu^+ - a\varepsilon\omega - \frac{1-a}{2^n}\varepsilon\omega, \quad A_n = [u > k_n] \cap B_n, \quad Y_n = \frac{|A_n|}{|B_n|}.$$

Apply (1.6) to $(u - k_n)_+$ over the pair of concentric balls $\tilde{B}_n \subset B_n$, for which $(1-\sigma) = 2^{-n}$, to get

$$\|\nabla(u - k_n)_+\|^p_{p,\tilde{B}_n} \le \frac{\gamma 2^{np}}{\rho^p}\|(u-k_n)_+\|^p_{p,B_n} + \gamma_*^p |A_n|^{1-\frac{p}{N}+p\delta}.$$

If $1 < p < N$, by the embedding (3.1) of Theorem 3.1 of Chapter 9

$$\begin{aligned}\left[\frac{(1-a)\varepsilon\omega}{2^{n+1}}\right]^p |A_{n+1}| &= (k_{n+1} - k_n)^p |A_{n+1}| \le \|(u-k_n)_+\zeta_n\|^p_{p,\tilde{B}_n} \\ &\le \|(u-k_n)_+\zeta_n\|^p_{p^*,\tilde{B}_n} |A_n|^{\frac{p}{N}} \le \|\nabla[(u-k_n)_+\zeta_n]\|^p_{p,\tilde{B}_n}|A_n|^{\frac{p}{N}} \\ &\le \left(\frac{\gamma 2^{np}}{\rho^p}\|(u-k_n)_+\|^p_{p,B_n} + \gamma\gamma_*^p |A_n|^{1-\frac{p}{N}+p\delta}\right)|A_n|^{\frac{p}{N}} \\ &\le \frac{\gamma 2^{np}}{\rho^p}\left(\frac{\varepsilon\omega}{2^n}\right)^p |A_n|^{1+\frac{p}{N}} + \gamma\gamma_*^p |A_n|^{1+p\delta}.\end{aligned}$$

From this, in dimensionless form, in terms of Y_n one derives

$$Y_{n+1} \le \frac{\gamma 2^{np}}{(1-a)^p}\left[Y_n^{1+\frac{p}{N}} + \left(\frac{\gamma_*\rho^{N\delta}}{\varepsilon\omega}\right)^p Y_n^{1+p\delta}\right] \le \frac{\gamma 2^{np}}{(1-a)^p} Y_n^{1+p\delta}$$

provided $\varepsilon\omega > \gamma_*\rho^{N\delta}$. It follows from these recursive inequalities that $\{Y_n\} \to 0$ as $n \to \infty$, provided (Lemma 15.1 of Chapter 9)

$$Y_1 = \frac{\big|[u > \mu^+ - \varepsilon\omega] \cap B_\rho\big|}{|B_\rho|} \le \frac{(1-a)^{1/\delta}}{\gamma^{1/p\delta}2^{1/p\delta^2}} \overset{\text{def}}{=} \nu. \tag{4.3}$$

Remark 4.1 This formula provides a precise dependence of ν on a and the data. In particular, ν is independent of ε.

5 Reducing the Measure of the Set Where u is Either Near μ^+ or Near μ^-

Proposition 5.1 *Assume that*

$$\big|[u \le \mu^+ - \tfrac{1}{2}\omega] \cap B_\rho\big| \ge \theta|B_\rho| \qquad (5.1)_+$$

for some $\theta \in (0,1)$. Then for every $\nu \in (0,1)$ there exists $\varepsilon \in (0,1)$ that can be determined a priori only in terms of the data and θ, and independent of ω, such that either $\varepsilon\omega \le \gamma_*\rho^{N\delta}$ or

$$\big|[u > \mu^+ - \varepsilon\omega] \cap B_\rho\big| \le \nu\big|B_\rho\big|. \qquad (5.2)_+$$

Similarly, if

$$\big|[u \ge \mu^- + \tfrac{1}{2}\omega] \cap B_\rho\big| \ge \theta |B_\rho| \tag{5.1$_-$}$$

for some $\theta \in (0,1)$*, then for every* $\nu \in (0,1)$ *there exists* $\varepsilon \in (0,1)$ *depending only on the data and* θ*, and independent of* ω*, such that either* $\varepsilon\omega \le \gamma_* \rho^{N\delta}$ *or*

$$\big|[u < \mu^- + \varepsilon\omega] \cap B_\rho\big| \le \nu \big|B_\rho\big|. \tag{5.2$_-$}$$

5.1 The Discrete Isoperimetric Inequality

Proposition 5.2 *Let* E *be a bounded convex open set in* $\mathbb{R}^N$*, let* $u \in W^{1,1}(E)$*, and assume that* $|[u=0]| > 0$*. Then*

$$\|u\|_1 \le \gamma(N) \frac{(\operatorname{diam} E)^{N+1}}{|[u=0]|} \|\nabla u\|_1. \tag{5.3}$$

Proof. For almost all $x \in E$ and almost all $y \in [u=0]$

$$|u(x)| = \Big| \int_0^{|y-x|} \frac{\partial}{\partial \rho} u(x + \mathbf{n}\rho) d\rho \Big| \le \int_0^{|y-x|} |\nabla u(x + \mathbf{n}\rho)| d\rho, \quad \mathbf{n} = \frac{x-y}{|x-y|}.$$

Integrating in dx over E and in dy over $[u=0]$ gives

$$|[u=0]|\, \|u\|_1 \le \int_E \Big\{ \int_{[u=0]} \int_0^{|y-x|} |\nabla u(x + \mathbf{n}\rho)| d\rho dy \Big\} dx.$$

The integral over $[u=0]$ is computed by introducing polar coordinates with center at x. Denoting by $R(x,y)$ the distance from x to ∂E along $\mathbf{n}$

$$\begin{aligned} &\int_{[u=0]} \int_0^{|y-x|} |\nabla u(x + \mathbf{n}\rho)| d\rho dy \\ &\qquad \le \int_0^{R(x,y)} s^{N-1} ds \int_{|\mathbf{n}|=1} \int_0^{R(x,y)} |\nabla u(x + \mathbf{n}\rho)| d\rho d\mathbf{n}. \end{aligned}$$

Combining these remarks, we arrive at

$$|[u=0]|\, \|u\|_1 \le \frac{1}{N} (\operatorname{diam} E)^N \int_E \int_E \frac{|\nabla u(y)|}{|x-y|^{N-1}} dy dx.$$

Inequality (5.3) follows from this, since

$$\sup_{y \in E} \int_E \frac{dx}{|x-y|^{N-1}} \le \omega_N \operatorname{diam} E. \qquad \blacksquare$$

For a real number ℓ and $u \in W^{1,1}(E)$, set

$$u_\ell = \begin{cases} \ell & \text{if } u > \ell \\ u & \text{if } u \le \ell. \end{cases}$$

Apply (5.3) to the function $(u_\ell - k)_+$ for $k < \ell$ to obtain

$$(\ell - k)|[u > \ell]| \le \gamma(N)\frac{(\operatorname{diam} E)^{N+1}}{|[u < k]|}\int_{[k<u<\ell]} |\nabla u| dx. \tag{5.4}$$

This is referred to as a discrete version of the isoperimetric inequality ([47]). A continuous version is in [77].

5.2 Proof of Proposition 5.1

We will establish $(5.2)_+$ starting from $(5.1)_+$. Set

$$k_s = \mu^+ - \frac{1}{2^s}\omega, \quad A_s = [u > k_s] \cap B_\rho, \quad \text{for } s = 1, 2, \dots, s_* \tag{5.5}$$

where s_* is a positive integer to be chosen. Apply (5.4) for the levels $k_s < k_{s+1}$ over the ball B_ρ. By virtue of $(5.1)_+$

$$\big|[u < k_s] \cap B_\rho\big| \ge \theta|B_\rho| \quad \text{for all } s \in \mathbb{N}.$$

Therefore

$$\begin{aligned}
\frac{\omega}{2^{s+1}}|A_{s+1}| &\le \frac{\gamma(N)}{\theta}\rho\int_{A_s - A_{s+1}} |\nabla u| dx \\
&\le \frac{\gamma(N)}{\theta}\rho\bigg(\int_{B_\rho} |\nabla(u - k_s)_+|^p dx\bigg)^{\frac{1}{p}} \big|A_s - A_{s+1}\big|^{\frac{p-1}{p}}
\end{aligned}$$

Take the p-power of both sides and estimate the term involving $\nabla(u - k_s)_+$ by making use of the DG classes (1.6) over the pair of balls $B_\rho \subset B_{2\rho}$ for which $(1 - \sigma) = \frac{1}{2}$. This gives

$$\begin{aligned}
\frac{\omega^p}{2^{sp}}|A_{s+1}|^p &\le \frac{\gamma^p\rho^p}{\theta^p}\bigg(\frac{\|(u - k_s)_+\|^p_{p,B_{2\rho}}}{\rho^p} + \gamma_*^p\rho^{N-p+Np\delta}\bigg)\big|A_s - A_{s+1}\big|^{p-1} \\
&\le \frac{\gamma^p\rho^N}{\theta^p}\frac{\omega^p}{2^{sp}}\bigg[1 + \bigg(\frac{2^s}{\omega}\gamma_*\rho^{N\delta}\bigg)^p\bigg]\big|A_s - A_{s+1}\big|^{p-1}.
\end{aligned}$$

Let $\varepsilon = 2^{-(s_*+1)}$ and stipulate that the term in $[\cdots]$ is majorized by 2. Then, after we divide through by $(\omega/2^s)^p$ and take the $\frac{1}{p-1}$ power of both sides, this inequality yields

$$|A_{s+1}|^{\frac{p}{p-1}} \le \Big(\frac{\gamma}{\theta}\Big)^{\frac{p}{p-1}}\rho^{\frac{N}{p-1}}\big|A_s - A_{s+1}\big|.$$

Add both sides over $s = 1, \dots, s_*$ and observe that the sum on the right-hand side can be majorized by a telescopic series, which in turn is majorized by $|B_\rho|$. On the left-hand side the sum is carried over the constant minorizing term $|A_{s_*+1}|$. Thus

$$\begin{aligned}s_*|A_{s_*+1}|^{\frac{p}{p-1}} &\le \sum_{s=1}^{s_*}|A_s|^{\frac{p}{p-1}}\\ &\le \left(\frac{\gamma}{\theta}\right)^{\frac{p}{p-1}}\rho^{\frac{N}{p-1}}\sum_{s=1}^{\infty}\left|A_s-A_{s+1}\right| \le \left(\frac{\gamma}{\theta}\right)^{\frac{p}{p-1}}|B_\rho|^{\frac{p}{p-1}}.\end{aligned}$$

From this

$$|A_{s_*}| \le \frac{1}{s_*^{\frac{p-1}{p}}}\frac{\gamma}{\theta}|B_\rho| \overset{\text{def}}{=} \nu|B_\rho|. \qquad \blacksquare$$

6 Proof of Theorem 3.1

Consider the assumption $(5.1)_\pm$ with $\theta = \frac{1}{2}$. Since $\omega \ge \omega(2\rho)$, by the definitions (3.1)

$$\left([u \le \mu^+ - \tfrac{1}{2}\omega] \cap B_\rho\right) \bigcup \left([u \ge \mu^- + \tfrac{1}{2}\omega] \cap B_\rho\right) \supset B_\rho.$$

Therefore not both of $(5.1)_\pm$ can be violated. Assuming the first is in force, fix the number ν as the one claimed by Proposition 4.1 for the choice $a = \frac{1}{2}$, and then, such a number being fixed, determine s_* and hence $\varepsilon = 2^{-(s_*+1)}$ by the procedure of Proposition 5.1. Then by Proposition 4.1, either $\varepsilon\omega \le \gamma\rho^{N\delta}$, or $(4.2)_+$ holds. The latter implies

$$\operatorname*{ess\,sup}_{B_{\frac{1}{2}\rho}} u \le \operatorname*{ess\,sup}_{B_{2\rho}} - \tfrac{1}{2}\varepsilon \operatorname*{ess\,osc}_{B_{2\rho}} u. \tag{6.1}$$

Now

$$-\operatorname*{ess\,inf}_{B_{\frac{1}{2}\rho}} u \le -\operatorname*{ess\,inf}_{B_{2\rho}} u.$$

Adding these inequalities gives

$$\omega(\tfrac{1}{2}\rho) \le \eta\omega(2\rho), \qquad \text{where } \eta = 1 - \frac{1}{2}\varepsilon. \tag{6.2}$$

Let $B_R(y) \subset E$ be fixed and set $\rho_n = 4^{-n}R$. The previous remarks imply that

$$\omega(\rho_{n+1}) \le \max\{\eta\omega(\rho_n)\,;\, \varepsilon^{-1}\gamma_*\rho_n^{N\delta}\} \tag{6.3}$$

and by iteration

$$\omega(\rho_{n+1}) \le \max\{\eta^n\omega(R)\,;\, \varepsilon^{-1}\gamma_*\rho_n^{N\delta}\}.$$

Compute

$$\rho_n = 4^{-n}R \implies -n = \ln\left(\frac{\rho_n}{R}\right)^{\frac{1}{\ln 4}} \implies \eta^n = \left(\frac{\rho_n}{R}\right)^\alpha \quad \text{for } \alpha = -\frac{\ln\eta}{\ln 4}. \qquad \blacksquare$$

7 Boundary DeGiorgi Classes: Dirichlet Data

Let ∂E be the finite union of portions of $(N-1)$-dimensional surfaces of class C^1, so that the trace of a function $u \in W^{1,p}(E)$ can be defined except possibly on an $(N-2)$-dimensional subset of ∂E. Given $\varphi \in W^{1-\frac{1}{p},p}(\partial E)$, the Dirichlet problem for the quasi-linear equation (1.1) consists in finding $u \in W^{1,p}(E)$ such that $\operatorname{tr}(u) = \varphi$ and u satisfies the PDE in the weak form (1.4), with the equality sign, for all $v \in W_o^{1,p}(E)$. Weak sub(super)-solutions of the Dirichlet problem are functions $u \in W^{1,p}(E)$ with $\operatorname{tr}(u) \le (\ge)\varphi$ and satisfying (1.4) for all non-negative $v \in W_o^{1,p}(E)$. If $\varphi \in C(\partial E)$, it is natural to ask whether a solution of the Dirichlet problem, whenever it exists, is continuous up the boundary ∂E. The issue builds on the Lebesgue counterexample of Section 7.1 of Chapter 2, and can be rephrased by asking what requirements are needed on ∂E for the interior continuity of functions in the DG classes to extend up to ∂E. Assume that ∂E satisfies the property of *positive geometric density*, that is, there exist $\beta \in (0,1)$ and $R > 0$ such that for all $y \in \partial E$

$$\big|B_\rho(y) \cap (\mathbb{R}^N - E)\big| \ge \beta \big|B_\rho\big| \quad \text{for all } 0 < \rho \le R. \tag{7.1}$$

Fix $y \in \partial E$, assume up to a possible translation that it coincides with the origin, and consider nested concentric balls $B_{\sigma\rho} \subset B_\rho$ for some $\rho > 0$ and $\sigma \in (0,1)$. Let $\varphi \in C(\partial E)$ and set

$$\begin{aligned} \varphi^+(\rho) &= \sup_{\partial E \cap B_\rho} \varphi, \qquad \varphi^-(\rho) = \inf_{\partial E \cap B_\rho} \varphi \\ \omega_\varphi(\rho) &= \varphi^+(\rho) - \varphi^-(\rho) = \operatorname*{osc}_{\partial E \cap B_\rho} \varphi. \end{aligned} \tag{7.2}$$

Let ζ be a non-negative, piecewise smooth cutoff function, that equals 1 on $B_{\sigma\rho}(y)$, vanishes outside $B_\rho(y)$, and such that $|\nabla \zeta| \le [(1-\sigma)\rho]^{-1}$, and let u be a local sub(super)-solution of the Dirichlet problem associated to (1.1) for the given φ. In the weak formulation (1.4), take as test functions v, the localized truncations $\pm\zeta^p(u-k)_\pm$. While ζ vanishes on ∂B_ρ, it does not vanish of $\partial E \cap B_\rho$; however

$$\begin{aligned} &\zeta^p(u-k)_+ \ \text{ is admissible if } \ k \ge \varphi^+(\rho) \\ &\zeta^p(u-k)_- \ \text{ is admissible if } \ k \le \varphi^-(\rho). \end{aligned} \tag{7.3}$$

Putting these choices in (1.4), all the calculations and estimates of Section 1 can be reproduced verbatim, with the understanding that the various integrals are now extended over $B_\rho \cap E$. However, since $\zeta^p(u-k)_\pm \in W_o^{1,p}(B_\rho \cap E)$, we may regard them as elements of $W_o^{1,p}(B_\rho)$ by defining them to be zero outside E. Then the same calculations lead to the inequalities (1.6), with the same stipulations that the various functions vanish outside E and the various integrals are extended over the full ball B_ρ. Given $\varphi \in C(\partial E)$, the boundary DeGiorgi classes $\mathrm{DG}_\varphi^\pm = \mathrm{DG}_\varphi^\pm(\partial E, p, \gamma, \gamma_*, \delta)$ are the collection of all $u \in W^{1,p}(E)$ such that for all $y \in \partial E$ and all pairs of balls $B_{\sigma\rho}(y) \subset B_\rho(y)$ the localized truncations $(u-k)_\pm$ satisfy (1.6) for all levels k subject to the restrictions (7.3). We further define $\mathrm{DG}_\varphi = \mathrm{DG}_\varphi^+ \cap \mathrm{DG}_\varphi^-$ and refer to these classes as homogeneous if $\gamma_* = 0$.

7.1 Continuity up to ∂E of Functions in the Boundary DG Classes (Dirichlet Data)

Let R be the parameter in the condition of positive geometric density (7.1). For $y \in \partial E$ consider concentric balls $B_\rho(y) \subset B_{2\rho}(y) \subset B_R(y)$ and set

$$\begin{aligned} \mu^+ &= \operatorname*{ess\,sup}_{B_{2\rho(y)}\cap E} u, \qquad \mu^- = \operatorname*{ess\,inf}_{B_{2\rho}(y)\cap E} u \\ \omega(2\rho) &= \mu^+ - \mu^- = \operatorname*{ess\,osc}_{B_{2\rho}(y)\cap E} u. \end{aligned} \tag{7.4}$$

Let also $\omega_\varphi(2\rho)$ be defined as in (7.2).

Theorem 7.1. *Let ∂E satisfy the condition of positive geometric density (7.1), and let $\varphi \in C(\partial E)$. Then every $u \in DG_\varphi$ is continuous up to ∂E, and there exist constants $C > 1$ and $\alpha \in (0,1)$, depending only on the data defining the DG_φ classes and the parameter β in (7.1), and independent of φ and u, such that for all $y \in \partial E$ and all balls $B_\rho(y) \subset B_R(y)$*

$$\omega(\rho) \le C \max\left\{\omega(R)\left(\frac{\rho}{R}\right)^\alpha ; \omega_\varphi(2\rho) ; \gamma_* \rho^{N\delta}\right\}. \tag{7.5}$$

The proof of this theorem is almost identical to that of the interior Hölder continuity, except for a few changes, which we outline next. First, Proposition 4.1 and its proof continue to hold, provided the levels $\varepsilon\omega$ satisfy (7.3). Next, Proposition 5.1 and its proof continue to be in force, provided the levels k_s in (5.5) satisfy the restriction (7.3) for all $s \ge 1$. Now either one of the inequalities

$$\mu^+ - \tfrac14\omega \ge \varphi^+, \qquad \mu^- + \tfrac14\omega \le \varphi^-$$

must be satisfied. Indeed, if both are violated

$$\mu^+ - \tfrac14\omega \le \varphi^+ \quad \text{and} \quad -\mu^- - \tfrac14\omega \le -\varphi^-.$$

Adding these inequalities gives

$$\omega(2\rho) \le 2\omega_\varphi(2\rho)$$

and there is nothing to prove. Assuming the first holds, then all levels k_s as defined in (5.5) satisfy the first of the restrictions (7.3) for $s \ge 2$ and thus are admissible. Moreover, $(u-k_2)_+$ vanishes outside E, and therefore

$$\left|[u \le \mu^+ - \tfrac14\omega] \cap B_\rho\right| \ge \beta|B_\rho|$$

where β is the parameter in the positive geometric density condition (7.1). From this, the procedure of Proposition 5.1 can be repeated with the understanding that $(u - k_s)_+$ are defined in the full ball B_ρ and are zero outside E. Proposition 5.1 now guarantees the existence of ε as in $(5.2)_+$ and then Proposition 4.1 ensures that (6.1) holds. ■

Remark 7.1 If φ is Hölder continuous, then u is Hölder continuous up to ∂E.

Remark 7.2 The arguments are local in nature and as such they require only local assumptions. For example, the positive geometric density (7.1) could be satisfied on only a portion of ∂E, open in the relative topology of ∂E, and φ could be continuous only on that portion of ∂E. Then the boundary continuity of Theorem 7.1 continues to hold only locally, on that portion of ∂E.

Corollary 7.1 *Let ∂E satisfy (7.1). A solution u of the Dirichlet problem for (1.1) for a datum $\varphi \in C(\partial E)$ is continuous in $\bar{E}$. If φ is Hölder continuous in ∂E, then u is Hölder continuous in $\bar{E}$. Analogous statements hold if ∂E satisfies (7.1) on an open portion of ∂E and if φ is continuous (Hölder continuous) on that portion of ∂E.*

8 Boundary DeGiorgi Classes: Neumann Data

Consider the quasi-linear Neumann problem

$$\begin{aligned} -\operatorname{div}\mathbf{A}(x,u,\nabla u) + B(x,u,\nabla u) &= 0 \qquad \text{in } E \\ \mathbf{A}(x,u,\nabla u)\cdot\mathbf{n} &= \psi \qquad \text{on } \partial E \end{aligned} \tag{8.1}$$

where $\mathbf{n}$ is the outward unit normal to ∂E. The functions $\mathbf{A}$ and B satisfy the structure (1.2), and the Neumann datum ψ satisfies

$$\psi \in L^q(\partial E), \qquad \text{where} \qquad \begin{cases} q = \dfrac{p}{p-1}\dfrac{N-1}{N} & \text{if } 1 < p < N \\ \text{any } q > 1 & \text{if } p = N. \end{cases} \tag{8.2}$$

A weak sub(super)-solution to (8.1) is a function $u \in W^{1,p}(E)$ such that

$$\int_E [\mathbf{A}(x,u,\nabla u)\nabla v + B(x,u,\nabla u)v]\,dx \le (\ge) \int_{\partial E} \psi v\,d\sigma \tag{8.3}$$

for all non-negative $v \in W^{1,p}(E)$, where $d\sigma$ is the surface measure on ∂E. All terms on the left-hand side are well defined by virtue of the structure conditions (1.2), whereas the boundary integral on the right-hand side is well defined by virtue of the trace inequalities of Proposition 8.2 of Chapter 9.

In defining boundary DG classes for the Neumann data ψ, fix $y \in \partial E$, assume without loss of generality that $y = 0$, and introduce a local change of coordinates by which $\partial E \cap B_R$ for some fixed $R > 0$ coincides with the hyperplane $x_N = 0$, and E lies locally in $\{x_N > 0\}$. Setting

$$B_\rho^+ = B_\rho \cap [x_N > 0] \quad \text{for all } \ 0 < \rho \le R$$

we require that all "concentric" $\frac{1}{2}$-balls $B_{\sigma\rho}^+ \subset B_\rho^+ \subset B_R^+$ be contained in E. Denote by ζ a non-negative piecewise smooth cutoff function that equals 1 on $B_{\sigma\rho}(y)$, vanishes outside $B_\rho(y)$, and such that $|\nabla\zeta| \le [(1-\sigma)\rho]^{-1}$. Notice that ζ vanishes on ∂B_ρ and not on ∂B_ρ^+. Let u be a local sub(super)-solution of (8.1) in the sense of (8.3), and in the latter take the test functions $v = \pm\zeta^p(u-k)_\pm \in W^{1,p}(E)$. Carrying on the same estimations as in Section 1, we arrive at integral inequalities analogous to (1.6) with the only difference that the various integrals are extended over $B_{\sigma\rho}^+$ and B_ρ^+, and that the right-hand side contains the boundary term arising from the right-hand side of (8.3). Precisely

$$\begin{aligned} \|\nabla(u-k)_\pm\zeta\|_{p,B_\rho^+}^p \le {} & \frac{\gamma}{(1-\sigma)^p\rho^p}\|(u-k)_\pm\|_{p,B_\rho^+}^p \\ & + \gamma_*^p |A_{k,\rho}^\pm|^{1-\frac{p}{N}+p\delta} + \Big| \int_{x_N=0} \psi(u-k)_\pm\zeta^p d\bar{x} \Big| \end{aligned} \tag{8.4}$$

where δ is given by (1.5), γ_*^p is defined in (1.7), the sets $A_{k,\rho}^\pm$ are redefined accordingly, and $\bar{x} = (x_1, \dots, x_{N-1})$. The requirement (8.2) merely ensures that (8.3) is well defined. The boundary DG classes for Neumann data ψ require a higher order of integrability of ψ. We assume that

$$\psi \in L^q(\partial E), \qquad \text{where} \qquad \begin{cases} q = \dfrac{N-1}{p-1}\left(\dfrac{N+\varepsilon}{N}\right) & \text{if } \ 1 < p < N \\ \\ \text{any } q > 1 & \text{if } \ p = N \end{cases} \tag{8.5}$$

for some $\varepsilon > 0$. Using such a q, define $\bar{p} > 1$ by

$$1 - \frac{1}{q} = \frac{1}{\bar{p}^*}\frac{N}{N-1}, \qquad \bar{p}^* = \frac{N\bar{p}}{N-\bar{p}}, \qquad \frac{1}{q} = \frac{\bar{p}-1}{\bar{p}}\frac{N}{N-1}.$$

One verifies that for these choices, $1 < \bar{p} < p \le N$ and the trace inequality (8.3) of Chapter 9 can be applied. With this stipulation, estimate the last integral as

$$\begin{aligned} \Big| \int_{x_N=0} & \psi(u-k)_\pm\zeta^p d\sigma \Big| \\ & \le \|\psi\|_{q;\partial E}\|(u-k)_\pm\zeta\|_{\bar{p}^*\frac{N-1}{N};\partial E} \\ & \le \|\psi\|_{q;\partial E}\big[\|\nabla[(u-k)_\pm\zeta]\|_{\bar{p}} + 2\gamma\|(u-k)_\pm\zeta\|_{\bar{p}}\big] \\ & \le \|\psi\|_{q;\partial E}\big[\|\nabla[(u-k)_\pm\zeta]\|_p + 2\gamma\|(u-k)_\pm\zeta\|_p\big]|A_{k,\rho}^\pm|^{\frac{1}{\bar{p}}-\frac{1}{p}} \\ & \le \frac{1}{2}\|\nabla(u-k)_\pm\zeta\|_{p,B_\rho^+}^p + \|(u-k)_\pm(\zeta+|\nabla\zeta|)\|_{p,B_\rho^+}^p \\ & \quad + \gamma(N,p)\|\psi\|_{q;\partial E}^{\frac{p}{p-1}}|A_{k,\rho}^\pm|^{(\frac{1}{\bar{p}}-\frac{1}{p})\frac{p}{p-1}}. \end{aligned}$$

Combining this with (8.4) and stipulating $\rho \le 1$ gives

$$\|\nabla(u-k)_\pm\|^p_{p,B^+_{\sigma\rho}} \le \frac{\gamma}{(1-\sigma)^p\rho^p}\|(u-k)_\pm\|^p_{p,B^+_\rho} + \gamma^p_{**}|A^\pm_{k,\rho}|^{1-\frac{p}{N}+p\delta} \tag{8.6}$$

where δ is given by (1.5), and

$$\gamma^p_{**} = \gamma(N,p)\big(\|f\|^p_{N+\varepsilon} + \|f_o\|^{\frac{p}{p-1}}_{\frac{N+\varepsilon}{p}} + \|\psi\|^{\frac{p}{p-1}}_{q;\partial E}\big). \tag{8.7}$$

Given $\psi \in L^q(\partial E)$ as in (8.5), the boundary DeGiorgi classes $\mathrm{DG}^\pm_\psi = \mathrm{DG}^\pm_\psi(\partial E, p, \gamma, \gamma_{**}, \delta)$ are the collection of all $u \in W^{1,p}(E)$ such that for all $y \in \partial E$ and all pairs of $\frac{1}{2}$-balls $B^+_{\sigma\rho}(y) \subset B^+_\rho(y)$ for $\rho < R$, the localized truncations $(u-k)_\pm$ satisfy (8.6). We further define $\mathrm{DG}_\psi = \mathrm{DG}^+_\psi \cap \mathrm{DG}^-_\psi$ and refer to these classes as homogeneous if $\gamma_{**} = 0$.

8.1 Continuity up to ∂E of Functions in the Boundary DG Classes (Neumann Data)

Having fixed $y \in \partial E$, assume after a flattening of ∂E about y that ∂E coincides with the hyperplane $x_N = 0$ within a ball $B_R(y)$. Consider the "concentric" $\frac{1}{2}$-balls $B^+_\rho(y) \subset B^+_{2\rho}(y) \subset B^+_R(y)$ and set

$$\mu^+ = \operatorname*{ess\,sup}_{B^+_{2\rho}(y)} u, \quad \mu^- = \operatorname*{ess\,inf}_{B^+_{2\rho}(y)} u, \quad \omega(2\rho) = \mu^+ - \mu^- = \operatorname*{ess\,osc}_{B^+_{2\rho}(y)} u. \tag{8.8}$$

Theorem 8.1. *Let ∂E be of class C^1 satisfying the segment property. Then every $u \in DG_\psi$ is continuous up to ∂E, and there exist constants $C > 1$ and $\alpha \in (0,1)$, depending only on the data defining the DG_ψ classes and the C^1 structure of ∂E, and independent of ψ and u, such that for all $y \in \partial E$ and all $\frac{1}{2}$-balls $B^+_\rho(y) \subset B^+_R(y)$*

$$\omega(\rho) \le C\max\left\{\omega(R)\left(\frac{\rho}{R}\right)^\alpha ; \gamma_{**}\rho^{N\delta}\right\}. \tag{8.9}$$

The proof of this theorem is almost identical to that of the interior Hölder continuity, the only difference being that we are working with "concentric" $\frac{1}{2}$-balls instead of balls. Proposition 4.1 and its proof continue to hold. Since $(u-k)_\pm\zeta$ do not vanish on ∂B^+_ρ, the embedding Theorem 2.1 of Chapter 9 is used instead of the multiplicative embedding. Next, Proposition 5.1 relies on the discrete isoperimetric inequality of Proposition 5.2, which holds for convex domains, and thus for $\frac{1}{2}$-balls. The rest of the proof is identical with the indicated change in the use of the embedding inequalities. ■

Remark 8.1 The regularity of ψ enters only in the requirement (8.5) through the constant γ_{**}.

Remark 8.2 The arguments are local in nature, and as such they require only local assumptions.

Corollary 8.1 *Let ∂E be of class C^1 satisfying the segment property. A weak solution u of the Neumann problem for (8.1) for a datum ψ satisfying (8.5), is Hölder continuous in $\bar{E}$. Analogous local statements are in force, if the assumptions on ∂E and ψ hold on portions of ∂E.*

9 The Harnack Inequality

Theorem 9.1 ([53, 48]). *Let $u \in DG(E, p, \gamma, \gamma_*, \delta)$ be non-negative. There exists a positive constant c_* that can be quantitatively determined a priori in terms of only the parameters N, p, γ and independent of u, γ_*, and δ such that for every ball $B_{4\rho}(y) \subset E$, either $u(y) \le c_*^{-1}\gamma_*\rho^{N\delta}$ or*

$$c_*\, u(y) \le \inf_{B_\rho(y)} u. \tag{9.1}$$

This inequality was first proved for non-negative harmonic functions (Section 5.1 of Chapter 2). Then it was shown to hold for non-negative solutions of quasi-linear elliptic equations of the type of (1.1) ([186, 235, 261]). It is quite remarkable that they continue to hold for non-negative functions in the DG classes, and it raises the still unsettled question of the structure of these classes, versus Harnack estimates, and weak forms of the maximum principle.

The first proof of Theorem 9.1 is in [53]. A different proof that avoids coverings is in [48]. This is the proof presented here, in view of its relative flexibility.

9.1 Proof of Theorem 9.1. Preliminaries

Fix $B_{4\rho}(y) \subset E$, assume $u(y) > 0$, and introduce the change of function and variables

$$w = \frac{u}{u(y)}, \qquad x \to \frac{x-y}{\rho}.$$

Then $w(0) = 1$, and w belongs to the DG classes relative to the ball B_4, with the same parameters as the original DG classes, except that γ_* is now replaced by

$$\Gamma_* = (2\rho)^{N\delta}\frac{\gamma_*}{u(y)}. \tag{9.2}$$

In particular, the truncations $(w-k)_\pm$ satisfy

$$\|\nabla(w-k)_\pm\|^p_{p,B_{\sigma r}(x_*)} \le \frac{\gamma}{(1-\sigma)^p r^p}\|(w-k)_\pm\|^p_{p,B_r(x_*)} + \Gamma_*^p |A^-_{k,r}|^{1-\frac{p}{N}+p\delta} \tag{9.3}$$

for all $B_r(x_*) \subset B_4$ and for all $k > 0$. By these transformations, (9.1) reduces to finding a positive constant c_* that can be determined a priori in terms of only the parameters of the original DG classes, such that

$$c_* \le \max\{\inf_{B_1} w\,;\, \Gamma_*\}. \tag{9.4}$$

9.2 Proof of Theorem 9.1. Expansion of Positivity

Proposition 9.1 *Let* $M>0$ *and* $B_{4r}(x_*)\subset B_4$*. If*

$$\big|[w\ge M]\cap B_r(x_*)\big|\ge\tfrac{1}{2}|B_r| \tag{9.5}$$

then for every $\nu\in(0,1)$ *there exists* $\varepsilon\in(0,1)$ *depending only on the data and* ν*, and independent of* $\varGamma_*$*, such that either* $\varepsilon M\le\varGamma_* r^{N\delta}$ *or*

$$\big|[w<2\varepsilon M]\cap B_{4r}(x_*)\big|\le\nu\big|B_{4r}\big|. \tag{9.6}$$

As a consequence, either $\varepsilon M\le\varGamma_* r^{N\delta}$ *or*

$$w\ge\varepsilon M\qquad in\ \ B_{2r}(x_*). \tag{9.7}$$

Proof. The assumption (9.5) implies that

$$\big|[w\ge M]\cap B_{4r}(x_*)\big|\ge\theta|B_{4r}|,\qquad\text{where }\ \theta=\frac{1}{2}\frac{1}{4^N}.$$

Then Proposition 5.1 applied for such a θ and for ρ replaced by $4r$ implies that (9.6) holds, for any prefixed $\nu\in(0,1)$. This in turn implies (9.7), by virtue of Proposition 4.1, applied with ρ replaced by $4r$. ■

Remark 9.1 Proposition 4.1 is a "shrinking" proposition, in that information on a ball B_ρ, yields information on a smaller ball $B_{\frac{1}{2}\rho}$. Proposition 9.1 is an "expanding" proposition in the sense that information on a ball $B_r(x_*)$ yields information on a larger ball $B_{2r}(x_*)$. This "expansion of positivity" is at the heart of the Harnack inequality (9.1).

9.3 Proof of Theorem 9.1

For $s\in[0,1)$ consider the balls B_s and the increasing families of numbers

$$M_s=\sup_{B_s}u,\qquad N_s=(1-s)^{-\beta}$$

where $\beta>0$ is to be chosen. Since $w\in L^\infty(B_2)$, the net $\{M_s\}$ is bounded. One verifies that

$$M_o=N_o=1,\qquad\lim_{s\to1}M_s<\infty,\qquad\text{and}\qquad\lim_{s\to1}N_s=\infty.$$

Therefore the equation $M_s=N_s$ has roots, and we denote by s_* the largest of these roots. Since w is continuous in B_2, there exists $x_*\in B_{s_*}$ such that

$$\sup_{B_{s_*}(x_*)}w=w(x_*)=(1-s_*)^{-\beta}.$$

Also, since s_* is the largest root of $M_s=N_s$

$$\sup_{B_R(x_*)} w \le \left(\frac{1-s_*}{2}\right)^{-\beta}, \qquad \text{where} \quad R = \frac{1-s_*}{2}.$$

By virtue of the Hölder continuity of w, in the form (3.2), for all $0 < r < R$ and for all $x \in B_r(x_*)$

$$\begin{aligned} w(x) - w(x_*) &\ge -C\Big\{\Big[\sup_{B_R(x_*)} w - \inf_{B_R(x_*)} w\Big]\left(\frac{r}{R}\right)^{\alpha} + \varGamma_* r^{N\delta}\Big\} \\ &\ge -C\Big[2^{\beta}(1-s_*)^{-\beta}\left(\frac{r}{R}\right)^{\alpha} + \varGamma_* r^{N\delta}\Big]. \end{aligned} \tag{9.8}$$

Next take $r = \epsilon_* R$, and then ϵ_* so small that

$$C\left\{2^{\beta}(1-s_*)^{-\beta}\epsilon_*^{\alpha} + \varGamma_*\epsilon_*^{N\delta}\left(\frac{1-s_*}{2}\right)^{N\delta}\right\} \le \frac{1}{2}(1-s_*)^{-\beta}.$$

The choice of ϵ_* depends on $C, \alpha, \varGamma_*, N, \delta$, which are quantitatively determined parameters; it depends also on β, which is still to be chosen; however the choice of ϵ_* can be made independent of s_*. For these choices

$$w(x) \ge w(x_*) - \frac{1}{2}(1-s_*)^{-\beta} = \frac{1}{2}(1-s_*)^{-\beta} \overset{\text{def}}{=} M$$

for all $x \in B_r(x_*)$. Therefore

$$\big|[w \ge M] \cap B_r(x_*)\big| \ge \tfrac{1}{2}\big|B_r\big|. \tag{9.9}$$

From this and Proposition 9.1, there exists $\varepsilon \in (0,1)$ that can be quantitatively determined in terms of only the nonhomogeneous parameters in the DG classes and is independent of β, r, $\varGamma_*$, and w such that either

$$\varepsilon M \le \varGamma_* r^{N\delta}, \qquad \text{where} \quad r = \tfrac{1}{2}\epsilon_*(1-s_*)$$

or

$$w \ge \varepsilon M \quad \text{on} \quad B_{2r}(x_*).$$

Iterating this process from the ball $B_{2^j r}(x_*)$ to the ball $B_{2^{j+1}r}(x_*)$ gives the recursive alternatives, either

$$\varepsilon^j M \le \varGamma_*(2^j r)^{N\delta} \qquad \text{or} \qquad w > \varepsilon^j M \quad \text{on} \;\; B_{2^{j+1}r}(x_*). \tag{9.10}$$

After n iterations, the ball $B_{2^{n+1}r}(x_*)$ will cover B_1 if n is so large that

$$2 \le 2^{n+1} r = 2^{n+1}\tfrac{1}{2}\epsilon_*(1-s_*) \le 4 \tag{9.11}$$

from which

$$2\varepsilon^n M = \varepsilon^n (1-s_*)^{-\beta} \le (2^{\beta}\varepsilon)^n \epsilon_*^{\beta} \le 2^{\beta}\varepsilon^n (1-s_*)^{-\beta} = 2^{\beta+1}\varepsilon^n M.$$

In these inequalities, all constants except s_* and β are quantitatively determined a priori in terms of only the nonhomogeneous parameters of the DG classes. The parameter ϵ_* depends on β but is independent of s_*. The latter is determined only qualitatively. The remainder of the proof consists in selecting β so that the qualitative parameter s_* is eliminated. Select β so large that $\varepsilon 2^\beta = 1$. Such a choice determines ϵ_*, and

$$\varepsilon^n M = \varepsilon^n \tfrac{1}{2}(1-s_*)^{-\beta} \ge 2^{-(\beta+1)}\epsilon_*^\beta \overset{\text{def}}{=} c_*.$$

Returning to (9.10), if the first alternative is violated for all $j = 1, 2, \dots, n$, then the second alternative holds recursively and gives

$$w \ge \varepsilon^n M \ge c_* \quad \text{in } B_1.$$

If the first alternative holds for some $j \in \{1, \dots, n\}$, then a fortiori it holds for $j = n$, which, taking into account the definition (9.2) of $\mathit{\Gamma}_*$ and (9.11), implies

$$c_* u(y) \le \gamma_* (2\rho)^{N\delta}. \qquad \blacksquare$$

10 Harnack Inequality and Hölder Continuity

The Hölder continuity of a function u in the DG classes in the form (3.2) has been used in an essential way in the proof of Theorem 9.1. For non-negative solutions of elliptic equations, the Harnack estimate can be established independent of the Hölder continuity, and indeed, the former implies the latter ([186]).

Let $\mu^\pm$ and $\omega(2\rho)$ be defined as in (3.1). Applying Theorem 9.1 to the two non-negative functions $w^+ = \mu^+ - u$ and $w^- = u - \mu^-$, gives either

$$\begin{aligned} \operatorname*{ess\,sup}_{B_\rho(y)} w^+ &= \mu^+ - \operatorname*{ess\,inf}_{B_\rho(y)} u \le c_*^{-1}\gamma_*\rho^{N\delta} \\ \operatorname*{ess\,sup}_{B_\rho(y)} w^- &= \operatorname*{ess\,sup}_{B_\rho(y)} u - \mu^- \le c_*^{-1}\gamma_*\rho^{N\delta} \end{aligned} \tag{10.1}$$

or

$$\begin{aligned} c_*(\mu^+ - \operatorname*{ess\,inf}_{B_\rho(y)} u) &\le \mu^+ - \operatorname*{ess\,sup}_{B_\rho(y)} u \\ c_*(\operatorname*{ess\,sup}_{B_\rho(y)} u - \mu^-) &\le \operatorname*{ess\,inf}_{B_\rho(y)} u - \mu^-. \end{aligned} \tag{10.2}$$

If either one of (10.1) holds, then

$$\omega(\rho) \le \omega(2\rho) \le c_*^{-1}\gamma_*\rho^{N\delta}. \tag{10.3}$$

Otherwise, both inequalities in (10.2) are in force. Adding them gives

$$c_*\omega(2\rho) + c_*\omega(\rho) \le \omega(2\rho) - \omega(\rho).$$

From this

$$\omega(\rho) \le \eta\omega(2\rho), \qquad \text{where} \qquad \eta = \frac{1-c_*}{1+c_*}. \tag{10.4}$$

The alternatives (10.3)–(10.4) yield recursive inequalities of the same form as (6.3), from which the Hölder continuity follows. These remarks raise the question whether the Harnack estimate for non-negative functions in the DG classes can be established independently of the Hölder continuity. The link between these two facts rendering them essentially equivalent, is the next lemma of real analysis.

11 Local Clustering of the Positivity Set of Functions in $W^{1,1}(E)$

For $R > 0$, denote by $K_R(y) \subset \mathbb{R}^N$ a cube of edge R centered at y and with faces parallel to the coordinate planes. If y is the origin on $\mathbb{R}^N$, write $K_R(0) = K_R$.

Lemma 11.1 ([54]) *Let $v \in W^{1,1}(K_R)$ satisfy*

$$\|v\|_{W^{1,1}(K_R)} \le \gamma R^{N-1} \qquad \textit{and} \qquad |[v>1]| \ge \nu|K_R| \tag{11.1}$$

for some $\gamma > 0$ and $\nu \in (0,1)$. Then for every $\nu_* \in (0,1)$ and $0 < \lambda < 1$, there exist $x_* \in K_R$ and $\epsilon_* = \epsilon_*(\nu, \nu_*, \lambda, \gamma, N) \in (0,1)$ such that

$$|[v>\lambda] \cap K_{\epsilon_* R}(x_*)| > (1-\nu_*)|K_{\epsilon_* R}|. \tag{11.2}$$

Remark 11.1 Roughly speaking, the lemma asserts that if the set where u is bounded away from zero occupies a sizable portion of K_R, then there exists at least one point x_* and a neighborhood $K_{\epsilon_* R}(x_*)$ where u remains large in a large portion of $K_{\epsilon_* R}(x_*)$. Thus the set where u is positive clusters about at least one point of K_R.

Proof (of Lemma 11.1). It suffices to establish the lemma for u continuous and $R = 1$. For $n \in \mathbb{N}$ partition K_1 into n^N cubes, with pairwise disjoint interior and each of edge $1/n$. Divide these cubes into two finite sub-collections $\mathbf{Q}^+$ and $\mathbf{Q}^-$ by

$$\begin{aligned} Q_j \in \mathbf{Q}^+ \quad &\Longleftrightarrow \quad |[v>1] \cap Q_j| > \tfrac{1}{2}\nu|Q_j| \\ Q_i \in \mathbf{Q}^- \quad &\Longleftrightarrow \quad |[v>1] \cap Q_i| \le \tfrac{1}{2}\nu|Q_i| \end{aligned}$$

and denote by $\#(\mathbf{Q}^+)$ the number of cubes in $\mathbf{Q}^+$. By the assumption

$$\sum_{Q_j \in \mathbf{Q}^+} |[v>1] \cap Q_j| + \sum_{Q_i \in \mathbf{Q}^-} |[v>1] \cap Q_i| > \nu|K_1| = \nu n^N |Q|$$

where $|Q|$ is the common measure of the Q_ℓ. From the definitions of $\mathbf{Q}^\pm$

$$\begin{aligned}\nu n^N &< \sum_{Q_j\in\mathbf{Q}^+}\frac{|[v>1]\cap Q_j|}{|Q_j|}+\sum_{Q_i\in\mathbf{Q}^-}\frac{|[v>1]\cap Q_i|}{|Q_i|}\\ &< \#(\mathbf{Q}^+)+\tfrac{1}{2}\nu(n^N-\#(\mathbf{Q}^+)).\end{aligned}$$

Therefore

$$\#(\mathbf{Q}^+)>\frac{\nu}{2-\nu}n^N. \tag{11.3}$$

Fix $\nu_*, \lambda\in(0,1)$. The integer n can be chosen depending on ν,ν_*,λ,γ, and N, such that

$$|[v>\lambda]\cap Q_j|\ge(1-\nu_*)|Q_j| \quad \text{for some } \; Q_j\in\mathbf{Q}^+. \tag{11.4}$$

This would establish the lemma for $\epsilon_*=1/n$. We first show that if Q is a cube in $\mathbf{Q}^+$ for which

$$|[v>\lambda]\cap Q|<(1-\nu_*)|Q| \tag{11.5}$$

then there exists a constant $c=c(\nu,\nu_*,\lambda,N)$ such that

$$\|v\|_{W^{1,1}(Q)}\ge c(\nu,\nu_*,\lambda,N)\frac{1}{n^{N-1}}. \tag{11.6}$$

From (11.5)

$$|[v\le\lambda]\cap Q|\ge\nu_*|Q| \qquad \text{and} \qquad \Big|\Big[v>\frac{1+\lambda}{2}\Big]\cap Q\Big|>\frac{1}{2}\nu|Q|.$$

For fixed $x\in[v\le\lambda]\cap Q$ and $y\in[v>(1+\lambda)/2]\cap Q$

$$\frac{1-\lambda}{2}\le v(y)-v(x)=\int_0^{|y-x|}\nabla u(x+t\mathbf{n})\cdot\mathbf{n}dt, \quad \mathbf{n}=\frac{y-x}{|x-y|}.$$

Let $R(x,\mathbf{n})$ be the polar representation of ∂Q with pole at x for the solid angle $\mathbf{n}$. Integrate the previous relation in dy over $[v>(1+\lambda)/2]\cap Q$. Minorize the resulting left-hand side, by using the lower bound on the measure of such a set, and majorize the resulting integral on the right-hand side by extending the integration over Q. Expressing such integration in polar coordinates with pole at $x\in[v\le\lambda]\cap Q$ gives

$$\begin{aligned}\frac{\nu(1-\lambda)}{4}|Q| &\le \int_{|\mathbf{n}|=1}\int_0^{R(x,\mathbf{n})} r^{N-1}\int_0^{|y-x|}|\nabla v(x+t\mathbf{n})|\,dt\,dr\,d\mathbf{n}\\ &\le N^{N/2}|Q|\int_{|\mathbf{n}|=1}\int_0^{R(x,\mathbf{n})}|\nabla v(x+t\mathbf{n})|\,dt\,d\mathbf{n}\\ &= N^{N/2}|Q|\int_Q\frac{|\nabla v(z)|}{|z-x|^{N-1}}dz.\end{aligned}$$

Now integrate in dx over $[u \le \lambda] \cap Q$. Minorize the resulting left-hand side using the lower bound on the measure of such a set, and majorize the resulting right-hand side, by extending the integration to Q. This gives

$$\begin{aligned}\frac{\nu\nu_*(1-\lambda)}{4N^{N/2}}|Q| &\le \|v\|_{W^{1,1}(Q)} \sup_{z\in Q}\int_Q \frac{1}{|z-x|^{N-1}}dx \\ &\le C(N)|Q|^{1/N}\|v\|_{W^{1,1}(Q)}\end{aligned}$$

for a constant $C(N)$ depending only on N, thereby proving (11.6).

If (11.4) does not hold for any cube $Q_j \in \mathbf{Q}^+$, then (11.6) is verified for all such Q_j. Adding (11.6) over such cubes and taking into account (11.3)

$$\frac{\nu}{2-\nu}c(\nu,\nu_*,\lambda,N)n \le \|u\|_{W^{1,1}(K_1)} \le \gamma. \qquad \blacksquare$$

Remark 11.2 While the lemma has been proved for cubes, by reducing the number ϵ_* if needed, we may assume without loss of generality that it continues to hold for balls.

12 A Proof of the Harnack Inequality Independent of Hölder Continuity

Introduce the same transformations of Section 9.1 and reduce the proof to establishing (9.4). Following the same arguments and notation of Section 9.3, for $\beta > 0$ to be chosen, let s_* be the largest root of $M_s = N_s$ and set

$$M_* = (1-s_*)^{-\beta} \qquad M^* = 2^\beta(1-s_*)^{-\beta}$$
$$R = \frac{1-s_*}{4} \qquad R_o = \frac{1+s_*}{2}$$

so that $M^* = 2^\beta M_*$ and $B_R(x) \subset B_{R_o}$ for all $x \in B_{s_*}$, and

$$\operatorname*{ess\,sup}_{B_R(x)} w \le M^* \quad \text{for all } x \in B_{s_*}.$$

Proposition 12.1 *There exists a ball $B_R(x) \subset B_{R_o}$ such that either*

$$M^* \le \Gamma_* R^{N\delta} \quad \Longrightarrow \quad u(y) \le \gamma_* \rho^{N\delta} \tag{12.1}$$

or

$$|[w > M^* - \varepsilon M^*] \cap B_R(x)| > \nu_a |B_R| \tag{12.2}$$

where

$$\nu_a = \frac{(1-a)^{1/\delta}}{\gamma^{1/\delta}2^{1/p\delta^2}}, \qquad a = \varepsilon = \sqrt{1-\frac{1}{2^\beta}} \tag{12.3}$$

and where γ is the quantitative constant appearing in (4.3) and dependent only on the inhomogeneous data of the DG classes.

Proof. If the first alternative (12.1) holds for some $B_R(x) \subset B_{R_o}$, there is nothing to prove. Thus assuming (12.1) fails for all such balls, if (12.2) holds for some of these balls, there is nothing to prove. Thus we may assume that (12.1) and (12.2) are both violated for all $B_R(x) \subset B_{R_o}$. Apply Proposition 4.1 with $\varepsilon = a$ and conclude, by the choice of a and ν_a, that

$$w < (1-a^2)M^* = M_* \quad \text{in all balls } B_{\frac{1}{2}R}(x) \subset B_{R_o}.$$

Thus $w < M_*$ in B_{s_*}, contradicting the definition of M_*. ■

A consequence is that there exists $B_R(x) \subset B_{R_o}$ such that

$$|[v > 1] \cap B_R(x)| > \nu |B_R|, \qquad \text{where} \quad v = \frac{w}{(1-a)M^*}. \tag{12.4}$$

Write the inequalities (9.3) for w_+ $(k = 0)$ over the pair of balls $B_R(x) \subset B_{2R}(x) \subset B_{R_o}$, and then divide the resulting inequalities by $[(1-a)M^*]^p$. Taking into account the definition (9.2) of Γ_*, and (12.2), this gives

$$\|\nabla v\|^p_{p,B_R(x)} \le \gamma 2^{p\beta/2} R^{N-p}.$$

From this

$$\|\nabla v\|_{1,B_R(x)} \le \gamma(\beta) R^{N-1}. \tag{12.5}$$

Thus the function v satisfies the assumptions of Lemma 11.1, with given and fixed constants $\gamma = \gamma(\text{data}, \beta)$ and $\nu = \nu_a(\text{data}, \beta)$. The parameter $\beta > 1$ has to be chosen. Applying the lemma for $\lambda = \nu = \frac{1}{2}$ yields the existence of $x_* \in B_R(x)$ and $\epsilon_* = \epsilon_*(\text{data}, \beta)$ such that

$$|[w > M] \cap B_r(x_*)| \ge \tfrac{1}{2}|B_r|, \qquad \text{where} \quad r = \epsilon_* R$$

and where $M = \frac{1}{4}M_*$. This is precisely (9.9), and the proof can now be concluded as before. ■

Remark 12.1 The Hölder continuity was used to ensure, starting from (9.8) that w is bounded below by M in a sizable portion of a small ball $B_{\epsilon_* R}(x_*)$ about x_*. In that process, the parameter ϵ_* had to be chosen in terms of the data and the still to be determined parameter β. Thus $\epsilon_* = \epsilon_*(\text{data}, \beta)$. The alternative proof based on Lemma 11.1, is intended to achieve the same lower bound on a sizable portion of $B_{\epsilon_* R}(x_*)$. The discussion has been conducted in order to trace the dependence of the various parameters on the unknown β. Indeed, also in this alternative argument, $\epsilon_* = \epsilon_*(\text{data}, \beta)$, but this is the only parameter dependent on β, whose choice can then be made by the very same argument, which from (9.9) leads to the conclusion of the proof.

11

LINEAR PARABOLIC EQUATIONS IN DIVERGENCE FORM WITH MEASURABLE COEFFICIENTS

1 Parabolic Spaces and Embeddings

Let E be a bounded domain in $\mathbb{R}^N$ with boundary ∂E of class C^1, and for $0 < T < \infty$ let E_T denote the cylindrical domain $E \times (0, T]$. The space $L^{r,q}(E_T)$ for $q, r \ge 1$ is the collection of functions f defined and measurable in E_T such that

$$\|f\|_{q,r;E_T} = \Big(\int_0^T \Big(\int_E |f|^q dx \Big)^{\frac{r}{q}} d\tau \Big)^{\frac{1}{r}} < \infty.$$

Also, $f \in L^{q,r}_{\text{loc}}(E_T)$, if for every compact subset $K \subset E$ and every sub-interval $[t_1, t_2] \subset (0, T]$

$$\int_{t_1}^{t_2} \Big(\int_K |f|^q dx \Big)^{\frac{r}{q}} d\tau < \infty.$$

Whenever $q = r$ we set $L^{q,q}(E_T) = L^q(E_T)$. These definitions are extended in the obvious way when either q or r is infinity.

We introduce spaces of functions, depending on $(x, t) \in E_T$, that exhibit different behavior in the space and time variables. These are spaces where solutions of parabolic equations are typically found.

The set of all functions that are continuous in E_T is denoted by $C(E_T)$. Given two points (x_1, t_1), $(x_2, t_2) \in E_T$, we define

$$d((x_1, t_1), (x_2, t_2)) \overset{\text{def}}{=} |x_1 - x_2| + |t_1 - t_2|^{\frac{1}{2}},$$

and for $\alpha \in (0, 1)$, we let $C^{\alpha, \frac{\alpha}{2}}(E_T)$ be the subspace of $C(E_T)$ consisting of all functions f such that the norm

$$\|f\|_{C^{\alpha,\frac{\alpha}{2}}(E_T)} = \sup_{(x,t)\in E_T} |f(x,t)| + \sup_{\substack{(x_i,t_i)\in E_T \\ (x_1,t_1)\neq(x_2,t_2)}} \frac{|f(x_1,t_1) - f(x_2,t_2)|}{[d((x_1,t_1),(x_2,t_2))]^\alpha}$$

© Springer Nature Switzerland AG 2023

E. DiBenedetto, U. Gianazza, *Partial Differential Equations*,
Cornerstones, https://doi.org/10.1007/978-3-031-46618-2_12

is finite. This is the space of the so-called *Hölder continuous* functions. Finally, $C^{2,1}(E_T)$ is the set of all continuous functions in E_T having continuous derivatives f_{x_i}, $f_{x_i x_j}$ with $i, j = 1, \dots, N$, f_t in E_T. The previous definitions can be extended from E_T to $\overline{E_T}$; moreover, exactly as we did above for $L^{q,r}(E_T)$ spaces, we say that

$$f \in C_{\mathrm{loc}}(E_T),\ C_{\mathrm{loc}}^{\alpha\frac{\alpha}{2}}(E_T),\ C_{\mathrm{loc}}^{2,1}(E_T),$$

whenever the previous definitions hold on any compact set $\mathcal{K} \subset E_T$.

Let $m, p \ge 1$ and consider the Banach spaces

$$\begin{aligned} V^{m,p}(E_T) &= L^\infty\big(0,T; L^m(E)\big) \cap L^p\big(0,T; W^{1,p}(E)\big), \\ V_o^{m,p}(E_T) &= L^\infty\big(0,T; L^m(E)\big) \cap L^p\big(0,T; W_o^{1,p}(E)\big) \end{aligned}$$

both equipped with the norm

$$\|v\|_{V^{m,p}(E_T)} = \operatorname*{ess\,sup}_{0<t<T} \|v(\cdot,t)\|_{m;E} + \|Dv\|_{p;E_T}.$$

When $m = p$, set $V_o^{p,p}(E_T) = V_o^p(E_T)$ and $V^{p,p}(E_T) = V^p(E_T)$. In the following we will be interested in $V^2(E_T)$ and $V_o^2(E_T)$. Both spaces are embedded in $L^q(E_T)$ for some $q > 2$. In a precise way we have the following.

Proposition 1.1 *There exists a constant γ depending only on N such that for every $v \in V_o^2(E_T)$*

$$\begin{aligned} \iint_{E_T} |v(x,t)|^q dx\, dt \le \gamma^q \Big(\iint_{E_T} |Dv(x,t)|^2 dx\, dt \Big) \\ \times \Big(\operatorname*{ess\,sup}_{0<t<T} \int_E |v(x,t)|^2 dx \Big)^{\frac{2}{N}}, \end{aligned} \tag{1.1}$$

where

$$q = 2\frac{N+2}{N}.$$

Moreover,

$$\|v\|_{q;E_T} \le \gamma \|v\|_{V^2(E_T)}. \tag{1.2}$$

Remark 1.1 The multiplicative inequality (1.1) and the embedding (1.2) continue to hold for functions $v \in V^2(E_T)$ such that

$$\int_E v(x,t)\, dx = 0 \qquad \text{for a.e. } t \in (0,T)$$

provided ∂E is piecewise smooth. In such a case the constant γ also depends on the structure of ∂E, but not on its size.

Proposition 1.2 *Assume that ∂E is piecewise smooth. There exists a constant γ depending only on N and the structure of E, such that for every $u \in V^2(E_T)$*

$$\|v\|_{q;E_T} \le \gamma \left(1 + \frac{T}{|E|^{\frac{2}{N}}}\right)^{\frac{1}{q}} \|v\|_{V^2(E_T)}, \tag{1.3}$$

where, again,

$$q = 2\frac{N+2}{N}.$$

The next corollaries follow from Propositions 1.1–1.2 by applying Hölder's inequality.

Corollary 1.1 *There exists a constant γ depending only on N, such that for every $v \in V_o^2(E_T)$,*

$$\|v\|_{2;E_T} \le \gamma \big|[|v| > 0]\big|^{\frac{1}{N+2}} \|v\|_{V^2(E_T)}.$$

Corollary 1.2 *Assume that ∂E is piecewise smooth. There exists a constant γ depending only on N and the structure of E, such that for every $v \in V^2(E_T)$,*

$$\|v\|_{2;E_T}^2 \le \gamma \left(1 + \frac{T}{|E|^{\frac{2}{N}}}\right)^{\frac{N}{N+2}} \big|[|v| > 0]\big|^{\frac{2}{N+2}} \|v\|_{V^2(E_T)}^2.$$

Working with $L^{q,r}(E_T)$ spaces, Propositions 1.1–1.2 take the following form

Proposition 1.3 *There exists a constant γ depending only on N, q, r, such that for every $v \in V_o^2(E_T)$,*

$$\|v\|_{q,r;E_T} \le \gamma \|v\|_{V^2(E_T)},$$

where the numbers $q, r \ge 1$ are linked by

$$\frac{1}{r} + \frac{N}{2q} = \frac{N}{4}, \tag{1.4}$$

and their admissible range is

$$\begin{aligned} &\text{if } N = 1, && q \in (2, \infty], && r \in [4, \infty); \\ &\text{if } N = 2, && q \in [2, \infty), && r \in \big(2, \infty\big]; \\ &\text{if } N > 2, && q \in \big[2, \tfrac{2N}{N-2}\big], && r \in [2, \infty]. \end{aligned} \tag{1.5}$$

Proposition 1.4 *Assume that ∂E is piecewise smooth. There exists a constant γ depending only on N, q, r and the structure of E, such that for every $v \in V^2(E_T)$,*

$$\|v\|_{q,r;E_T} \le \gamma \left(1 + \frac{T}{|E|^{\frac{2}{N}}}\right)^{\frac{1}{r}} \|v\|_{V^2(E_T)}, \tag{1.6}$$

where q and r satisfy (1.4)–(1.5).

We conclude this section by stating a Lemma concerning the truncated functions $(v-k)_\pm$.

Lemma 1.1 *Let $v \in V^2(E_T)$. Then, $(v-k)_\pm \in V^2(E_T)$ for all $k \in \mathbb{R}$. Assume in addition that ∂E is piecewise smooth and that the trace of $v(\cdot,t)$ on ∂E is essentially bounded and*

$$\operatorname*{ess\,sup}_{0<t<T} \|v(\cdot,t)\|_{\infty;\partial E} \le M \qquad \textit{for some } \ M>0.$$

Then, $(v-k)_\pm \in V_o^2(E_T)$ for all $k \ge M$.

Besides $V^2(E_T)$ and $V_o^2(E_T)$, in the sequel we will also need the following functional spaces:

$$\begin{aligned}
\tilde{V}^2(E_T) &= C([0,T]; L^2(E)) \cap L^2(0,T; W^{1,2}(E)),\\
\tilde{V}_o^2(E_T) &= C([0,T]; L^2(E)) \cap L^2(0,T; W_o^{1,2}(E)),\\
W^2(E_T) &= W^{1,2}([0,T]; L^2(E)) \cap L^2(0,T; W^{1,2}(E)),\\
W_o^2(E_T) &= W^{1,2}([0,T]; L^2(E)) \cap L^2(0,T; W_o^{1,2}(E)).
\end{aligned}$$

Finally, $W_q^{k,m}(E_T)$ for integers k, m, and $q>1$ is the Banach space consisting of the elements of $L^q(E_T)$ having k generalized space derivatives and m generalized time derivatives also in $L^q(E_T)$ (see Section 1c of Chapter 9 for higher order generalized derivatives).

Estimates analogous to (1.6) can be derived for the norm $\|u\|_{q,r;\Sigma}$, where $\Sigma = \partial E \times (0,T]$.

Proposition 1.5 *Assume that ∂E is piecewise smooth. There exists a constant γ depending only on N, q, r and the structure of E, such that for every $v \in V^2(E_T)$,*

$$\|v\|_{q,r;\Sigma} \le \gamma \left[1 + \left(\frac{T^{\frac{N}{2}}}{|E|}\right)^{\frac{1}{2}-\frac{N-1}{Nq}}\right] \|v\|_{V^2(E_T)},$$

where the numbers $q, r \ge 1$ are linked by

$$\frac{1}{r} + \frac{N-1}{2q} = \frac{N}{4},$$

and their admissible range is

$$\begin{aligned}
&\textit{if } \ N=2, \quad q \in [1,\infty), &&r \in \big(2,\infty\big];\\
&\textit{if } \ N>2, \quad q \in \big[\tfrac{2(N-1)}{N}, \tfrac{2(N-1)}{N-2}\big], &&r \in [2,\infty].
\end{aligned}$$

1.1 Steklov Averages

Let $v \in L^1(E_T)$ and let $0 < h < T$. The Steklov averages $v_h(\cdot,t)$ and $v_{\bar{h}}(\cdot,t)$ are defined by

$$v_h = \begin{cases} \dfrac{1}{h}\displaystyle\int_t^{t+h} v(\cdot,\tau)d\tau & \text{for } t \in (0, T-h], \\ \\ 0, & \text{for } t > T-h. \end{cases}$$

$$v_{\bar{h}} = \begin{cases} \dfrac{1}{h}\displaystyle\int_{t-h}^{t} v(\cdot,\tau)d\tau & \text{for } t \in (h, T], \\ \\ 0, & \text{for } t < h. \end{cases}$$

Lemma 1.2 *Let $v \in L^{q,r}(E_T)$. Then, as $h \to 0$, $v_h \to v$ in $L^{q,r}(E_{T-\varepsilon})$ for every $\varepsilon \in (0,T)$. If $v \in C(0,T;L^q(E))$, then $v_h(\cdot,t) \to v(\cdot,t)$ in $L^q(E)$ for every $t \in (0, T-\varepsilon)$ for all $\varepsilon \in (0,T)$.*

A similar statement holds for $v_{\bar{h}}$. The proof of the lemma is straightforward from the theory of L^p spaces.

2 Weak Formulations

Denote by (a_{ij}) an $N \times N$ symmetric matrix with entries $a_{ij} \in L^\infty(E_T)$, and satisfying the *ellipticity condition*

$$\lambda|\xi|^2 \le a_{ij}(x,t)\xi_i\xi_j \le \Lambda|\xi|^2 \tag{2.1}$$

for all $\xi \in \mathbb{R}^N$ and all $(x,t) \in E_T$, for some $0 < \lambda \le \Lambda$. The number Λ is the least upper bound of the eigenvalues of (a_{ij}) in E_T, and λ is their greatest lower bound. A vector-valued function $\mathbf{f} = (f_1, \dots, f_N) : E_T \to \mathbb{R}^N$ is said to be in $L^p_{\text{loc}}(E_T)$, for some $p \ge 1$, if all the components $f_j \in L^p_{\text{loc}}(E_T)$. Given a scalar function $f \in L^1_{\text{loc}}(E_T)$ and a vector-valued function $\mathbf{f} \in L^1_{\text{loc}}(E_T)$, consider the *formal* partial differential equation in divergence form (Section 3.1 of the Preliminaries)

$$u_t - \big(a_{ij}u_{x_j}\big)_{x_i} = f - \operatorname{div}\mathbf{f} \quad \text{in } E_T. \tag{2.2}$$

Expanding formally the indicated derivatives gives a PDE of the type of (3.1) of Chapter 1, which, in view of the ellipticity condition (2.1), admits one family of real characteristic surfaces (Section 3 of Chapter 1). In this formal sense, (2.2) is a *second-order parabolic* equation.

Now multiply (2.1) by an arbitrary smooth function $\varphi = \varphi(x,t)$ which vanishes on $\partial E \times (0,T)$, integrate both sides over E_T, and carry out an integration by parts in the terms containing the coefficients a_{ij}. As a result, we obtain

$$\iint_{E_T} (u_t\varphi + a_{ij}u_{x_j}\varphi_{x_i})dx\,dt = \iint_{E_T} (f\varphi + f_i\varphi_{x_i})dx\,dt. \tag{2.3}$$

As it is easy to find out, (2.2) and (2.3) are equivalent, if all the terms a_{ij}, f, f_i are regular.

The above formulation contains the term u_t, which we want to get rid of (the motivation for doing so lies in the possibility of proving suitable existence and uniqueness theorems, as we will show in the next paragraphs). If we perform a further formal integration by parts with respect to the variable t, we finally arrive at

$$\int_E u\varphi\Big|_0^T dx + \iint_{E_T} (-u\varphi_t + a_{ij}u_{x_j}\varphi_{x_i})dx\,dt = \iint_{E_T} (f\varphi + f_i\varphi_{x_i})dx\,dt. \tag{2.4}$$

Equation (2.4) is well defined, provided we make the right integrability assumptions on u and ∇u. Moreover, we also need to give meaning to $\displaystyle\int_E u\varphi\Big|_0^T dx$. Leaving these issues aside for a moment, we say that (2.4) is the *weak formulation* of (2.3).

3 The Homogeneous Dirichlet Problem

Consider the homogeneous Cauchy–Dirichlet problem

$$\begin{aligned} u_t - \big(a_{ij}u_{x_j}\big)_{x_i} &= f - \operatorname{div}\mathbf{f} && \text{in } E_T,\\ u &= 0 && \text{on } \partial E\times(0,T),\\ u(\cdot,0) &= u_o && \text{in } E, \end{aligned} \tag{3.1}$$

where

$$\mathbf{f}\in L^2(E_T),\ \ u_o\in L^2(E),\ \ f\in L^{q,r}(E_T), \tag{3.2}$$

with

$$\frac{1}{r}+\frac{N}{2q} = 1+\frac{N}{4} \ \text{ and } \begin{cases} q\in\left[\dfrac{2N}{N+2},2\right],\ r\in[1,2] \text{ for } N\ge 3,\\ q\in(1,2],\ r\in[1,2) \text{ for } N=2,\\ q\in[1,2],\ r\in\left[1,\dfrac{4}{3}\right] \text{ for } N=1. \end{cases} \tag{3.3}$$

The PDE is meant in the weak sense by requiring that $u\in V_o^2(E_T)$, and such that for almost all $\tau\in[0,T]$ and $\forall\,\varphi\in W_o^2(E_T)$ the following identity is satisfied

$$\begin{aligned} &\int_E u(x,\tau)\varphi(x,\tau)dx - \int_E u_o(x)\varphi(x,0)dx - \int_0^\tau\int_E u\varphi_t dx\,dt\\ &+\int_0^\tau\int_E a_{ij}u_{x_j}\varphi_{x_i}dx\,dt = \int_0^\tau\int_E (f_i\varphi_{x_i}+f\varphi)dx\,dt. \end{aligned} \tag{3.4}$$

It is a matter of straightforward computations to show that if the integrability conditions (3.2)–(3.3) are satisfied, then all the integrals in (3.4) are finite for any functions u and φ in the indicated classes.

There is a different but equivalent way to define a weak solution of the homogeneous Cauchy–Dirichlet problem (3.1): we say that $u \in V_o^2(E_T)$ is a weak solution of (3.1) if

$$\begin{aligned}
&-\int_0^T\int_E u\varphi_t dx\,dt + \int_0^T\int_E a_{ij}u_{x_j}\varphi_{x_i}dx\,dt \\
&= \int_0^T\int_E (f_i\varphi_{x_i} + f\varphi)dx\,dt + \int_E u_o(x)\varphi(x,0)dx.
\end{aligned} \tag{3.5}$$

for all $\varphi \in W_o^2(E_T)$, which vanish for $t = T$.

Proposition 3.1 *The two notions of solutions given in* (3.4) *and* (3.5) *are equivalent.*

Proof. That (3.4) implies (3.5) is trivial. Let us show that the opposite implication is also true. In the weak formulation (3.5), consider the test function

$$\varphi_\epsilon(x,t) = \zeta(x,t)\eta_\epsilon(t),$$

where $\zeta \in W_o^2(E_T)$, and η_ϵ is a piecewise linear function, which is equal to 1 for $t \le \tau - \epsilon$, vanishes for $t \ge \tau$, and is equal to $\dfrac{\tau - t}{\epsilon}$ for $t \in (\tau - \epsilon, \tau)$. If we pass to the limit as $\epsilon \to 0$, it is apparent that

$$\begin{aligned}
\int_0^T\int_E a_{ij}u_{x_j}\varphi_{\epsilon,x_i}dx\,dt &\quad\to\quad \int_0^\tau\int_E a_{ij}u_{x_j}\zeta_{x_i}dx\,dt, \\
\int_0^T\int_E (f_i\varphi_{\epsilon,x_i} + f\varphi_\epsilon)dx\,dt &\quad\to\quad \int_0^\tau\int_E (f_i\zeta_{x_i} + f\zeta)dx\,dt, \\
\int_E u_o(x)\varphi_\epsilon(x,0)dx &\quad\to\quad \int_E u_o(x)\zeta(x,0)dx.
\end{aligned}$$

As for the first term in (3.5), we have

$$-\int_0^T\int_E u\varphi_{\epsilon,t}dx\,dt = -\int_0^T\int_E u\zeta_t\eta_\epsilon dx\,dt + \frac{1}{\epsilon}\int_{\tau-\epsilon}^\tau\int_E u\zeta dx\,dt.$$

Since the function

$$w(t) = \int_E u(x,t)\zeta(x,t)dx$$

is summable on $[0,T]$, we conclude that, possibly up to a subsequence,

$$-\int_0^T\int_E u\varphi_{\epsilon,t}dx\,dt \quad\to\quad -\int_0^\tau\int_E u\zeta_t dx\,dt + \int_E u(x,\tau)\zeta(x,\tau)dx.$$

■

Remark 3.1 Formulation (3.5) will be useful in the proof of existence of solutions.

We can also give a formulation in terms of the Steklov averages. Namely, let $0 < h < T$. In the weak formulation (3.4) take as test function the Steklov average

$$\varphi_{\bar{h}}(x,t) = \frac{1}{h}\int_{t-h}^{t} \hat{\varphi}(x,\tau)\, d\tau,$$

where $\hat{\varphi}$ is an arbitrary element of $W_o^2(E_T)$, which vanishes for $t \geq T-h$ and for $t \leq 0$. If we take $\tau = T$ in (3.4), the first two terms on the left-hand side vanish because of the assumption on $\hat{\varphi}$, whereas the third one is transformed in the following way

$$-\iint_{E_T} u\varphi_{\bar{h}t}\, dx\, dt = -\iint_{E_T} u_h\varphi_t\, dx\, dt = \iint_{E_T} u_{ht}\varphi\, dx\, dt.$$

Indeed, for functions v, $w \in L^2(-h,T)$, one of which vanishes in $(-h,0)$ and $(T-h,T)$, we have

$$\int_0^T vw_{\bar{h}}\, dt = \int_0^{T-h} v_h w\, dt,$$

where we recall from Section 1.1 the definition of v_h. Moreover, we have interchanged the order of integration with respect to t and τ. In all the other terms, in a similar way, we transfer the averaging from φ to the corresponding factor, taking into account that averaging and differentiation with respect to x commute. We conclude

$$\iint_{E_{T-h}} \left[u_{ht}\varphi + (a_{ij}u_{x_j})_h\varphi_{x_i}\right] dx\, dt = \iint_{E_{T-h}} (f_h\varphi + f_{i,h}\varphi_{x_i})\, dx\, dt.$$

By a density argument, it is not difficult to see that the previous equality holds for any function φ that vanishes for $t > t_1$ with $t_1 \leq T-h$ and belongs to $\tilde{V_o^2}(E_T)$. Therefore, we end up with the weak formulation

$$\iint_{E_{t_1}} \left[u_{ht}\varphi + (a_{ij}u_{x_j})_h\varphi_{x_i}\right] dx\, dt = \iint_{E_{t_1}} (f_h\varphi + f_{i,h}\varphi_{x_i})\, dx\, dt \tag{3.6}$$

for any function $\varphi \in \tilde{V_o^2}(E_{t_1})$, provided that $t_1 \leq T-h$.

4 The Energy Inequality

We have the following result, which is important in itself, and at the same time plays a fundamental role in the existence proof for (3.1)–(3.3).

Proposition 4.1 *Let* $u \in \tilde{V_o^2}(E_T)$ *be a solution of the homogeneous Cauchy–Dirichlet problem* (3.1) *with data satisfying* (3.2)–(3.3). *Then, the* Energy Inequality

$$\|u\|_{V^2(E_T)} \le \tilde{\gamma}\Big[\|u_o\|_{2;E} + \|\mathbf{f}\|_{2;E_T} + \|f\|_{q,r;E_T}\Big] \tag{4.1}$$

holds true, where $\tilde{\gamma} > 0$ *depends only on* N *and* λ.

Proof. In (3.6) choose $\varphi = u_h$, take into account that

$$\iint_{E_{t_1}} u_{ht} u_h dx\, dt = \frac{1}{2}\int_E u_h^2(x,t)dx\Big|_{t=0}^{t=t_1}$$

and let $h \to 0$. The regularity of all the functions guarantees that this can be done, and yields

$$\frac{1}{2}\int_E u^2(x,t)dx\Big|_{t=0}^{t=t_1} + \int_0^{t_1}\int_E a_{ij}u_{x_j}u_{x_i}dx\, dt = \int_0^{t_1}\int_E (fu + f_i u_{x_i})dx\, dt.$$

The convergence of the term $\displaystyle\int_E u_h^2(x,t)\Big|_{t=0}^{t=t_1}$ is ensured by the continuity of u with respect to t in $L^2(E)$, which we have assumed.

By the ellipticity condition (2.1) and the integrability conditions (3.2)–(3.3), if q' and r' denote the Hölder conjugate exponent of q and r, we obtain

$$\begin{aligned}
&\frac{1}{2}\int_E u^2(x,t_1)dx + \lambda\int_0^{t_1}\int_E |\nabla u|^2 dx\, dt \\
&\qquad \le \frac{1}{2}\int_E u^2(x,0)dx + \left(\int_0^{t_1}\int_E |\mathbf{f}|^2 dx\, dt\right)^{\frac{1}{2}}\left(\int_0^{t_1}\int_E |\nabla u|^2 dx\, dt\right)^{\frac{1}{2}} \\
&\qquad + \left(\int_0^{t_1}\left(\int_E |f|^q dx\right)^{\frac{r}{q}} dt\right)^{\frac{1}{r}}\left(\int_0^{t_1}\left(\int_E |u|^{q'} dx\right)^{\frac{r'}{q'}} dt\right)^{\frac{1}{r'}} \\
&\qquad \le \frac{1}{2}\int_E u^2(x,0)dx + \gamma\left[\|\mathbf{f}\|_{2;E_{t_1}} + \|f\|_{q,r;E_{t_1}}\right]\|u\|_{V^2(E_{t_1})},
\end{aligned}$$

since by Proposition 1.3 we have $u \in L^{q',r'}(E_T)$ and $\|u\|_{q',r';E_{t_1}} \le \gamma\|u\|_{V^2(E_{t_1})}$. Hence,

$$\|u\|^2_{V^2(E_{t_1})} \le \tilde{\gamma}\Big[\|u_o\|_{2;E} + \|\mathbf{f}\|_{2;E_{t_1}} + \|f\|_{q_1,r_1;E_{t_1}}\Big]\|u\|_{V^2(E_{t_1})}.$$

Now, choose $t_1 = T$ on the right-hand side, and by the arbitrariness of t_1 take the supremum on the left-hand side, to conclude.

5 Existence of Solutions of the Homogeneous Cauchy–Dirichlet Problem (3.1) by Galerkin Approximations

We finally come to the solvability of the homogeneous Dirichlet problem (3.1).

Theorem 5.1. *If conditions* (3.2)–(3.3) *are satisfied, then Problem* (3.1) *has a solution in* $V_o^2(E_T)$.

Proof. Let $\{w_k\}$ be a countable, complete, orthonormal system for $W_o^{1,2}(E)$. The existence of a such a system follows by the same argument as in Section 7 of Chapter 9. For the sake of simplicity, we assume it to be normalized with respect to the inner product in $L^2(E)$, that is $(w_i, w_j) = \delta_{ij}$. We look for an approximate solution u^M of (3.1) of the form

$$u^M(x,t) = \sum_{k=1}^{M} c_k^M(t) w_k(x),$$

where the coefficients $c_k^M(t) = (u^M(x,t), w_k(x))$ are determined from the Cauchy problem

$$\frac{d}{dt}\int_E u^M w_k dx + \int_E a_{ij} u^M_{x_j} w_{k,x_i} dx = \int_E (f_i w_{k,x_i} + f w_k)\, dx, \tag{5.1}$$

$$c_k^M(0) = \int_E u_o w_k dx, \tag{5.2}$$

with $k = 1, \dots, M$. Conditions (5.1) are a system of linear ordinary differential equations, which we can rewrite as

$$\frac{d}{dt} c_k^M(t) + A_{kl} c_l^M(t) = F_k(t), \quad k = 1, \dots, M, \tag{5.3}$$

where

$$A_{kl} = \int_E a_{ij} w_{k,x_i} w_{l,x_j}\, dx,$$

$$F_k = \int_E (f_i w_{k,x_i} + f w_k)\, dx,$$

and both belong to $L^2(0,T)$, as can be easily seen from (2.1) and (3.2)–(3.3). In Section 5c we discuss a general result, which guarantees that under these conditions, (5.3)–(5.2) have a unique solution; we can then conclude that the approximate solutions u^M are uniquely determined for any M.

Let us first show that the coefficients c_k^M are equibounded. If we multiply each of (5.1) by c_k^M, sum what we have obtained over k from 1 to M, and integrate with respect to t from 0 to t_1 for any $t_1 \in (0,T)$, we obtain

$$\frac{1}{2}\int_E |u^M|^2(x,t) dx\Big|_0^{t_1} + \int_0^{t_1}\int_E a_{ij} u^M_{x_j} u^M_{x_i} dx\, dt = \int_0^{t_1}\int_E (f_i u^M_{x_i} + f u^M) dx\, dt.$$

We can then work as in the proof of the Energy Inequality of Section 4 and conclude that

$$\|u^M\|_{V^2(E_T)} \le \gamma\Big[\|\mathbf{f}\|_{2;E_T} + \|f\|_{q,r;E_T} + \|u_o\|_{2;E}\Big], \tag{5.4}$$

where γ depends only on N, λ, q, r, and we have taken into account that

$$\|u^M(x,0)\|^2_{2;E} = \sum_{k=1}^{M} |c_k^M(0)|^2 \le \|u_o\|^2_{2;E}.$$

From (5.4) it follows that the coefficients $c_k^M = c_k^M(t)$ are all bounded by the same quantity for any k and for any M.

Let us now prove that for fixed k and arbitrary $M \ge k$ the coefficients c_k^M are equicontinuous. It suffices to show that

$$|c_k^M(t+h) - c_k^M(t)| \le \epsilon(h,k),$$

where $\epsilon(h,k)$ tends to zero as $h \to 0$, and depends on h and k, but does not depend on M. From (5.1)

$$|c_k^M(t+h) - c_k^M(t)| \le \int_t^{t+h} \int_E \Big[|a_{ij} u^M_{x_j} w_{k,x_i}| + |f_i w_{k,x_i}| + |f w_k| \Big] dx\,dt.$$

By (2.1) and (3.2)–(3.3) we have

$$\begin{aligned}
&\int_t^{t+h} \int_E |a_{ij} u^M_{x_j} w_{k,x_i}| dx\,dt \\
&\le \Lambda \left[\int_t^{t+h} \int_E |\nabla u^M|^2 dx\,dt \right]^{\frac{1}{2}} \left[\int_E |\nabla w_k|^2 dx \right]^{\frac{1}{2}} h^{\frac{1}{2}},
\end{aligned}$$

$$\int_t^{t+h} \int_E |f_i w_{k,x_i}| dx\,dt \le \left[\int_t^{t+h} \int_E \sum_{i=1}^{N} |f_i|^2 dx\,dt \right]^{\frac{1}{2}} \left[\int_E |\nabla w_k|^2 dx \right]^{\frac{1}{2}} h^{\frac{1}{2}},$$

$$\int_t^{t+h} \int_E |f w_k| dx\,dt \le \gamma \left[\int_t^{t+h} \left(\int_E |f|^q dx \right)^{\frac{r}{q}} dt \right]^{\frac{1}{r}} \left[\int_E |\nabla w_k|^2 dx \right]^{\frac{1}{2}} h^{\frac{1}{r'}}.$$

Since all the integrals on the right-hand sides tend to zero as h tends to zero, we conclude that

$$|c_k^M(t+h) - c_k^M(t)| \le \epsilon(h) \|\nabla w_k\|_{2;E}.$$

We can now apply the Ascoli–Arzelà Theorem, and for any fixed k, select a subsequence $\{c_k^{M_i}\}$ that converges uniformly on $[0,T]$ to some continuous function c_k, $k \ge 1$ as M_i tends to ∞. Notice that, in general, the sequence $\{M_i\}$ depends on k. However, such a dependence can be dispensed with, owing to a diagonal selection process.

With these coefficients c_k we build a function

$$u(x,t) = \sum_{k=1}^{\infty} c_k(t) w_k(x).$$

We claim that $u \in V_o^2(E_T)$ and is a weak solution of (3.1).

If we let

$$u^{M_i} = \sum_{k=1}^{M_i} c_k^{M_i} w_k,$$

it is matter of straightforward computations to check that, as $M_i \to \infty$,

$$\begin{aligned} u^{M_i} &\to u \quad \text{weakly in } \ L^2(E) \ \text{ uniformly with respect to } t, \\ u^{M_i} &\to u \quad \text{weakly in } \ L^2(E_T), \\ \nabla u^{M_i} &\to \nabla u \quad \text{weakly in } \ L^2(E_T). \end{aligned}$$

Moreover, by the properties of the weak convergence, u will satisfy

$$\|u\|_{V^2(E_T)} \le \gamma \Big[\|\mathbf{f}\|_{2;E_T} + \|f\|_{q,r;E_T} + \|u_o\|_{2;E} \Big],$$

just as u^M, and we conclude that $u \in V_o^2(E_T)$.

In order to show that u is a weak solution, multiply each (5.1) by a smooth function $d_k(t)$, which vanishes for $t = T$, sum over k from 1 to $M' \le M$, and integrate with respect to t from 0 to T. If we let

$$\varPhi^{M'}(x,t) = \sum_{k=1}^{M'} d_k(t) w_k(x),$$

after an integration by parts we have

$$\begin{aligned} \int_0^T \int_E \Big[- u^M \varPhi_t^{M'} + a_{ij} u_{x_j}^M \varPhi_{x_i}^{M'} \Big] dx\, dt = \int_0^T \int_E \Big(f_i \varPhi_{x_i}^{M'} + f \varPhi^{M'} \Big)\, dxdt \\ + \int_E u^M(x,0) \varPhi^{M'}(x,0) dx. \end{aligned}$$

By the weak convergence of the subsequence $\{u^{M_i}\}$ studied above, we can pass to the limit with respect to M and obtain

$$\begin{aligned} \int_0^T \int_E \Big[- u \varPhi_t^{M'} + a_{ij} u_{x_j} \varPhi_{x_i}^{M'} \Big] dx\, dt = \int_0^T \int_E \Big(f_i \varPhi_{x_i}^{M'} + f \varPhi^{M'} \Big)\, dxdt \\ + \int_E u_o(x) \varPhi^{M'}(x,0) dx, \end{aligned}$$

where we have taken into account that

$$\int_E u^M(x,0) \varPhi^{M'}(x,0) dx \ \to \ \int_E u_o(x) \varPhi^{M'}(x,0) dx.$$

Now, by Lemma 4.12 of Chapter II of Ladyzhenskaya et al. [151], the functions $\varPhi^{M'}$ are dense in the space of all functions $\varphi \in W_o^2(E_T)$, which vanish at $t = T$. Therefore, by density we can conclude that

$$\begin{aligned}\int_0^T\int_E\Big[-u\varphi_t+a_{ij}u_{x_j}\varphi_{x_i}\Big]dx\,dt=\int_0^T\int_E(f_i\varphi_{x_i}+f\varphi)\,dxdt&\\+\int_E u_o(x)\varphi(x,0)dx,&\end{aligned}\tag{5.5}$$

for all $\varphi\in W^2_o(E_T)$, which vanish at $t=T$. By Proposition 3.1 we conclude that u is a weak solution. ■

Proposition 5.1 *Under the same assumptions as in Theorem 5.1, any solution* $u\in V^2_o(E_T)$ *of Problem* (3.1) *is strongly continuous in* t *in the* $L^2(E)$ *norm.*

Proof. By the previous construction, $u\in L^\infty(0,T;L^2(E))$; now we show that it is actually more regular. We let

$$F_i=f_i-a_{ij}u_{x_j},$$

so that we can rewrite (5.5) as

$$-\int_0^T\int_E u\varphi_t dx\,dt-\int_E u_o(x)\varphi(x,0)dx=\int_0^T\int_E(F_i\varphi_{x_i}+f\varphi)\,dx\,dt.$$

We now extend u, F_i, f onto the infinite cylinder $E_\infty=E\times(-\infty,+\infty)$ by setting

$$u^*=\begin{cases}u(x,t) & t\in[0,T]\\ u(x,-t) & t\in[-T,0)\\ 0 & |t|>T\end{cases},$$

$$F_i^*=\begin{cases}F_i(x,t) & t\in[0,T]\\ -F_i(x,-t) & t\in[-T,0)\\ 0 & |t|>T\end{cases},$$

$$f^*=\begin{cases}f(x,t) & t\in[0,T]\\ -f(x,-t) & t\in[-T,0)\\ 0 & |t|>T\end{cases}.$$

By the previous definitions we have

$$-\iint_{E_\infty}u^*\varphi_t\,dx\,dt=\iint_{E_\infty}(F_i^*\varphi_{x_i}+f^*\varphi)\,dx\,dt$$

for any $\varphi\in W^2_o(E_T)$, which vanishes for $|t|\ge T$. Choosing $\omega=\omega(t)$, a smooth function that equals 1 for $t\in[-(T-\delta),T-\delta]$ and 0 for $|t|\ge T$, and setting $\varphi(x,t)=\omega(t)\Phi(x,t)$, where $\Phi\in W^2_o(E_\infty)$, yields

$$-\iint_{E_\infty}u^*\omega\Phi_t\,dx\,dt=\iint_{E_\infty}(F_i^*\omega\Phi_{x_i}+f^*\omega\Phi+u^*\omega_t\Phi)\,dx\,dt.$$

If we let $v = u^*\omega$, then

$$-\iint_{E_\infty} v\Phi_t \, dx\, dt = \iint_{E_\infty} (F_i^*\omega\Phi_{x_i} + (f^*\omega + u^*\omega_t)\Phi)\, dx\, dt.$$

Consider $\eta = \eta(x,t)$ a function in $L^2(0,T; W_o^{1,2}(E))$ and in the above identity choose as Φ the Steklov average $\eta_{\bar{h}}$. Shifting the average from the second factor to the first one, as we have already done in the proof of the Energy Inequality of Section 4, yields

$$-\iint_{E_\infty} v_h\eta_t \, dx\, dt = \iint_{E_\infty} [(F_i^*\omega)_h\eta_{x_i} + (f^*\omega + u^*\omega_t)_h\eta]\, dx\, dt.$$

Let $\eta(x,t) = \chi(t)\zeta(x)$, where χ is a smooth function of t, and $\zeta \in W_o^{1,2}(E)$. We can then rewrite

$$\begin{aligned} -\int_{-\infty}^{\infty} \chi_t \, dt \int_E v_h(x,t)\zeta(x) dx & \\ &= \int_{-\infty}^{\infty} \chi dt \int_E [(F_i^*\omega)_h\zeta_{x_i} + (f^*\omega + u^*\omega_t)_h\zeta]\, dx, \end{aligned}$$

$$-\int_{-\infty}^{\infty} \chi_t(v_h,\zeta)dt = \int_{-\infty}^{\infty} \chi\, [((F_i^*\omega)_h, \zeta_{x_i}) + ((f^*\omega + u^*\omega_t)_h, \zeta)]\, dt,$$

where $(\cdot,\cdot)$ stands for the inner product in $L^2(E)$. Taking into account the notion of distributional derivative,

$$\frac{d}{dt}(v_h,\zeta) = ((F_i^*\omega)_h, \zeta_{x_i}) + ((f^*\omega + u^*\omega_t)_h, \zeta)\,.$$

By its definition, v_h is strongly continuous in the $L^2(E)$-norm with respect to time, and therefore, a fortiori, the inner product (v_h,ζ) is continuous in time. Moreover, for almost all $t \in (-\infty,\infty)$,

$$\frac{d}{dt}(v_h,\zeta) = (v_{h,t},\zeta)$$

and

$$(v_{h,t},\zeta) = ((F_i^*\omega)_h, \zeta_{x_i}) + ((f^*\omega + u^*\omega_t)_h, \zeta)\,.$$

Choosing two different values h_1, h_2 gives

$$\begin{aligned} &(v_{h_1,t} - v_{h_2,t}, \zeta) \\ &= ((F_i^*\omega)_{h_1} - (F_i^*\omega)_{h_2}, \zeta_{x_i}) + ((f^*\omega + u^*\omega_t)_{h_1} - (f^*\omega + u^*\omega_t)_{h_2}, \zeta)\,. \end{aligned}$$

Taking $\zeta = v_{h_1} - v_{h_2}$, which is an admissible choice owing to the space regularity of v_h, and integrating with respect to time in an arbitrary interval $[t_1, t_2]$, gives

$$\frac{1}{2}\|v_{h_1} - v_{h_2}\|_{2;E}\Big|_{t_1}^{t_2} = \int_{t_1}^{t_2} ((F_i^*\omega)_{h_1} - (F_i^*\omega)_{h_2}, \zeta_{x_i})\, dt \\ + \int_{t_1}^{t_2} ((f^*\omega + u^*\omega_t)_{h_1} - (f^*\omega + u^*\omega_t)_{h_2}, \zeta)\, dt.$$

The right-hand side tends to zero as both h_1 and h_2 tend to zero. Therefore, choosing $t_1 = -\infty$ and t_2 arbitrary, it follows that we have strong convergence in $L^2(E)$, as $h \to 0$, uniformly in $t \in (-\infty, \infty)$. Consequently, the limit function v is equivalent to a strongly continuous function in the $L^2(E)$-norm with respect to t, and by the definition of v, the same must hold for u.

Notice that, strictly speaking, we have proved the continuity up to $T - \delta$, with $\delta > 0$, but this is immaterial, since we could first extend u to $T + \delta$, and then apply the previous argument to this extension. ■

6 Uniqueness of Solutions of the Homogeneous Cauchy–Dirichlet Problem (3.1)

Theorem 6.1. *The homogeneous Cauchy–Dirichlet problem* (3.1) *provided with conditions* (3.2)–(3.3) *admits at most one weak solution* $u \in V_o^2$.

Proof. It is a direct consequence of Theorem 5.1. Assume by contradiction that two solutions u_1 and u_2 exist, with $u_1 \not\equiv u_2$. By the linearity of the homogeneous Cauchy–Dirichlet problem, their difference $u \in \tilde{V}_o^2(E_T)$ would satisfy Problem (3.1) with $\mathbf{f} = f = u_o = 0$. By (4.1), it follows that $u \equiv 0$. ■

7 Traces of Functions on $\Sigma \overset{\text{def}}{=} \partial E \times (0, T]$

In Chapter 9 we characterized the traces on ∂E of functions in $W^{1,p}(E)$, and at the same time we studied the extension from ∂E to E of functions defined in proper Sobolev spaces. Such results were instrumental in solving the elliptic inhomogeneous Dirichlet Problem.

In order to deal with the same problem in the parabolic context, we need to consider similar trace and extension theorems in space-time cylinders and on their lateral boundary. Given a bounded set E with boundary ∂E, we denote the lateral boundary $\partial E \times (0, T)$ of the cylinder E_T with Σ.

We are not going to present the theory in full generality. Here, we concentrate on an approach limited to Hilbert spaces, as developed, for example, in Chapter 4 of Lions and Magenes [176]. For related comments, see Section 7c in the Complements.

In order to properly define the spaces we are interested in, we need some introductory material. For $\theta \in (0, 1)$, consider the so-called Slobodeckij seminorm, which we have already briefly considered in Section 8.3 of Chapter 9. We let

$$[f]_{\theta;E} \overset{\text{def}}{=} \left(\int_E \int_E \frac{|f(x)-f(y)|^2}{|x-y|^{N+2\theta}} dx\, dy \right)^{\frac{1}{2}}.$$

We have already introduced Sobolev spaces of higher, integer order in Section 1c of Chapter 9. We now extend the definition. Indeed, for $r > 0$, $r \notin \mathbb{N}$, let $\lfloor r \rfloor$ be its integer part, and set $\theta = r - \lfloor r \rfloor$. The space $W^{r,2}(E)$ is defined as

$$W^{r,2} \overset{\text{def}}{=} \left\{ f \in W^{\lfloor r \rfloor,2}(E) : \sup_{|\alpha|=\lfloor r \rfloor} [D^\alpha f]_{\theta;E} < \infty \right\}.$$

It is a Hilbert space endowed with the norm

$$\|f\|_{r,2;E} = \left(\|f\|^2_{\lfloor r \rfloor,2;E} + \left(\sup_{|\alpha|=\lfloor r \rfloor} [D^\alpha f]_{\theta;E} \right)^2 \right)^{\frac{1}{2}}.$$

For a given Hilbert space H, we now consider $W^{s,2}(0,T;H)$: if $s \in \mathbb{N}$, then

$$W^{s,2}(0,T;H) = \{ v \in L^2(0,T;H) : \frac{\partial v}{\partial t}, \dots, \frac{\partial^s v}{\partial t^s} \in L^2(0,T;H) \},$$

and if $s > 0$ but $s \notin \mathbb{N}$, as we have just done for $W^{r,s}(E)$, we let $k \overset{\text{def}}{=} \lfloor s \rfloor$, and define

$$W^{s,2}(0,T;H) = \Big\{ v \in W^{k,2}(0,T;H) : \\ \int_0^T \int_0^T \frac{\left\| \frac{\partial^k v(\cdot,t)}{\partial \sigma^k} - \frac{\partial^k v(\cdot,\tau)}{\partial \sigma^k} \right\|^2_{2;H}}{|t-\tau|^{1+2(s-k)}} dt\, d\tau < \infty \Big\},$$

where $\frac{\partial^k v(\cdot,t)}{\partial \sigma^k}$ and $\frac{\partial^k v(\cdot,\tau)}{\partial \sigma^k}$ stand for time derivatives of order k of v, evaluated respectively at time t and at time τ. Finally, let r and s be two non-negative real numbers. For a bounded open set E, whose boundary ∂E is of class $C^{1,1}$ and satisfies the segment property (see Section 8.1 of Chapter 9) we define

$$H^{r,s}(E_T) = L^2(0,T;W^{r,2}(E)) \cap W^{s,2}(0,T;L^2(E)),$$

which is a Hilbert space with the norm

$$\|u\|_{H^{r,s}(E_T)} = \left(\int_0^T \|u(t)\|^2_{W^{r,2}(E)} dt + \|u\|^2_{W^{s,2}(0,T;L^2(E))} \right)^{\frac{1}{2}}.$$

The previous definition extends in a straightforward way to the case where E_T is replaced by its lateral boundary Σ: it suffices to use the spaces $L^2(\partial E)$ and $W^{r,2}(\partial E)$, instead of $L^2(E)$ and $W^{r,2}(E)$. Therefore,

$$H^{r,s}(\Sigma) \overset{\text{def}}{=} L^2(0,T;W^{r,2}(\partial E)) \cap W^{s,2}(0,T;L^2(\partial E)).$$

As in the elliptic case, it turns out that the differential properties of the boundary values of functions from spaces $H^{r,s}(E_T)$ and of certain derivatives of theirs can be exactly described in terms of the spaces $H^{\mu_j,\nu_j}(\Sigma)$. Let $\frac{\partial}{\partial n}$ denote the normal derivative on Σ oriented toward the interior of E_T. We have

Theorem 7.1. *For* $u \in H^{r,s}(E_T)$ *with* $r > \frac{1}{2}$, $s \geq 0$, *we may define* $\dfrac{\partial^j u}{\partial n^j}$ *on* Σ *if* $j < r - \frac{1}{2}$, $j \geq 0$ *is an integer; moreover,* $\dfrac{\partial^j u}{\partial n^j} \in H^{\mu_j,\nu_j}(\Sigma)$, *where*

$$\frac{\mu_j}{r} = \frac{\nu_j}{s} = \frac{r-j-\frac{1}{2}}{r}, \quad \textit{and} \;\; \nu_j = 0 \;\; \textit{if}\; s = 0. \tag{7.1}$$

Finally, the mapping $u \to \dfrac{\partial^j u}{\partial n^j}$ *is linear and continuous from* $H^{r,s}(E_T)$ *to* $H^{\mu_j,\nu_j}(\Sigma)$.

Similarly, for $s > \frac{1}{2}$, $r \geq 0$ *we may define* $\dfrac{\partial^k u}{\partial t^k}$ *on* E *if* $k < s - \frac{1}{2}$, $k \geq 0$ *is an integer, and* $\dfrac{\partial^k u}{\partial t^k}(x,0) \in W^{p_k,2}(E)$, *where*

$$p_k = \frac{r}{s}\left(s - k - \frac{1}{2}\right).$$

The mapping $u \to \dfrac{\partial^k u}{\partial t^k}(x,0)$ *is linear and continuous from* $H^{r,s}(E_T)$ *to* $H^{p_k}(E)$.

Conversely, given $r > \frac{1}{2}$ *and* $s \geq 0$, $\mu_j > 0$ *and* $\nu_j \geq 0$, *which satisfy* (7.1), *and* $v \in H^{\mu_j,\nu_j}(\Sigma)$, *there exists* $u \in H^{r,s}(E_T)$ *such that* $\dfrac{\partial^j u}{\partial n^j} = v$.

Proof. See Section 7c of the Complements. ■

Remark 7.1 We omit any extension with regard to the value of u on E, since in the applications to come we are only interested in the boundary values.

8 The Inhomogeneous Dirichlet Problem

Assume that ∂E is of class $C^{1,1}$ and satisfies the segment property. Given $\mathbf{f}$ and f satisfying (3.2)–(3.3) and $\varphi \in H^{\frac{1}{2},\frac{1}{2}}(\Sigma)$, consider the Cauchy–Dirichlet problem

$$\begin{aligned} u_t - \left(a_{ij}u_{x_j}\right)_{x_i} &= f - \operatorname{div}\mathbf{f} && \text{in } E_T \\ u(x,t) &= \varphi && \text{on } \Sigma \\ u(\cdot,0) &= u_o && \text{in } E. \end{aligned} \tag{8.1}$$

By Theorem 7.1, if we take $r = s = 1$, $j = k = 0$, and $\mu_j = \nu_j = \frac{1}{2}$, there exists $v \in H^{1,1}(E_T)$ such that

$$v \in L^2(E_T), \quad v_t \in L^2(E_T), \quad \nabla v \in L^2(E_T), \quad v(\cdot,0) \in W^{\frac{1}{2},2}(E), \quad v\Big|_{\Sigma} = \varphi.$$

Let $v_o \overset{\mathrm{def}}{=} v(\cdot,0)$. A solution of (8.1) is sought of the form $u = w+v$, where $w \in V_o^2(E_T)$ is the unique weak solution of the auxiliary, homogeneous Cauchy–Dirichlet problem

$$\begin{aligned} w_t - \big(a_{ij}w_{x_j}\big)_{x_i} &= \tilde{f} - \operatorname{div}\tilde{\mathbf{f}} && \text{in } E_T \\ w(x,t) &= 0 && \text{on } \Sigma \\ w(\cdot,0) &= w_o && \text{in } E, \end{aligned}$$

where

$$\tilde{f} = f - v_t, \quad \tilde{f}_j = f_j - a_{ij}v_{x_i}, \quad w_o = u_o - v_o.$$

Since $v_t \in L^2(E_T)$, if we choose q, r as in (3.3), we have

$$\|v_t\|_{q,r;E_T} \le T^{1-\frac{r}{2}}|E|^{\frac{1}{r}-\frac{q}{2r}}\|v_t\|_{2;E},$$

and therefore, $\tilde{f} \in L^{q,r}(E_T)$; in a similar way, we have $\tilde{\mathbf{f}} \in L^2(E_T)$, and $w_o \in L^2(E)$. Hence, we conclude

Theorem 8.1. *Assume that ∂E is of class $C^{1,1}$ and satisfies the segment property. For every* $\mathbf{f}$ *and* f *satisfying* (3.2)–(3.3) *and $\varphi \in H^{\frac{1}{2},\frac{1}{2}}(\Sigma)$, the inhomogeneous Cauchy–Dirichlet problem (8.1) has a unique weak solution $u \in V^2(E_T)$.*

Remark 8.1 The class $H^{\frac{1}{2},\frac{1}{2}}(\Sigma)$ where the boundary datum φ is assumed, is not the most general one. In this context we are not interested in the minimal regularity assumptions on the data, which ensure the existence of a unique solution.

9 The Neumann Problem

Assume that ∂E is of class C^1 and satisfies the segment property. Given $\mathbf{f}$ and f satisfying (3.2)–(3.3), consider the *formal* Cauchy–Neumann problem

$$\begin{aligned} u_t - \big(a_{ij}u_{x_j}\big)_{x_i} &= f - \operatorname{div}\mathbf{f} && \text{in } E_T \\ (a_{ij}u_{x_i} - f_j)\,n_j &= \psi && \text{on } \Sigma \\ u(\cdot,0) &= u_o && \text{in } E, \end{aligned} \tag{9.1}$$

where $\mathbf{n} = (n_1,\dots,n_N)$ is the outward unit normal to ∂E and $\psi \in L^{q_2,r_2}(\Sigma)$ for some proper (q_2,r_2). If $a_{ij} = \delta_{ij}$, $\mathbf{f} = f = 0$, and ψ is sufficiently regular, this is precisely the Neumann problem (1.3) of Chapter 5. Since $a_{ij} \in L^\infty(E_T)$ and $f \in L^{q,r}(E_T)$, $\mathbf{f} \in L^2(E_T)$, neither the PDE nor the boundary condition in (9.1) are well defined in the classical sense, and they have to be interpreted in some weak form, as we have just done for the Cauchy–Dirichlet problem.

We first make proper hypotheses on the integrability of ψ, namely we assume that

$$\psi \in L^{q_2,r_2}(\Sigma), \qquad \text{where } \frac{1}{r_2} + \frac{N-1}{2q_2} = \frac{N}{4} + \frac{1}{2} \qquad \text{and}$$
$$\begin{cases} q_2 \in \left[\frac{2(N-1)}{N}, \frac{2(N-1)}{N-2}\right], \ r_2 \in [1,2], & \text{if } N > 2 \\ q_2 > 1, \ r_2 \in [1,2) & \text{if } N = 2 \\ q_2 = \frac{4}{3}, \ r_2 = \frac{4}{3}, & \text{if } N = 1. \end{cases} \tag{9.2}$$

By a weak solution of the Cauchy–Neumann problem (9.1), under assumptions (3.2)–(3.3) and (9.2), we mean a function $u \in V^2(E_T)$ satisfying the identity

$$\begin{aligned} -\iint_{E_T} u\varphi_t \,dx\,dt + \iint_{E_T} a_{ij}u_{x_j}\varphi_{x_i} dx\,dt &= \iint_{E_T} (f\varphi + f_i\varphi_{x_i})dx\,dt \\ &+ \iint_{\Sigma} \psi\varphi \,d\sigma\,dt + \int_E u_o(x)\varphi(x,0)dx \end{aligned} \tag{9.3}$$

for any function $\varphi \in W^2(E_T)$, which vanishes for $t = T$. We formally obtain the previous relation, by multiplying the first of (9.1) by φ, performing integration by parts with respect to x and t, and taking into account the second and third of (9.1). It is easy to check that by (9.2), the surface integral $\iint_\Sigma \psi\varphi \,d\sigma\,dt$ is well defined, whereas for the remaining terms, we are under the same conditions that we considered for the Cauchy–Dirichlet problem.

As we did before, we can give an equivalent definition of a weak solution, requiring that $u \in V^2(E_T)$ satisfies

$$\begin{aligned} &\int_E u(x,\tau)\varphi(x,\tau)dx - \int_0^\tau \int_E u\varphi_t dx\,dt + \int_0^\tau \int_E a_{ij}u_{x_j}\varphi_{x_i} dx\,dt \\ &= \int_0^\tau \int_E (f\varphi + f_i\varphi_{x_i})dx\,dt + \int_0^\tau \int_{\partial E} \psi\varphi d\sigma\,dt + \int_E u_o(x)\varphi(x,0)dx \end{aligned} \tag{9.4}$$

for almost any $\tau \in (0,T)$ and for any function $\varphi \in W^2(E_T)$.

In order to prove the existence of a weak solution, we apply Galerkin's method. In this instance, we require $\{w_k\}$ to be a countable, complete system for $W^{1,2}(E)$ (not for $W_o^{1,2}(E)$ as for the homogeneous Cauchy–Dirichlet problem), and we assume that $(w_i, w_j) = \delta_{ij}$, where $(\cdot,\cdot)$ is the inner product in $L^2(E)$.

We look for an approximate solution $u^M = u^M(x,t)$, given by

$$u^M(x,t) = \sum_{k=1}^{M} c_k^M(t) w_k(x),$$

where the coefficients $c_k^M(t) = (u^M, w_k)$ are determined from the Cauchy problem

$$\frac{d}{dt}\int_E u^M w_k dx + \int_E a_{ij} u_{x_j}^M w_{k,x_i} dx = \int_E (f_i w_{k,x_i} + f w_k) dx + \int_{\partial E} \psi w_k d\sigma,$$
$$c_k^M(0) = \int_E u_o w_k dx.$$

Then, we proceed as with the homogeneous Cauchy–Dirichlet problem, and we conclude that there exists at least one weak solution $u \in V^2(E_T)$. As before, we also have that $u \in \tilde{V}^2(E_T)$. Therefore, we have proved

Theorem 9.1. *Let ∂E be of class C^1 and satisfy the segment property. Let $\mathbf{f}$ and f satisfy* (3.2)–(3.3), *and ψ satisfy (9.2). Then, the Cauchy–Neumann problem (9.1) admits a weak solution $u \in \tilde{V}^2(E_T)$.*

9.1 The Energy Inequality for the Neumann Problem

Proposition 9.1 *Let ∂E be of class C^1 and satisfy the segment property. Let $u \in \tilde{V}^2(E_T)$ be a solution of the Cauchy–Neumann problem* (9.1) *with f, $\mathbf{f}$ satisfying* (3.2)–(3.3), *and ψ satisfying* (9.2). *Then, the* Energy Inequality

$$\begin{aligned} \|u\|_{V^2(E_T)} \le \gamma \Bigg[& \|u_o\|_{2;E} + \|\mathbf{f}\|_{2;E_T} + \left(1 + \frac{T}{|E|^{\frac{2}{N}}}\right)^{\frac{1}{r'}} \|f\|_{q,r;E_T} \\ & + \left(1 + \left(\frac{T}{|E|^{\frac{2}{N}}}\right)^{\frac{1}{r_2'} - \frac{1}{2}}\right) \|\psi\|_{q_2,r_2;\Sigma} \Bigg] \end{aligned} \tag{9.5}$$

holds true, where $\gamma > 0$ depends only on N, λ, q, r, q_2, r_2.

Proof. In (9.4) take as test function the Steklov average

$$\varphi_{\bar{h}}(x,t) = \frac{1}{h}\int_{t-h}^{h} \hat{\varphi}(x,\tau) d\tau,$$

where $\hat{\varphi}$ is an arbitrary element of $W^2(E_T)$, which vanishes for $t \ge T - h$ and for $t \le 0$. The regularity of all the involved quantities allows us to work as in Sections 3–4, so that for any $t_1 \in (0, T]$ we obtain

$$\begin{aligned} \frac{1}{2}\int_E u^2(x,t)dx\Big|_{t=0}^{t=t_1} &+ \iint_{E_{t_1}} a_{ij} u_{x_j} u_{x_i} dx\, dt \\ &= \iint_{E_{t_1}} (fu + f_i u_{x_i}) dx\, dt + \int_0^{t_1} \int_{\partial E} \psi u \, d\sigma\, dt. \end{aligned}$$

The ellipticity condition (2.1) and Hölder's inequality yield

$$\begin{aligned}
&\frac{1}{2}\int_E u^2(x,t_1)\,dx+\lambda\iint_{E_{t_1}}|\nabla u|^2 dx\,dt\\
\leq&\frac{1}{2}\int_E u_o^2\,dx+\left(\iint_{E_{t_1}}|\mathbf{f}|^2 dx\,dt\right)^{\frac{1}{2}}\left(\iint_{E_{t_1}}|\nabla u|^2 dx\,dt\right)^{\frac{1}{2}}\\
&+\left(\int_0^{t_1}\left(\int_E|f|^q\,dx\right)^{\frac{r}{q}}dt\right)^{\frac{1}{r}}\left(\int_0^{t_1}\left(\int_E|u|^{q'}dx\right)^{\frac{r'}{q'}}dt\right)^{\frac{1}{r'}}\\
&+\left(\int_0^{t_1}\left(\int_{\partial E}|\psi|^{q_2}d\sigma\right)^{\frac{r_2}{q_2}}dt\right)^{\frac{1}{r_2}}\left(\int_0^{t_1}\left(\int_{\partial E}|u|^{q_2'}\,d\sigma\right)^{\frac{r_2'}{q_2'}}dt\right)^{\frac{1}{r_2'}}\\
=&I+II+III+IV.
\end{aligned}$$

We now proceed with the estimates of I, II, III, IV; relying on Propositions 1.4 and 1.5, and on the first of (9.2) gives

$$\begin{aligned}
I&\leq\frac{1}{2}\|u_o\|_{2;E}\|u\|_{V^2(E_T)};\\
II&\leq\|\mathbf{f}\|_{2;E_T}\|u\|_{2;E_T};\\
III&\leq\|f\|_{q,r;E_T}\|u\|_{q',r';E_T}\leq\gamma\left(1+\frac{T}{|E|^{\frac{2}{N}}}\right)^{\frac{1}{r'}}\|f\|_{q,r;E_T}\|u\|_{V^2(E_T)};\\
IV&\leq\|\psi\|_{q_2,r_2;\Sigma}\|u\|_{q_2',r_2';\Sigma}\leq\gamma\left(1+\left(\frac{T^{\frac{N}{2}}}{|E|}\right)^{\frac{1}{2}-\frac{N-1}{Nq_2'}}\right)\|\psi\|_{q_2,r_2;\Sigma}\|u\|_{V^2(E_T)}\\
&=\gamma\left(1+\left(\frac{T}{|E|^{\frac{2}{N}}}\right)^{\frac{1}{r_2'}-\frac{1}{2}}\right)\|\psi\|_{q_2,r_2;\Sigma}\|u\|_{V^2(E_T)}.
\end{aligned}$$

Collecting all these estimates and taking the supremum over t_1 on the left-hand side, we conclude. ■

As a direct consequence of Proposition 9.1 we have the following.

Theorem 9.2. *The Cauchy–Neumann problem* (9.1) *provided with conditions* (9.2) *admits at most one weak solution* $u\in V^2$.

Proof. Assume by contradiction that two solutions u_1 and u_2 exist, with $u_1\not\equiv u_2$. By the linearity of the Cauchy–Neumann problem, their difference $u\in\tilde{V}^2(E_T)$ would satisfy Problem (9.1) with $\mathbf{f}=f=u_o=\psi=0$. By (9.5), it follows that $u\equiv 0$. ■

9.2 A Variant of Problems (3.1) and (9.1)

We can also consider the following mixed problem

$$\begin{aligned} u_t - \big(a_{ij}u_{x_j}\big)_{x_i} &= f - \operatorname{div}\mathbf{f} && \text{in } E_T \\ (a_{ij}u_{x_i} - f_j)\, n_j &= \psi && \text{on } \Sigma_1 \\ u &= 0 && \text{on } \Sigma_2 \\ u(\cdot,0) &= u_o && \text{in } E, \end{aligned} \tag{9.6}$$

where ∂E is of class C^1 and satisfies the segment property, $\Sigma_1 = (\partial E)_1 \times (0,T)$, $\Sigma_2 = (\partial E)_2 \times (0,T)$, with $(\partial E)_1 \cup (\partial E)_2 = \partial E$, and $(\partial E)_1 \cap (\partial E)_2 = \emptyset$, $\mathbf{f}$ and f satisfy (3.2)–(3.3), and ψ satisfies (9.2).

We let

$$W^{1,2}_{\gamma_o(\partial E)_2}(E) = \{v \in W^{1,2}(E) : \ \gamma_o v = 0 \ \text{ on } \ (\partial E)_2\},$$

where $\gamma_o v$ is the trace of v on ∂E.

We say that

$$u \in C([0,T]; L^2(E)) \cap L^2(0,T; W^{1,2}_{\gamma_o(\partial E)_2}(E))$$

is a weak solution of (9.6) if

$$\begin{aligned} -\iint_{E_T} u\varphi_t \, dx\, dt + \iint_{E_T} a_{ij}u_{x_j}\varphi_{x_i} dx\, dt &= \iint_{E_T} (f\varphi + f_i\varphi_{x_i})\, dx\, dt \\ &\quad + \iint_{\Sigma_1} \psi\varphi \, d\sigma\, dt + \int_E u_o(x)\varphi(x,0)dx \end{aligned}$$

for any function

$$\varphi \in W^{1,2}([0,T]; L^2(E)) \cap L^2(0,T; W^{1,2}_{\gamma_o(\partial E)_2}(E)),$$

which vanishes for $t = T$. This can be seen as a combination of (3.5) and (9.3). An equivalent formulation can be similarly obtained combining (3.4) and (9.4). The existence of a weak solution is proved by Galerkin's method, the only difference being in the choice of the countable, complete, system for the subspace of $W^{1,2}(E)$ of the functions that vanish on $(\partial E)_2$. The existence of such a system can be ensured by the arguments discussed in Sections 19–20 of Chapter 7 of DiBenedetto [50].

10 A Priori $L^\infty(E_T)$ Estimates for Solutions of the Cauchy–Dirichlet Problem (8.1)

A weak sub(super)-solution of the Cauchy–Dirichlet problem (8.1) is a function $u \in \tilde{V}^2(E_T)$, such that $u(\cdot,t) \le (\ge)\varphi(\cdot,t)$ on ∂E in the sense of the traces of functions in $W^{1,2}(E)$ for a.e. $t \in (0,T)$, and such that

$$\begin{aligned}&\int_E u(x,\tau)\zeta(x,\tau)dx - \int_E u_o(x)\zeta(x,0)dx - \int_0^\tau\int_E u\zeta_t\,dx\,dt\\&+\int_0^\tau\int_E a_{ij}u_{x_j}\zeta_{x_i}dx\,dt \le (\ge) \int_0^\tau\int_E (f_i\zeta_{x_i}+f\zeta)dx\,dt,\end{aligned}\tag{10.1}$$

for almost all $\tau\in[0,T]$ and for all non-negative $\zeta\in W^2_o(E_T)$. It is apparent that a function $u\in\tilde{V}^2(E_T)$ is a weak solution of the Cauchy–Dirichlet problem (8.1), if and only if it is both a weak sub- and super-solution of that problem.

Proposition 10.1 *Let $u\in\tilde{V}^2(E_T)$ be a weak sub-solution of (8.1) for $N\ge 2$. For $1<p<\frac{N+2}{N}$ let*

$$r=r(p)=2\frac{p}{p-1},\qquad s=s(p)=\frac{2p(N+2)}{(N+4)p-(N+2)},\tag{10.2}$$

and assume

$$\begin{aligned}&u_{o,+}\in L^\infty(E),\quad \varphi_+\in L^\infty(\Sigma),\\&\mathbf{f}\in L^{r(p)}(E_T),\quad f_+\in L^{s(p)}(E_T).\end{aligned}\tag{10.2$_+$}$$

Then, $u_+\in L^\infty(E_T)$ and there exists a constant C_p that can be determined a priori only in terms of λ, N, p, and the constant γ in the parabolic embedding (1.2)–(1.3), such that

$$\begin{aligned}\operatorname*{ess\,sup}_{E_T} u_+ \le& C_p\max\Big\{\operatorname*{ess\,sup}_E u_{o,+};\operatorname*{ess\,sup}_\Sigma\varphi_+;\big[\|\mathbf{f}\|_{r(p);E_T}\\&+\|f_+\|_{s(p);E_T}\big]|E_T|^{\frac{\delta}{2}}\Big\},\end{aligned}\tag{10.3$_+$}$$

where

$$\delta=\frac{1}{p}+\frac{2}{N+2}-1.$$

A similar statement holds for super-solutions. Precisely:

Proposition 10.2 *Let $u\in\tilde{V}^2(E_T)$ be a weak super-solution of (8.1) for $N\ge 2$. For $1<p<\frac{N+2}{N}$ let $r=r(p)$ and $s=s(p)$ as in* (10.2), *and assume*

$$\begin{aligned}&u_{o,-}\in L^\infty(E),\quad \varphi_-\in L^\infty(\Sigma),\\&\mathbf{f}\in L^{r(p)}(E_T),\quad f_-\in L^{s(p)}(E_T).\end{aligned}\tag{10.2$_-$}$$

Then, $u_-\in L^\infty(E_T)$ and

$$\begin{aligned}\operatorname*{ess\,sup}_{E_T} u_- \le& C_p\max\Big\{\operatorname*{ess\,sup}_E u_{o,-};\operatorname*{ess\,sup}_\Sigma\varphi_-;\big[\|\mathbf{f}\|_{r(p);E_T}\\&+\|f_-\|_{s(p);E_T}\big]|E_T|^{\frac{\delta}{2}}\Big\}\end{aligned}\tag{10.3$_-$}$$

for the same constants C_p and δ as in Proposition 10.1.

Remark 10.1 The integrability exponents $r(p)$ and $s(p)$ are both decreasing with respect to p. In particular,

$$\lim_{p\to\frac{N+2}{N}} r(p) = N+2, \qquad \lim_{p\to\frac{N+2}{N}} s(p) = \frac{N+2}{2}.$$

Remark 10.2 The constant C_p tends to infinity and δ tends to zero, as $p \to \frac{N+2}{N}$. Indeed, the propositions are false for $p = \frac{N+2}{N}$.

11 Proof of Propositions 10.1–10.2

It suffices to establish Proposition 10.1. Let $u \in \tilde{V^2}(E_T)$ be a weak sub-solution of the Cauchy–Dirichlet problem (8.1), in the sense of (10.1) for all non-negative $\zeta \in W_o^2(E_T)$. Let

$$k \ge \max\{\|\varphi_+\|_{\infty,\Sigma}, \|u_{o,+}\|_{\infty,E}\} \tag{11.1}$$

be chosen and consider the Steklov average u_h of u; according to Lemma 1.1 it is apparent that $(u_h(\cdot,t)-k)_+ \in W_o^{1,2}(E)$ for all $t \in (0, T-h)$, and therefore, the test function

$$\zeta(x,t) = (u_h(x,t)-k)_+$$

is admissible in the weak formulation (10.1).

If we rewrite (10.1) in terms of the Steklov average u_h, and take ζ as above, we have

$$\begin{aligned}\iint_{E_{t_1}} u_{h,t}(u_h(x,t)-k)_+ dx\,dt + \iint_{E_{t_1}} (a_{ij}u_{x_j})_h((u_h(x,t)-k)_+)_{x_i} dx\,dt \\ \le \iint_{E_{t_1}} [f_{i,h}((u_h(x,t)-k)_+)_{x_i} + f_{+,h}(u_h(x,t)-k)_+]\, dx\,dt,\end{aligned}$$

that is

$$\begin{aligned}\frac{1}{2}\iint_{E_{t_1}} \partial_t((u_h(x,t)-k)_+)^2 dx\,dt + \iint_{E_{t_1}} (a_{ij}u_{x_j})_h((u_h(x,t)-k)_+)_{x_i} dx\,dt \\ \le \iint_{E_{t_1}} [f_{i,h}((u_h(x,t)-k)_+)_{x_i} + f_{+,h}(u_h(x,t)-k)_+]\, dx\,dt,\end{aligned}$$

where t_1 is any value in $(0, T-h)$. Integrating with respect to time, and taking (11.1) into account gives

$$\begin{aligned}\frac{1}{2}\int_E (u_h-k)_+^2(x,t_1)dx + \iint_{E_{t_1}} (a_{ij}u_{x_j})_h((u_h(x,t)-k)_+)_{x_i} dx\,dt \\ \le \iint_{E_{t_1}} [f_{i,h}((u_h(x,t)-k)_+)_{x_i} + f_{+,h}(u_h(x,t)-k)_+]\, dx\,dt.\end{aligned}$$

Passing to the limit as $h \to 0$, taking the supremum with respect to t_1 over $(0, T]$, and using the ellipticity condition (2.1) yields

$$\begin{aligned}\frac{1}{2} \sup_{0<t<T} \int_E (u-k)_+^2(x,t)dx + \lambda \iint_{E_T} |\nabla(u-k)_+|^2 dx\,dt \\ \le \iint_{E_T} \left[f_i((u-k)_+)_{x_i} + f_+(u-k)_+\right] dx\,dt.\end{aligned}$$

Since

$$\begin{aligned}&\iint_{E_T} f_i((u-k)_+)_{x_i} dx\,dt \\ &\qquad \le \frac{\lambda}{2} \iint_{E_T} |\nabla(u-k)_+|^2 dx\,dt + \frac{1}{2\lambda} \iint_{E_T} \sum_{i=1}^N |f_i|^2 \chi_{[u>k]} dx\,dt,\end{aligned}$$

we obtain

$$\begin{aligned}&\sup_{0<t<T} \int_E (u-k)_+^2(x,t)dx + \iint_{E_T} |\nabla(u-k)_+|^2 dx\,dt \\ &\qquad \le \gamma_1 \iint_{E_T} f_+(u-k)_+ dx\,dt + \gamma_1 \iint_{E_T} \sum_{i=1}^N |f_i|^2 \chi_{[u>k]} dx\,dt.\end{aligned}$$

Moreover, by Proposition 1.1

$$\begin{aligned}\iint_{E_T} f_+(u-k)_+\, dx\,dt &\le \|f_+\chi_{[u>k]}\|_{2\frac{N+2}{N+4};E_T} \, \|(u-k)_+\|_{2\frac{N+2}{N};E_T} \\ &\le \gamma \|(u-k)_+\|_{V^2(E_T)} \, \|f_+\chi_{[u>k]}\|_{2\frac{N+2}{N+4};E_T} \\ &\le \frac{1}{4\gamma_1} \|(u-k)_+\|_{V^2(E_T)}^2 + C(\gamma,\gamma_1) \|f_+\chi_{[u>k]}\|_{2\frac{N+2}{N+4};E_T}^2.\end{aligned}$$

Hence,

$$\begin{aligned}&\sup_{0<t<T} \int_E (u-k)_+^2(x,t)dx + \iint_{E_T} |\nabla(u-k)_+|^2 dx\,dt \\ &\le \gamma_2 \iint_{E_T} \sum_{i=1}^N |f_i|^2 \chi_{[u>k]} dx\,dt + \gamma_2 \left(\iint_{E_T} f_+^{2\frac{N+2}{N+4}} \chi_{[u>k]} dx\,dt \right)^{\frac{N+4}{N+2}}.\end{aligned} \tag{11.2}$$

For any $p \in \left(1, \frac{N+2}{N}\right)$ we have

$$\iint_{E_T} \sum_{i=1}^N |f_i|^2 \chi_{[u>k]} dx\,dt \le \left(\iint_{E_T} |\mathbf{f}|^{2p'} dx\,dt \right)^{\frac{1}{p'}} \left|A_k\right|^{\frac{1}{p}},$$

where $A_k \overset{\text{def}}{=} \{(x,t) \in E_T : \ u(x,t) > k\}$; by Hölder's inequality for $q = p\frac{N+4}{N+2}$ and $q' = \frac{p(N+4)}{p(N+4)-(N+2)}$ we obtain

$$\begin{aligned}\Big(\iint_{E_T} f_+^{2\frac{N+2}{N+4}}\chi_{[u>k]}dx\,dt\Big)^{\frac{N+4}{N+2}}&\\ \le\Big(\iint_{E_T} f_+^{\frac{2(N+2)p}{p(N+4)-(N+2)}}dx\,dt\Big)^{\frac{p(N+4)-(N+2)}{(N+2)p}}&\big|A_k\big|^{\frac{1}{p}}.\end{aligned}$$

Therefore,

$$\begin{aligned}\sup_{0<t<T}\int_E(u-k)_+^2(x,t)dx&+\iint_{E_T}|\nabla(u-k)_+|^2dx\,dt\\ \le&\gamma_2\big|A_k\big|^{\frac{1}{p}}\bigg[\Big(\iint_{E_T}|\mathbf{f}|^{2p'}dx\,dt\Big)^{\frac{1}{p'}}\\ &+\Big(\iint_{E_T} f_+^{\frac{2(N+2)p}{p(N+4)-(N+2)}}dx\,dt\Big)^{\frac{p(N+4)-(N+2)}{(N+2)p}}\bigg]=\gamma_2C_f\big|A_k\big|^{\frac{1}{p}},\end{aligned}$$

where we have set

$$C_f=\Big(\iint_{E_T}|\mathbf{f}|^{2p'}dx\,dt\Big)^{\frac{1}{p'}}+\Big(\iint_{E_T} f_+^{\frac{2(N+2)p}{p(N+4)-(N+2)}}dx\,dt\Big)^{\frac{p(N+4)-(N+2)}{(N+2)p}}.$$

Let

$$k_n=k\Big(2-\frac{1}{2^{n-1}}\Big),\qquad A_{k_n}=[u>k_n],\qquad n=1,2,\dots,$$

and rewrite the previous estimate in terms of k_n; this yields

$$\sup_{0<t<T}\int_E(u-k_n)_+^2(x,t)dx+\iint_{E_T}|\nabla(u-k_n)_+|^2dx\,dt\le\gamma C_f|A_n|^{\frac{1}{p}},\tag{11.3}$$

where $A_n=A_{k_n}$. Moreover,

$$\begin{aligned}\iint_{E_T}(u-k_n)_+^2dx\,dt&\ge\iint_{E_T}(u-k_n)_+^2\chi_{[u>k_{n+1}]}dx\,dt\\ &\ge(k_n-k_{n+1})^2|A_{n+1}|=\frac{k^2}{4^n}|A_{n+1}|,\end{aligned}\tag{11.4}$$

and, by the parabolic embedding of Proposition 1.1

$$\begin{aligned}&\iint_{E_T}(u-k_n)_+^{2\frac{N+2}{N}}dx\,dt\\ &\qquad\le\gamma\iint_{E_T}|\nabla(u-k_n)_+|^2dx\,dt\Big(\sup_{0<t<T}\int_E(u-k_n)_+^2dx\Big)^{\frac{2}{N}}.\end{aligned}$$

Therefore, taking into account (11.3) and the previous two estimates

$$\frac{k^2}{4^n}|A_{n+1}|\le\iint_{E_T}(u-k_n)_+^2dx\,dt$$

$$
\begin{aligned}
&\le \left(\iint_{E_T} (u-k_n)_+^{2\frac{N+2}{N}} dx\,dt \right)^{\frac{N}{N+2}} |A_n|^{\frac{2}{N+2}} \\
&\le \left(\sup_{0<t<T} \int_E (u-k_n)_+^2(x,t)dx \right)^{\frac{2}{N+2}} \\
&\quad \cdot \left(\iint_{E_T} |\nabla(u-k_n)_+|^2 dx\,dt \right)^{\frac{N}{N+2}} |A_n|^{\frac{2}{N+2}} \\
&\le \gamma_3\, C_f\, |A_n|^{\frac{1}{p}}\, |A_n|^{\frac{2}{N+2}} = \gamma_3\, C_f\, |A_n|^{\frac{1}{p}+\frac{2}{N+2}}.
\end{aligned}
$$

By our assumption on p, it is apparent that $\frac{1}{p}+\frac{2}{N+2}>1$. If we set $\frac{1}{p}+\frac{2}{N+2}=1+\delta$, with $\delta=\delta(p,N)$, we obtain

$$
|A_{n+1}| \le \gamma_3 \frac{4^n}{k^2} C_f\, |A_n|^{1+\delta}, \qquad \forall\, n\in\mathbb{N}.
$$

If $|A_n|\to 0$ as $n\to\infty$, then $u\le 2k$ a.e. in E_T. By Lemma 15.1 of Chapter 9 this occurs if

$$
|A_1| \le |E_T| \le \gamma_3^{-1/\delta} 4^{-1/\delta^2} C_f^{-1/\delta} k^{2/\delta},
$$

and this is in turn satisfied if k is chosen from

$$
k = 2^{1/\delta}\gamma_3^{1/2}\, C_f^{1/2}\, |E_T|^{\delta/2},
$$

which finally implies

$$
u \le \gamma_4(\lambda,N,p,\gamma)|E_T|^{\frac{\delta}{2}} \left[\left(\iint_{E_T} |\mathbf{f}|^{r(p)} dx\,dt \right)^{\frac{1}{r(p)}} + \left(\iint_{E_T} f_+^{s(p)} dx\,dt \right)^{\frac{1}{s(p)}} \right].
$$

Taking into account the original assumptions on k, we conclude. ■

Remark 11.1 In (11.2), the last term on the right-hand side can be estimated in a slightly different way, namely

$$
\begin{aligned}
&\left(\iint_{E_T} f_+^{2\frac{N+2}{N+4}} \chi_{[u>k]} dx\,dt \right)^{\frac{N+4}{N+2}} \\
&\qquad \le \left(\iint_{E_T} f_+^{2\frac{N+2}{N+4}q'} dx\,dt \right)^{\frac{N+4}{(N+2)q'}} \left|A_k\right|^{\frac{N+4}{(N+2)q}} \\
&\qquad \le \left(\iint_{E_T} f_+^{2\frac{N+2}{N+4}q'} dx\,dt \right)^{\frac{N+4}{(N+2)q'}} \left|A_k\right|^{\frac{1}{q}} \left|E_T\right|^{\frac{2}{(N+2)q}},
\end{aligned}
$$

where now $q>1$, whereas in (11.2) $q>p\frac{N+4}{N+2}$ with $p>1$.

We can relabel q with p, and then proceed as before. This is particularly useful when considering with this different estimate the limiting case of $p\to 1$, which entails that both $\mathbf{f}$ and f_+ end up belonging to $L^\infty(E_T)$. In such a case we have

$$\begin{aligned}\operatorname*{ess\,sup}_{E_T} u_+ \le & C_1 \max\Big\{ \operatorname*{ess\,sup}_{E} u_{o,+}; \operatorname*{ess\,sup}_{\partial E} \varphi_+; \Big[\|\mathbf{f}\|_{\infty;E_T} \\ & +|E_T|^{\frac{1}{N+2}} \|f_+\|_{\infty;E_T}\Big] |E|^{\frac{1}{N+2}} \Big\}\end{aligned}$$

where the constant C_1 is the stable limit of C_p as $p \to 1$. Moreover, apart from change from N to $N+2$, we have a perfect correspondence with the analogous estimates of Section 15 of Chapter 9.

Remark 11.2 The fact that most elliptic estimates can be repeated almost verbatim in the parabolic framework up to a change from N to $N+2$ is not a casual fact, but it is a direct consequence of the so-called *parabolic* scaling, as we shall more clearly see in Chapter 12.

12 A Priori $L^\infty(E_T)$ Estimates for Solutions of the Neumann Problem (9.1)

A weak sub(super)-solution of the Neumann problem (9.1) is a function $u \in \tilde{V}^2(E_T)$ satisfying

$$\begin{aligned}&\int_E u(x,\tau)\zeta(x,\tau)dx - \int_E u_o(x)\zeta(x,0)dx \\ &- \int_0^\tau \int_E u\zeta_t \, dx\, dt + \int_0^\tau \int_E a_{ij} u_{x_i} \zeta_{x_j} \, dx\, dt \\ &\le (\ge) \int_0^\tau \int_E (f_j \zeta_{x_j} + f\zeta) dx\, dt + \int_0^\tau \int_{\partial E} \psi z \, d\sigma\, dt\end{aligned} \tag{12.1}$$

for all non-negative test functions $\zeta \in W^2(E_T)$. A function $u \in \tilde{V}^2(E_T)$ is a weak solution of the Cauchy–Neumann problem (9.1), if and only if it is both a weak sub- and super-solution of that problem.

Proposition 12.1 *Let ∂E be of class C^1 and satisfy the segment property. Let $u \in \tilde{V}^2(E_T)$ be a weak sub-solution of (9.1) for $N \ge 2$. For $1 < p < \frac{N+2}{N+1}$ let*

$$\begin{aligned}&q = q(p) = \frac{N-1}{N}\frac{p}{p-1}, \qquad r(p) = \frac{p}{p-1}, \\ &s = s(p) = \frac{2p(N+2)}{(N+4)p - (2-p)(N+2)},\end{aligned} \tag{12.2}$$

and assume

$$\begin{aligned}&u_{o,+} \in L^\infty(E), \quad \psi_+ \in L^{q(p),r(p)}(\Sigma), \\ &\mathbf{f} \in L^{r(p)}(E_T), \quad f_+ \in L^{s(p)}(E_T).\end{aligned} \tag{12.3$_+$}$$

Then, $u_+ \in L^\infty(E_T)$, and there exists a constant C_p that can be determined a priori only in terms of λ, N, p, T, the constant γ in Proposition 1.2, the *constant γ in the embeddings of Theorem 2.1 of Chapter 9, the constant γ of the trace inequality of Proposition 8.2 of Chapter 9, and the structure of ∂E through the parameters h and ω of its cone condition such that*

$$\begin{aligned}&\operatorname*{ess\,sup}_{E_T} u_+\\ \le &C_p \max\left\{\operatorname*{ess\,sup}_{E} u_{o,+};\left(1+\frac{T}{|E|^{\frac{2}{N}}}\right)^{\frac{N}{2(N+2)}}|E_T|^{\frac{\delta}{2}}\left[\|\psi_+\|_{q(p),r(p);\Sigma}\right.\right.\\ &\left.\left.+\|\mathbf{f}\|_{r(p);E_T}+\left(1+\frac{T}{|E|^{\frac{2}{N}}}\right)^{\frac{N}{2(N+2)}}\|f_+\|_{s(p);E_T}\right]\right\},\end{aligned}\tag{12.4$_+$}$$

where

$$\delta=\frac{2}{p}+\frac{2}{N+2}-2.\tag{12.5}$$

Proposition 12.2 *Let ∂E be of class C^1 and satisfy the segment property. Let $u\in \tilde{V}^2(E_T)$ be a weak sub-solution of (9.1) for $N\ge 2$. Assume that*

$$\begin{aligned}&u_{o,-}\in L^\infty(E),\quad \psi_-\in L^{q(p),r(p)}(\Sigma),\\ &\mathbf{f}\in L^{r(p)}(E_T),\quad f_-\in L^{s(p)}(E_T),\end{aligned}\tag{12.3$_-$}$$

for the same $q(p)$, $r(p)$, $s(p)$ as in (12.2). *Then, $u_-\in L^\infty(E_T)$ and*

$$\begin{aligned}&\operatorname*{ess\,sup}_{E_T} u_+\\ \le &C_p \max\left\{\operatorname*{ess\,sup}_{E} u_{o,-};\left(1+\frac{T}{|E|^{\frac{2}{N}}}\right)^{\frac{N}{2(N+2)}}|E_T|^{\frac{\delta}{2}}\left[\|\psi_-\|_{q(p),r(p);\Sigma}\right.\right.\\ &\left.\left.+\|\mathbf{f}\|_{r(p);E_T}+\left(1+\frac{T}{|E|^{\frac{2}{N}}}\right)^{\frac{N}{2(N+2)}}\|f_-\|_{s(p);E_T}\right]\right\}\end{aligned}\tag{12.4$_-$}$$

where the parameters C_p, δ are the same as in (12.4)$_+$ and (12.5).

Remark 12.1 The constant C_p depends on the embedding constants of Theorem 2.1 of Chapter 9. As such, they depend on the structure of ∂E through the parameters h and ω of its cone condition. Because of this dependence, C_p tends to ∞ as either $h\to 0$ or $\omega\to 0$.

Remark 12.2 The integrability exponents $q(p)$, $r(p)$ and $s(p)$ are all decreasing with respect to p. In particular,

$$\lim_{p\to\frac{N+2}{N+1}} q(p)=\frac{(N-1)(N+2)}{N},\qquad \lim_{p\to\frac{N+2}{N+1}} r(p)=N+2,$$
$$\lim_{p\to\frac{N+2}{N+1}} s(p)=\frac{N+2}{2},$$

and as far as $\mathbf{f}$ and f are concerned, we find the same limiting conditions as in Propositions 10.1–10.2.

Remark 12.3 The constant C_p tends to infinity and δ tends to zero, as $p \to \frac{N+2}{N+1}$. Indeed, the propositions are false for $p = \frac{N+2}{N+1}$.

13 Proof of Propositions 12.1–12.2

It suffices to establish Proposition 12.1. Let $u \in \tilde{V^2}(E_T)$ be a sub-solution of the Cauchy–Neumann problem (9.1), in the sense of (12.1). Let $k \geq \operatorname{ess\,sup}_E u_{o,+}$ to be chosen.

Working with Steklov averages, and then passing to the limit with respect to the parameter h as in the previous section, we have

$$\begin{aligned}
&\sup_{0<t<T} \frac{1}{2} \int_E (u-k)_+^2(x,t)dx + \lambda \iint_{E_T} |\nabla(u-k)_+|^2 dx\,dt \\
&\leq \iint_{E_T} (f_i((u-k)_+)_{x_i} + f_+(u-k)_+)dx\,dt + \iint_{\Sigma} \psi_+(u-k)_+ d\sigma\,dt.
\end{aligned}$$

Now we have to estimate the right-hand side, taking into account that $(u(\cdot,t)-k)_+ \in W^{1,2}(E)$ for almost every $t \in (0,T]$, but in general it does not have zero trace. Since

$$\begin{aligned}
&\iint_{E_T} f_i((u-k)_+)_{x_i} dx\,dt \\
&\qquad \leq \frac{\lambda}{2} \iint_{E_T} |\nabla(u-k)_+|^2 dx\,dt + \frac{1}{2\lambda} \iint_{E_T} \sum_{i=1}^N |f_i|^2 \chi_{[u>k]} dx\,dt,
\end{aligned}$$

we have

$$\begin{aligned}
&\sup_{0<t<T} \int_E (u-k)_+^2(x,t)dx + \iint_{E_T} |\nabla(u-k)_+|^2 dx\,dt \\
&\leq \gamma_1 \iint_{E_T} |\mathbf{f}|^2 \chi_{[u>k]} dx\,dt + \gamma_1 \iint_{E_T} f_+(u-k)_+ dx\,dt \\
&+ \gamma_1 \iint_{\Sigma} \psi_+(u-k)_+ d\sigma\,dt.
\end{aligned}$$

By Proposition 1.2

$$\begin{aligned}
\iint_{E_T} f_+(u-k)_+ dx\,dt &\leq \|f_+\chi_{[u>k]}\|_{2\frac{N+2}{N+4};E_T} \|(u-k)_+\|_{2\frac{N+2}{N};E_T} \\
&\leq \gamma \left(1 + \frac{T}{|E|^{\frac{2}{N}}}\right)^{\frac{N}{2(N+2)}} \|(u-k)_+\|_{V^2(E_T)} \|f_+\chi_{[u>k]}\|_{2\frac{N+2}{N+4};E_T} \\
&\leq \frac{1}{4\gamma_1} \|(u-k)_+\|^2_{V^2(E_T)} \\
&\quad + C(\gamma,\gamma_1) \left(1 + \frac{T}{|E|^{\frac{2}{N}}}\right)^{\frac{N}{(N+2)}} \|f_+\chi_{[u>k]}\|^2_{2\frac{N+2}{N+4};E_T},
\end{aligned}$$

which gives

$$\begin{aligned}&\sup_{0<t<T}\int_E (u-k)_+^2(x,t)dx+\iint_{E_T}|\nabla(u-k)_+|^2dx\,dt\\&\le\gamma_2\iint_{E_T}|\mathbf{f}|^2\chi_{[u>k]}dx\,dt+\gamma_1\iint_{\Sigma}\psi_+(u-k)_+d\sigma\,dt\\&+\gamma_2\left(1+\frac{T}{|E|^{\frac{2}{N}}}\right)^{\frac{N}{N+2}}\left[\iint_{E_T}f_+^{2\frac{N+2}{N+4}}\chi_{[u>k]}dx\,dt\right]^{\frac{N+4}{N+2}}.\end{aligned}$$

By Remark 12.2, $q>N-1$ and by assumption $1<p<\frac{N+2}{N+1}$; therefore, we can determine p and p^* from the following conditions

$$1-\frac{1}{q}=\frac{1}{p^*},\quad p^*=\frac{Np}{N-p},\quad \frac{1}{q}=\frac{p-1}{p}\frac{N}{N-1}.$$

We can the apply the trace inequality (8.3) of Chapter 9, to obtain

$$\begin{aligned}&\int_0^T\int_{\partial E}\psi_+(u-k)_+d\sigma\,dt\le\int_0^T\|\psi_+(\cdot,t)\|_{q;\partial E}\|(u(\cdot,t)-k)_+\|_{p^*\frac{N-1}{N};\partial E}\,dt\\&\le\int_0^T\|\psi_+(\cdot,t)\|_{q;\partial E}\left[\|\nabla(u-k)_+\|_{p;E}+2\gamma\|(u-k)_+\|_{p;E}\right]dt\\&\le\int_0^T\|\psi_+(\cdot,t)\|_{q;\partial E}\left[\|\nabla(u-k)_+\|_{2;E}+2\gamma\|(u-k)_+\|_{2;E}\right]|A_k(t)|^{\frac{1}{p}-\frac{1}{2}}dt\\&\le\frac{1}{4}\iint_{E_T}|\nabla(u-k)_+|^2dx\,dt+\gamma\int_0^T\|\psi_+(\cdot,t)\|^2_{q;\partial E}|A_k(t)|^{\frac{2}{p}-1}dt\\&\quad+\frac{1}{4T}\iint_{E_T}(u-k)_+^2dx\,dt+\gamma\int_0^T\|\psi_+(\cdot,t)\|^2_{q;\partial E}|A_k(t)|^{\frac{2}{p}-1}dt\\&\le\frac{1}{4}\left[\sup_{0<t<T}\int_E(u-k)_+^2(x,t)dx+\iint_{E_T}|\nabla(u-k)_+|^2dx\,dt\right]\\&\quad+\gamma\left(\int_0^T\|\psi_+(\cdot,t)\|^{\frac{p}{p-1}}_{q;\partial E}dt\right)^{\frac{2(p-1)}{p}}|A_k|^{\frac{2-p}{p}},\end{aligned}$$

where $A_k(t)=\{x\in E:\ \ u(x,t)>k\}$ and $A_k=\{(x,t)\in E_T:\ \ u(x,t)>k\}$. Hence, collecting all the terms yields

$$\begin{aligned}&\sup_{0<t<T}\int_E(u-k)_+^2(x,t)dx+\iint_{E_T}|\nabla(u-k)_+|^2dx\,dt\\&\le\gamma_2\left(1+\frac{T}{|E|^{\frac{2}{N}}}\right)^{\frac{N}{N+2}}\left(\iint_{E_T}f_+^{2\frac{N+2}{N+4}}\chi_{[u>k]}dx\,dt\right)^{\frac{N+4}{N+2}}\\&\quad+\gamma_2\iint_{E_T}|\mathbf{f}|^2\chi_{[u>k]}dx\,dt+\gamma_3\left(\int_0^T\|\psi_+(\cdot,t)\|^{\frac{p}{p-1}}_{q;\partial E}\,dt\right)^{\frac{2(p-1)}{p}}|A_k|^{\frac{2-p}{p}}.\end{aligned}$$

Moreover,

$$\iint_{E_T} |\mathbf{f}|^2 \chi_{[u>k]} dx\,dt \le |A_k|^{\frac{2-p}{p}} \left[\iint_{E_T} |\mathbf{f}|^{\frac{p}{p-1}} dx\,dt\right]^{\frac{2(p-1)}{p}},$$

and

$$\begin{aligned}&\left(\iint_{E_T} f_+^{2\frac{N+2}{N+4}} \chi_{[u>k]} dx\,dt\right)^{\frac{N+4}{N+2}} \\ &\le |A_k|^{\frac{2-p}{p}} \left[\iint_{E_T} f_+^{\frac{2p(N+2)}{p(N+4)-(2-p)(N+2)}} dx\,dt\right]^{\frac{p(N+4)-(2-p)(N+2)}{p(N+2)}}.\end{aligned} \tag{13.1}$$

Hence,

$$\sup_{0<t<T} \int_E (u-k)_+^2(x,t)dx + \iint_{E_T} |\nabla(u-k)_+|^2 dx\,dt \le \gamma C_{f,\psi} |A_k|^{\frac{2-p}{p}},$$

where

$$\begin{aligned}C_{f,\psi} \overset{\text{def}}{=} &\left[\iint_{E_T} |\mathbf{f}|^{\frac{p}{p-1}} dx\,dt\right]^{\frac{2(p-1)}{p}} \\ &+ \left(1 + \frac{T}{|E|^{\frac{2}{N}}}\right)^{\frac{N}{N+2}} \left[\iint_{E_T} f_+^{\frac{2p(N+2)}{p(N+4)-(2-p)(N+2)}} dx\,dt\right]^{\frac{p(N+4)-(2-p)(N+2)}{p(N+2)}} \\ &+ \left(\int_0^T \|\psi_+(\cdot,t)\|_{q;\partial E}^{\frac{p}{p-1}} dt\right)^{\frac{2(p-1)}{p}}.\end{aligned}$$

Let

$$k_n = k\Big(2 - \frac{1}{2^{n-1}}\Big), \qquad A_{k_n} = [u > k_n], \qquad n = 1, 2, \dots,$$

and rewrite the previous estimate in terms of k_n. By Proposition 1.2

$$\begin{aligned}&\iint_{E_T} (u-k_n)_+^2 dx\,dt \\ &\quad\le \left(\iint_{E_T} (u-k_n)_+^{2\frac{N+2}{N}} dx\,dt\right)^{\frac{N}{N+2}} |A_n|^{\frac{2}{N+2}} \\ &\quad\le \gamma \left(1 + \frac{T}{|E|^{\frac{2}{N}}}\right)^{\frac{N}{N+2}} \left[\sup_{0<t<T} \int_E (u-k_n)_+^2(x,t)dx \right. \\ &\qquad\quad \left. + \iint_{E_T} |\nabla(u-k_n)_+|^2 dx\,dt\right] |A_n|^{\frac{2}{N+2}},\end{aligned}$$

where $A_n = A_{k_n}$, and therefore,

$$\iint_{E_T} (u-k_n)_+^2 dx\,dt \le \gamma C_{f,\psi} \left(1+\frac{T}{|E|^{\frac{2}{N}}}\right)^{\frac{N}{N+2}} |A_n|^{\frac{2-p}{p}+\frac{2}{N+2}}.$$

By the assumption on p, we have $\frac{2-p}{p}+\frac{2}{N+2}>1$. We set

$$\frac{2-p}{p}+\frac{2}{N+2}=1+\delta \;\Rightarrow\; \delta \overset{\text{def}}{=} \frac{2}{p}+\frac{2}{N+2}-2.$$

As we have already seen in (11.4)

$$\iint_{E_T} (u-k_n)_+^2 dx\,dt \ge \frac{k^2}{4^n}|A_{n+1}|;$$

and therefore,

$$|A_{n+1}| \le \gamma C_{f,\psi}\left(1+\frac{T}{|E|^{\frac{2}{N}}}\right)^{\frac{N}{N+2}} \frac{4^n}{k^2}|A_n|^{1+\delta}.$$

We can now apply Lemma 15.1 of Chapter 9, and conclude that $u \le 2k$ a.e. in E_T, provided we have

$$|A_1| \le |E_T| \le \gamma^{-1/\delta} 4^{-1/\delta^2} C_{f,\psi}^{-1/\delta} k^{2/\delta} \left(1+\frac{T}{|E|^{\frac{2}{N}}}\right)^{-\frac{N}{\delta(N+2)}},$$

and this is in turn satisfied if k is chosen from

$$k = 2^{\frac{1}{\delta}} \gamma^{\frac{1}{2}} |E_T|^{\frac{\delta}{2}} C_{f,\psi}^{\frac{1}{2}} \left(1+\frac{T}{|E|^{\frac{2}{N}}}\right)^{\frac{N}{2(N+2)}},$$

which implies

$$\begin{aligned}\operatorname*{ess\,sup}_{E_T} u \le{}& \gamma\left(1+\frac{T}{|E|^{\frac{2}{N}}}\right)^{\frac{N}{2(N+2)}} |E_T|^{\frac{\delta}{2}} \left\{ \left[\iint_{E_T} |\mathbf{f}|^{r(p)} dx\,dt\right]^{\frac{1}{r(p)}} \right.\\ &+ \left(1+\frac{T}{|E_T|^{\frac{2}{N}}}\right)^{\frac{N}{2(N+2)}} \left[\iint_{E_T} f_+^{s(p)} dx\,dt\right]^{\frac{1}{s(p)}} \\ &\left. + \left[\int_0^T \left(\int_{\partial E} \|\psi_+(\cdot,t)\|^{q(p)} d\sigma\right)^{\frac{r(p)}{q(p)}} dt\right]^{\frac{1}{s(p)}} \right\}.\end{aligned}$$

If we take into account the original stipulation on k, we conclude. ■

Remark 13.1 In (13.1), the last term on the right-hand side can be estimated in a slightly different way, namely

$$\left(\iint_{E_T} f_+^{2\frac{N+2}{N+4}} \chi_{[u>k]} dx\,dt\right)^{\frac{N+4}{N+2}}$$

$$
\begin{aligned}
&\leq \left(\iint_{E_T} f_+^{2\frac{N+2}{N+4}q'} \chi_{[u>k]} dx\, dt \right)^{\frac{N+4}{(N+2)q'}} \left| A_k \right|^{\frac{N+4}{(N+2)q}} \\
&\leq \left(\iint_{E_T} f_+^{2\frac{N+2}{N+4}q'} \chi_{[u>k]} dx\, dt \right)^{\frac{N+4}{(N+2)q'}} \left| A_k \right|^{\frac{2-p}{p}} \left| E_T \right|^{\frac{N+4}{(N+2)q} - \frac{2-p}{p}},
\end{aligned}
$$

where $q > 1$.

We can relabel q with p, and then proceed as before. This is particularly useful when considering with this different estimate the limiting case of $p \to 1$, which entails that $\mathbf{f}$, f_+, ψ_+ all end up belonging to $L^\infty(E_T)$. In such a case we have

$$
\begin{aligned}
&\operatorname*{ess\,sup}_{E_T} u_+ \\
&\leq C_1 \max \left\{ \operatorname*{ess\,sup}_{E} u_{o,+} ; \left(1 + \frac{T}{|E|^{\frac{2}{N}}} \right)^{\frac{N}{2(N+2)}} |E_T|^{\frac{1}{N+2}} \left[\|\psi_+\|_{\infty;\Sigma} \right.\right. \\
&\quad \left.\left. + \|\mathbf{f}\|_{\infty;E_T} + \left(1 + \frac{T}{|E|^{\frac{2}{N}}} \right)^{\frac{N}{2(N+2)}} |E_T|^{\frac{1}{N+2}} \|f_+\|_{\infty;E_T} \right] \right\},
\end{aligned}
$$

where the constant C_1 is the stable limit of C_p as $p \to 1$. Moreover, apart from change from N to $N+2$, we have a perfect correspondence with the analogous estimates of Section 16 of Chapter 9.

14 Miscellaneous Remarks on Further Regularity

A function $u \in C_{\text{loc}}(0,T; L^2_{\text{loc}}(E)) \cap L^2_{\text{loc}}(0,T; W^{1,2}_{\text{loc}}(E))$ is a *local* weak solution to (2.2), irrespective of possible boundary data, if for every compact set $\mathcal{K} \subset E$ and for every sub-interval $[t_1, t_2] \subset (0,T]$

$$
\int_{\mathcal{K}} u\varphi dx \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_{\mathcal{K}} a_{ij} u_{x_j} \varphi_{xi} dx\, dt = \int_{t_1}^{t_2} \int_{\mathcal{K}} \left[f\varphi + f_i \varphi_{x_i} \right] dx\, dt
$$

for all $\varphi \in C_{\text{loc}}(0,T; L^2(\mathcal{K})) \cap L^2_{\text{loc}}(0,T; W^{1,2}_o(\mathcal{K}))$. On the data $\mathbf{f}$ and f assume

$$
\mathbf{f} \in L^r(E_T), \qquad f_\pm \in L^s(E_T), \qquad \text{with } r > N+2, \; s > \frac{N+2}{2}. \tag{14.1}
$$

The set of parameters $\{N, \lambda, \Lambda, r, s, \|\mathbf{f}\|_{r;E_T}, \|f\|_{s;E_T}\}$ are the data, and we say that a constant $C, \gamma, \dots$ depends on the data if it can be quantitatively determined a priori in terms of these quantities only. Continue to assume that the boundary ∂E is of class C^1 and with the segment property, and let $\partial_p E_T \overset{\text{def}}{=} \Sigma \cup (E \times \{0\})$. For a compact set $\mathcal{K} \subset E_T$ define

$$
\operatorname{dist}(\mathcal{K}, \partial_p E_T) = \inf_{\substack{(x,t)\in\mathcal{K} \\ (y,s)\in\partial_p E_T}} \left(|x-y| + |t-s|^{\frac{1}{2}} \right).
$$

Theorem 14.1. *Let* $u \in C_{\mathrm{loc}}(0,T;L^2_{\mathrm{loc}}(E)) \cap L^2_{\mathrm{loc}}(0,T;W^{1,2}_{\mathrm{loc}}(E))$ *be a local weak solution of (2.2) and let (14.1) hold. Then,* u *is locally bounded and locally Hölder continuous in* E_T*. Moreover, for every compact set* $\mathcal{K} \subset E_T$*, there exist positive constants* $\gamma_{\mathcal{K}} > 1$ *and* $\alpha \in (0,1)$*, depending only on the data, such that*

$$|u(x_1,t_1) - u(x_2,t_2)| \le \gamma_{\mathcal{K}} \left(\frac{|x_1 - x_2| + |t_1 - t_2|^{\frac{1}{2}}}{\operatorname{dist}(\mathcal{K}, \partial_p E_T)} \right)^{\alpha}$$

for every pair of points (x_1,t_1)*,* $(x_2,t_2) \in \mathcal{K}$*.*

Theorem 14.2. *Let* $u \in \tilde{V}^2(E_T)$ *be a solution of the Cauchy–Dirichlet problem (8.1), with* $\mathbf{f}$ *and* f *satisfying (14.1),* $u_o \in C^{\alpha_o}(\bar{E})$ *for some exponent* $\alpha_o \in (0,1)$*,* $\varphi \in H^{\frac{1}{2},\frac{1}{2}}(\Sigma)$ *and Hölder continuous in* Σ *for some exponent* $\epsilon \in (0,1)$*. Then,* u *is Hölder continuous in* $\overline{E_T}$ *and there exist constants* $\gamma > 1$ *and* $\alpha \in (0,1)$*, depending only upon the data, the* C^1 *structure of* ∂E*, the Hölder norm* $|||\varphi|||_{\epsilon;\Sigma}$*, and the Hölder norm* $|||u_o|||_{\alpha_o;\bar{E}}$ *such that*

$$|u(x_1,t_1) - u(x_2,t_2)| \le \gamma \left(|x_1 - x_2| + |t_1 - t_2|^{\frac{1}{2}} \right)^{\alpha}$$

for every pair of points (x_1,t_1)*,* $(x_2,t_2) \in \overline{E_T}$*.*

Theorem 14.3. *Let* $u \in \tilde{V}^2(E_T)$ *be a solution of the Cauchy–Neumann problem (9.1), with* $\mathbf{f}$ *and* f *satisfying (12.1),* $\psi \in L^{N+1+\sigma}(\partial E)$ *for some* $\sigma \in (0,1)$*,* $u_o \in C^{\alpha_o}(\bar{E})$ *for some exponent* $\alpha_o \in (0,1)$*. Then,* u *is Hölder continuous in* $\overline{E_T}$*, and there exist constants* γ *and* $\alpha \in (0,1)$*, depending on the data, the* C^1 *structure of* ∂E*,* $\|\psi\|_{N+1+\sigma;\partial E}$*, and the Hölder norm* $|||u_o|||_{\alpha_o;\bar{E}}$ *such that*

$$|u(x_1,t_1) - u(x_2,t_2)| \le \gamma \left(|x_1 - x_2| + |t_1 - t_2|^{\frac{1}{2}} \right)^{\alpha}$$

for every pair of points (x_1,t_1)*,* $(x_2,t_2) \in \overline{E_T}$*.*

The precise structure of these estimates in terms of the initial condition u_o, Dirichlet data φ, or Neumann data ψ, as well as the dependence on the structure of ∂E, is specified in more general theorems for functions in the parabolic DeGiorgi classes (Theorem 8.1 and Theorem 9.1 of the next Chapter). These are the key, seminal facts in the theory of regularity of solutions of parabolic equations in divergence form. They can be used, by boot-strap arguments, to establish further regularity on the solutions, whenever further regularity is assumed on the coefficients.

15 Gaussian Bounds on the Fundamental Solution

In Section 2 of Chapter 5 we studied the fundamental solution of the heat equation, and in particular, in Theorem 2.1 we discussed the role it plays in solving the Cauchy problem in $\mathbb{R}^N$. The linearity of the heat equation is a central feature in these estimates.

When considering (2.2) with $f = \mathbf{f} = 0$ in $\mathbb{R}^N \times (0, T)$, since the linearity is preserved, it is quite natural to define a fundamental solution in this more general framework as well, and inquire about its structure.

In other words, we seek a smooth function $(t, x, y) \mapsto \Gamma(x, t; y)$ defined on $\mathbb{R}^N \times (0, +\infty) \times \mathbb{R}^N$ such that, for every $y \in \mathbb{R}^N$, $\Gamma(x, t; y)$ is a solution of

$$(\partial_t - \operatorname{div} a_{ij}(x,t)\nabla)\, u = 0, \tag{15.1}$$

satisfies

$$\int_{\mathbb{R}^N} \Gamma(x, t; y)\, dx = 1$$

$\forall\, y \in \mathbb{R}^N$, $\forall\, t > 0$, and for any $\varphi \in C_0^\infty\left(\mathbb{R}^N\right)$

$$u(x,t) = \int_{\mathbb{R}^N} \Gamma(x, t; y)\varphi(y)\, dy$$

tends to $\varphi(x)$ as $t \to 0$. In other words, Γ allows us to solve the Cauchy problem for (15.1) in $\mathbb{R}^N$. When the initial condition is assigned not at time 0 but at a general time s, the corresponding fundamental solution depends on s too, and we write $\Gamma(x, t; y, s)$. As a straightforward consequence of its definition, we have

$$\Gamma(x, t; y, s) = \int_{\mathbb{R}^N} \Gamma(x, t; z, \tau)\Gamma(z, \tau; y, s)\, dz.$$

This is usually referred to as *reproducing property*. We assume that

$$a_{ij} \in C^\infty(\mathbb{R}^N \times (0, T)) \cap L^\infty(\mathbb{R}^N \times (0, T))$$

and the $N \times N$ matrix a_{ij} satisfies the structure (2.1). The C^∞ regularity of the coefficients a_{ij} allows us to easily justify all the computations to follow, but otherwise plays no role.

Whenever we need to highlight the correspondence with the matrix a_{ij}, we will write Γ_a.

Quite surprisingly, it turns out that the estimates above and below for Γ very strongly resemble the fundamental solution of the heat equation. Indeed, there exists a positive constant C, which depends only on λ, Λ, N, such that for $s < t$ we have

$$\frac{\exp\left(-\frac{C|x-y|^2}{t-s}\right)}{C(t-s)^{\frac{N}{2}}} \le \Gamma(x, t; y, s) \le \frac{C\exp\left(-\frac{|x-y|^2}{C(t-s)}\right)}{(t-s)^{\frac{N}{2}}}.$$

Upper and lower bounds on Γ were first shown by Aronson ([11]), and then by Davies ([44]). In the first case, the proof relies on the parabolic Harnack inequality (see Chapter 12), whereas in the second instance such an inequality is not used. Anyway, in both examples, the connection with Nash's ideas, developed in his foundational paper [190], is not clear.

Here, we derive the bound above on Γ, following Fabes' interpretation of the original ideas by Nash [68, 190] (see also Fabes and Stroock [69]). For the bound below, again we rely on Nash's paper, combining the presentation given in Fabes [68] and Semenov [229].

As shown, for example, in Fabes and Stroock [69], once the estimates on the fundamental solution Γ are available, the local Hölder continuity of its local solutions, and the Harnack inequality can be proven. We refrain from studying such properties in this chapter; we will deal with them in Chapter 12, in the more general context of parabolic DeGiorgi classes. Here, we limit ourselves to the bounds on Γ, which represent a very interesting issue in themselves.

In both coming sections, whenever we refer to the data, we mean the set $\{\lambda, \Lambda, N\}$. Moreover, we denote with ∂_t and ∂_{x_i} the partial derivatives with respect to t and x_i respectively.

15.1 The Gaussian Upper Bound

We state and prove a number of introductory results, before coming to the main statement.

Proposition 15.1 *There exists a positive constant γ, depending only on the data, such that for $s < t$ and all $x \in \mathbb{R}^N$ we have*

$$\int_{\mathbb{R}^N} |x-y|\Gamma(x,t;y,s)\,dy \le \gamma(t-s)^{\frac{1}{2}}.$$

In order to prove Proposition 15.1, we need some introductory Lemmata.

Lemma 15.1 *There exists a positive constant γ, depending only on the data, such that for $s < t$ and all x, $y \in \mathbb{R}^N$ we have*

$$\Gamma(x,t;y,s) \le \frac{\gamma}{(t-s)^{\frac{N}{2}}}.$$

Proof. Fix $t > 0$, let $p(x,t) = \Gamma(x,t;0,0)$, and set

$$u(t) = \int_{\mathbb{R}^N} p^2(x,t)\,dx.$$

Differentiating, we have

$$\begin{aligned} u'(t) &= 2\int_{\mathbb{R}^N} (\partial_t p)\,p\,dx = 2\int_{\mathbb{R}^N} p\,\mathrm{div}\,(a_{ij}\nabla p)\;dx \\ &= -2\int_{\mathbb{R}^N} \left(a_{ij}\partial_{x_j}p\,\partial_{x_i}p\right)\,dx \le -2\lambda\int_{\mathbb{R}^N} |\nabla p|^2\,dx. \end{aligned}$$

If we rely on the so-called Nash inequality (see Section 15c of the Complements for the proof)

$$\left(\int_{\mathbb{R}^N} p^2\,dx\right)^{1+\frac{2}{N}} \le \gamma(N)\left[\int_{\mathbb{R}^N} |\nabla p|^2\,dx\right]\left[\int_{\mathbb{R}^N} p\,dx\right]^{\frac{4}{N}}, \tag{15.2}$$

and take into account that $\int_{\mathbb{R}^N} p\,dx = 1$, we have

$$\begin{aligned}
&u'(t) \le -2\frac{\lambda}{\gamma(N)}u^{1+\frac{2}{N}}, \qquad \int_{u(t/2)}^{u(t)} \frac{du}{u^{1+2/N}} \le -\gamma\int_{t/2}^{t} ds,\\
&\frac{N}{2}u^{-2/N}(t/2) - \frac{N}{2}u^{-2/N}(t) \le -\gamma\frac{t}{2}, \qquad \gamma t + \frac{N}{2}u^{-2/N}(t/2) \le \frac{N}{2}u^{-2/N}(t),\\
&\gamma t + u^{-2/N}(t/2) \le \frac{1}{u^{2/N}(t)}, \qquad u^{2/N} \le \frac{1}{\gamma t + u^{-2/N}(t/2)},
\end{aligned}$$

and therefore,

$$u \le \gamma t^{-N/2} \quad \Leftrightarrow \quad \int_{\mathbb{R}^N} \Gamma^2(x,t;0,0)\,dx \le \gamma t^{-N/2}.$$

By the translation invariance and the reproducing property, we easily obtain

$$\int_{\mathbb{R}^N} \Gamma^2(x,t;y,s)\,dy \le \gamma(t-s)^{-N/2}$$

whenever $s < t$, and also

$$\int_{\mathbb{R}^N} \Gamma^2(x,t;y,s)\,dx \le \gamma(t-s)^{-N/2}.$$

Since for $s < t$

$$\Gamma(x,t;y,s) = \int_{\mathbb{R}^N} \Gamma\Big(x,t;z,\frac{t+s}{2}\Big)\Gamma\Big(z,\frac{t+s}{2};y,s\Big)\,dz,$$

by the Hölder inequality

$$\begin{aligned}
\Gamma(x,t;y,s) &\le \left(\int_{\mathbb{R}^N} \Gamma^2\Big(x,t;z,\frac{t+s}{2}\Big)\,dz\right)^{1/2}\left(\int_{\mathbb{R}^N} \Gamma^2\Big(z,\frac{t+s}{2};y,s\Big)\,dz\right)^{1/2}\\
&\le \gamma(t-s)^{-N/2}.
\end{aligned}$$

■

Lemma 15.2 *There exists a constant $\gamma > 0$, depending only on the data, such that for all $y \in \mathbb{R}^N$,*

$$\int_{\mathbb{R}^N} |x-y|\Gamma(x,t;y,s)\,dx \le \gamma(t-s)^{1/2}.$$

Proof. As in the proof of Lemma 15.1, we can take $y = 0$, $s = 0$. Moreover, as before, we let $p(x,t) = \mathit{\Gamma}(x,t;0,0)$, and define

$$M_1(t) = \int_{\mathbb{R}^N} |x|\, p(x,t)\, dx.$$

We have

$$\begin{aligned} M_1'(t) &= \int_{\mathbb{R}^N} |x| \partial_t p(x,t)\, dx = \int_{\mathbb{R}^N} |x| \operatorname{div}\left(a_{ij}\partial_{x_i} p\right)\, dx \\ &= -\int_{\mathbb{R}^N} a_{ij}\partial_{x_i} p \frac{x_j}{|x|}\, dx = -\int_{\mathbb{R}^N} a_{ij}\partial_{x_i} p \frac{1}{\sqrt{p}}\sqrt{p}\frac{x_j}{|x|}\, dx \\ &\le \left(\int_{\mathbb{R}^N} a_{ij} \frac{\partial_{x_i} p\, \partial_{x_j} p}{p}\, dx\right)^{1/2} \left(\int_{\mathbb{R}^N} a_{ij} \frac{x_i}{|x|}\frac{x_j}{|x|}\, p\, dx\right)^{1/2} \\ &= C\left(\int_{\mathbb{R}^N} \frac{|\nabla p|^2}{p}\, dx\right)^{1/2}, \end{aligned}$$

where the constant $C > 0$ depends only on the data. Now, we introduce the entropy

$$Q(t) = -\int_{\mathbb{R}^N} p(x,t)\, \ln p(x,t)\, dx.$$

If we differentiate, we have

$$\begin{aligned} Q'(t) &= -\int_{\mathbb{R}^N} \left(\partial_t p\, \ln p + \partial_t p\right)\, dx \\ &= -\int_{\mathbb{R}^N} \left[\operatorname{div}\left(a_{ij}\partial_{x_j} p\right)\, \ln p + \operatorname{div}\left(a_{ij}\partial_{x_j} p\right)\right]\, dx \\ &= \int_{\mathbb{R}^N} a_{ij} \frac{\partial_{x_j} p \partial_{x_i} p}{p}\, dx. \end{aligned}$$

Hence, by the ellipticity condition

$$\lambda \int_{\mathbb{R}^N} \frac{|\nabla p|^2}{p}\, dx \le Q'(t) \le \mathit{\Lambda} \int_{\mathbb{R}^N} \frac{|\nabla p|^2}{p}\, dx,$$

and we conclude that

$$M_1'(t) \le C\left[Q'(t)\right]^{1/2} = C\left(Q'(t) t \frac{1}{t}\right)^{1/2} \le C t^{1/2}\left[Q'(t) + \frac{1}{t}\right]. \tag{15.3}$$

If we integrate from $\epsilon > 0$ to t, we have

$$\begin{aligned} M_1(t) - M_1(\epsilon) \le& C\int_\epsilon^t \tau^{1/2} Q'(\tau)\, d\tau + C\int_\epsilon^t \frac{1}{\tau^{1/2}}\, d\tau \\ \le& 2C\left[t^{1/2} - \epsilon^{1/2}\right] + C\left(Q(t) t^{1/2} - Q(\epsilon)\epsilon^{1/2}\right) \end{aligned}$$

$$-\frac{C}{2}\int_{\epsilon}^{t}\frac{Q(\tau)}{\sqrt{\tau}}\,d\tau,$$

with $C>0$. If we let as usual $Q_{-}\overset{\text{def}}{=}\max\{0;-Q\}$, we obtain

$$M_1(t)-M_1(\epsilon)\le 2Ct^{1/2}+CQ(t)t^{1/2}+\frac{C}{2}\int_{\epsilon}^{t}\frac{Q_{-}(\tau)}{\sqrt{\tau}}\,d\tau. \tag{15.4}$$

We need an estimate for Q_{-}. Since from Lemma 15.1 we have

$$\Gamma(x,t;0,0)=p(x,t)\le\gamma t^{-N/2},$$

we get

$$-\ln p\ge -C+\frac{N}{2}\ln t$$

and also

$$\begin{aligned}Q(t)&=-\int_{\mathbb{R}^N}p(x,t)\,\ln p(x,t)\,dx\\&\ge\int_{\mathbb{R}^N}\left[\min_{x\in\mathbb{R}^N}(-\ln p(x,t))\right]p(x,t)\,dx\\&\ge\int_{\mathbb{R}^N}\left[-C+\frac{N}{2}\ln t\right]p(x,t)\,dx=-C+\frac{N}{2}\ln t,\end{aligned}$$

because $\int_{\mathbb{R}^N}p(x,t)\,dx=1$. Therefore, we have proved that

$$Q(t)\ge -C+\frac{N}{2}\ln t, \tag{15.5}$$

which implies that

$$Q_{-}(t)\le C+\frac{N}{2}|\ln t|. \tag{15.6}$$

If we insert (15.6) in (15.4) and let $t\to 0^{+}$, we conclude that

$$M_1(t)\le 2Ct^{1/2}+CQ(t)t^{1/2}+\frac{C}{2}\int_{0}^{t}\frac{|\ln\tau|}{\sqrt{\tau}}\,d\tau.$$

Now we want to estimate M_1 from below. If we consider

$$g:(0,+\infty)\to\mathbb{R},\qquad\text{defined by}\qquad g(p)=p\,\ln p+\sigma p,$$

since $g'(p)=\ln p+\sigma+1$, we immediately obtain

$$\min_{p\in(0,+\infty)}g(p)=-e^{-\sigma-1}.$$

Hence, if we choose $\sigma=a|x|+b$, we have

$$p(x,t)\,\ln p(x,t) + (a|x|+b)p(x,t) \geq -e^{-b-1}e^{-a|x|}$$

and also, if we integrate over $\mathbb{R}^N$,

$$\int_{\mathbb{R}^N} [p(x,t)\,\ln p(x,t) + (a|x|+b)p(x,t)]\,dx \geq -e^{-b-1}\int_{\mathbb{R}^N} e^{-a|x|}\,dx,$$

which implies

$$\begin{aligned} -Q(t) + aM_1(t) + b &\geq -e^{-b-1}\int_{|\omega|=1} d\omega \int_0^\infty e^{-a\rho}\rho^{N-1}\,d\rho \\ &= -\frac{e^{-b-1}}{a^N}\int_{|\omega|=1} d\omega \int_0^\infty e^{-a\rho}(a\rho)^{N-1} d(a\rho), \end{aligned}$$

that is,

$$-Q(t) + aM_1(t) + b \geq -C_N e^{-b} a^{-N}.$$

If we choose $a = \dfrac{1}{M_1(t)}$, $e^{-b} = a^N$, we obtain

$$-Q(t) + 1 - N\ln\frac{1}{M_1(t)} \geq -C_N,$$

that is,

$$Q(t) \leq C_N + 1 + N\ln M_1(t),$$

which yields

$$CM_1 \geq e^{\frac{1}{N}Q(t)}.$$

Hence, we have the following estimates

$$Q(t) \geq -C + \frac{N}{2}\ln t; \tag{15.7}$$

$$M_1(t) \leq 2Ct^{1/2} + CQ(t)t^{1/2} + \frac{C}{2}\int_0^t \frac{|\ln\tau|}{\sqrt{\tau}}d\tau; \tag{15.8}$$

$$CM_1(t) \geq e^{\frac{1}{N}Q(t)}. \tag{15.9}$$

From (15.3) and (15.9), we have

$$\gamma_1 e^{\frac{1}{N}Q(t)} \leq M_1(t) \leq \gamma_2 \int_0^t [Q'(\tau)]^{1/2}\,d\tau.$$

Now define $NR(t) = Q(t) + C - \dfrac{N}{2}\ln t$. By what we have just seen, $NR \geq 0$. Moreover, $Q'(t) = NR'(t) + \dfrac{N}{2t}$. Hence,

$$\gamma_1 e^{R(t) - \frac{C}{N} + \frac{1}{2}\ln t} \leq M_1(t) \leq N^{1/2}\gamma_2 \int_0^t \left(R'(\tau) + \frac{1}{2\tau}\right)^{1/2} d\tau,$$

$$\gamma_3 \sqrt{t}\, e^{R(t)} \le M_1(t) \le \gamma_2 \left[\int_0^t \left(\frac{1}{2\tau} \right)^{1/2} d\tau + \int_0^t \sqrt{\frac{\tau}{2}} R'(\tau)\, d\tau \right],$$

$$\gamma_3 \sqrt{t}\, e^{R(t)} \le M_1(t) \le \gamma_2 \left[\sqrt{2t} + \sqrt{\frac{t}{2}}\, R(t) - \int_0^t \frac{1}{2\sqrt{2}} \frac{R(\tau)}{\sqrt{\tau}}\, d\tau \right],$$

where we have used that when a and $a+b$ are positive, $(a+b)^{\frac{1}{2}} \le a^{\frac{1}{2}} + \dfrac{b}{2a^{\frac{1}{2}}}$; this yields

$$\gamma_3 \sqrt{t}\, e^{R(t)} \le M_1(t) \le \gamma_2 \sqrt{t} \left[\sqrt{2} + \frac{1}{\sqrt{2}} R(t) \right].$$

Since the right-hand side grows slower than the left-hand one, we conclude that $R(t)$ has to be bounded above and

$$\gamma_4 \sqrt{t} \le M_1(t) \le \gamma_5 \sqrt{t}.$$

■

Remark 15.1 Since

$$\gamma_1 e^{\frac{1}{N} Q(t)} \le M_1(t),$$

we have

$$\gamma_1 e^{\frac{1}{N} Q(t)} \le \gamma_5 t^{1/2} \quad \Rightarrow \quad Q(t) \le \gamma_8 + \frac{N}{2} \ln t.$$

Hence, we can say that

$$-C + \frac{N}{2} \ln t \le Q(t) \le C + \frac{N}{2} \ln t.$$

Proof of Proposition 15.1 – The bounds above and below on $M_1(t)$ proved in Lemma 15.2 amount to the bound required by Proposition 15.1, again because of the properties of the fundamental solution. ■

Proposition 15.2 *For each $j \in \mathbb{N}$, there exists $C_j > 0$, depending only on the data and j, such that*

$$M_j(t) \overset{def}{=} \int_{\mathbb{R}^N} |x|^j\, \Gamma(x,t;0,0)\, dx \le C_j t^{j/2}.$$

Proof. It obviously suffices to take $j \ge 2$. As before, we let $p(x,t) = \Gamma(x,t;0,0)$. If we differentiate, we obtain

$$\begin{aligned} M_j'(t) &= \int_{\mathbb{R}^N} |x|^j\, \partial_t p\, dx = \int_{\mathbb{R}^N} |x|^j \operatorname{div}\left(a_{ik} \partial_{x_k} p\right) dx \\ &= -j \int_{\mathbb{R}^N} a_{ik} \partial_{x_k} p\, |x|^{j-1} \frac{x_i}{|x|}\, dx \\ &\le C_j \int_{\mathbb{R}^N} |x|^{j-1}\, |\nabla p|\, dx = C_j \int_{\mathbb{R}^N} |x|^{\frac{j}{2}}\, |x|^{j/2-1} \frac{|\nabla p|}{\sqrt{p}} \sqrt{p}\, dx \end{aligned}$$

$$\leq C_j \left(\int_{\mathbb{R}^N} |x|^j \, p(x,t) \, dx \right)^{1/2} \left(\int_{\mathbb{R}^N} |x|^{j-2} \frac{|\nabla p|^2}{p} \, dx \right)^{1/2}$$
$$= C_j M_j^{1/2} \left(\int_{\mathbb{R}^N} |x|^{j-2} \, \frac{|\nabla p|^2}{p} \, dx \right)^{1/2} .$$

Hence,

$$\sqrt{M_j(t)} \leq C_j \int_0^t \left(\int_{\mathbb{R}^N} |x|^{j-2} \, \frac{|\nabla p|^2}{p} \, dx \right)^{1/2} \, ds. \tag{15.10}$$

If we now define $Q_j(t) = - \int_{\mathbb{R}^N} |x|^j p \ln p \, dx$, we easily obtain

$$\begin{aligned} Q_j' &= - \int_{\mathbb{R}^N} |x|^j \left[\partial_t p \, \ln p + \partial_t p \right] \, dx \\ &= - \int_{\mathbb{R}^N} |x|^j \, \left[\operatorname{div} \left(a_{ik} \partial_{x_k} p \right) \right] (1 + \ln p) \, dx \\ &= j \int_{\mathbb{R}^N} |x|^{j-2} \, a_{i_k} \partial_{x_k} p \, x_i \, (1 + \ln p) \, dx + \int_{\mathbb{R}^N} |x|^j \, a_{ik} \frac{\partial_{x_k} p \, \partial_{x_i} p}{p} \, dx. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_{\mathbb{R}^N} |x|^j \, \frac{|\nabla p|^2}{p} \, dx &\leq \Lambda Q_j' + \Lambda^2 j \int_{\mathbb{R}^N} |x|^{j-1} |\nabla p| (1 + |\ln p|) \, dx \\ &= \Lambda Q_j' + \Lambda^2 j \int_{\mathbb{R}^N} |x|^{\frac{j-1}{2}} \sqrt{p} (1 + |\ln p|) \, |x|^{\frac{j-1}{2}} \, \frac{|\nabla p|}{\sqrt{p}} \, dx \\ &\leq \Lambda Q_j' \\ &\quad + \Lambda^2 j \left[\int_{\mathbb{R}^N} |x|^{j-1} \frac{|\nabla p|^2}{p} \, dx \right]^{1/2} \left[\int_{\mathbb{R}^N} |x|^{j-1} p (1 + |\ln p|)^2 \, dx \right]^{1/2} \\ &\leq \Lambda Q_j' \\ &\quad + C_j \Lambda \left[\int_{\mathbb{R}^N} |x|^{j-1} \frac{|\nabla p|^2}{p} \, dx \right]^{1/2} \left[M_{j-1} + \int_{\mathbb{R}^N} |x|^{j-1} p |\ln p|^2 \, dx \right]^{1/2} . \end{aligned}$$

We conclude that

$$\begin{aligned} \int_{\mathbb{R}^N} |x|^j \frac{|\nabla p|^2}{p} \, dx &\leq \Lambda Q_j' \\ &\quad + C_j \left[\int_{\mathbb{R}^N} |x|^{j-1} \frac{|\nabla p|^2}{p} \, dx + M_{j-1} + \int_{\mathbb{R}^N} |x|^{j-1} p |\ln p|^2 \, dx \right] . \end{aligned}$$

If we iterate, we have

$$\begin{aligned} \int_{\mathbb{R}^N} |x|^j \frac{|\nabla p|^2}{p} \, dx \leq \sum_{k \leq j} C_{k_j} \, Q_k'(t) + \sum_{k<j} \tilde{C}_{k_j} \, M_k(t) \\ + \sum_{k<j} \tilde{C}_{k_j} \int_{\mathbb{R}^N} |x|^k p \, |\ln p|^2 \, dx \end{aligned} \tag{15.11}$$

where $C_{k_j} > 0$, $\tilde{C}_{k_j} \geq 0$ depend only on k, j, and the data. Inserting (15.11) in (15.10) yields

$$\begin{aligned} M_j^{1/2}(t) \leq \int_0^t \Big[\sum_{k \leq j-2} C_{k_j} Q_k'(s) + \sum_{k<j-2} \tilde{C}_{k_j} M_k(s) \\ + \sum_{k<j-2} \tilde{C}_{k_j} \int_{\mathbb{R}^N} |x|^k \, p \, | \ln p|^2 \, dx \Big]^{1/2} ds. \end{aligned} \tag{15.12}$$

Now, we need to control the right-hand side of (15.12). This is given by the following result.

Lemma 15.3 *Set*

$$P_k(t) = \int_{\mathbb{R}^N} |x|^k \, p(x,t) | \ln p(x,t)|^2 \, dx.$$

Then there exists a positive constant C_k, depending only on k, and the data, such that

$$P_k(t) \leq C_k \left(1 + \ln^2 t\right) \left(M_k(t) + \frac{1}{\sqrt{t}} M_{k+1}(t) + t^{k/2} \right).$$

Proof. Since

$$\Gamma_a(x,t;0,0) = \Gamma_a \left(\sqrt{t} \frac{x}{\sqrt{t}}, t \cdot 1; 0, 0 \right) = t^{-N/2} \Gamma_{a(\sqrt{t} \cdot \,, t \cdot\,)} \left(\frac{x}{\sqrt{t}}, 1; 0, 0 \right),$$

it suffices to assume $t = 1$. We let $p(x) = \Gamma(x,1;0,0)$, and we want to show that

$$\int_{\mathbb{R}^N} |x|^k \, p(x) | \ln p(x)|^2 \, dx \leq C_k \left[M_k(1) + M_{k+1}(1) + 1 \right].$$

Since p is bounded by Lemma 15.1, we have

$$\begin{aligned} \int_{\mathbb{R}^N \cap \{p > 1/e\}} |x|^k \, p | \ln p|^2 \, dx &\leq C \int_{\mathbb{R}^N} |x|^k \, p^2 \, dx \\ &\leq \tilde{C} \int_{\mathbb{R}^N} |x|^k \, p \, dx = \tilde{C} M_k(1). \end{aligned}$$

Moreover, for any $r \in \mathbb{R}_+$

$$\max_{0 \leq p \leq 1/e} p \left[\ln^2 p - r \right] = e^{-\sqrt{1+r}-1} [2\sqrt{1+r} + 2].$$

Hence,

$$\int_{\mathbb{R}^N \cap \{0 < p < 1/e\}} |x|^k p \, | \ln p|^2 \, dx$$

$$\begin{aligned}
&= \int_{\mathbb{R}^N\cap\{0<p<1/e\}} |x|^k p \left[|\ln p|^2 - |x| + |x|\right] dx \\
&= \int_{\mathbb{R}^N\cap\{0<p<1/e\}} |x|^k p \left[|\ln p|^2 - |x|\right] dx + M_{k+1}(1) \\
&\le C \int_{\mathbb{R}^N\cap\left\{0<p<\frac{1}{e}\right\}} |x|^k e^{-\sqrt{1+|x|}}\, dx + M_{k+1}(1) \\
&\le C_k + M_{k+1}(1).
\end{aligned}$$

■

If we now go back to (15.12), we have

$$M_j^{1/2}(t) \le \int_0^t \Big[\sum_{k\le j-2} C_{k_j} Q_k'(s) + \sum_{0\le k<j-2} \tilde{C}_{k_j}\left(M_k(s) + P_k(s)\right) \Big]^{1/2} ds.$$

Now we work by induction, recalling that $M_o(t) \equiv 1$, $M_1(t) \le C_1 t^{1/2}$. Take $j \ge 2$, and assume that $M_k(t) \le C_k t^{k/2}$ for all $0 \le k < j$. Relying on all the previous estimates, we have

$$M_j^{1/2}(t) \le \int_0^t \Big[\sum_{k\le j-2} C_{k_j} Q_k'(s) + C_j \sum_{0\le k<j-2} s^{k/2}\left(1 + \ln^2 s\right) \Big]^{1/2} ds.$$

If we take $t = 1$, we have

$$\begin{aligned}
M_j^{1/2}(1) \le &\int_0^1 \Big[\sum_{k\le j-2} C_{kj} Q_k'(s) + C_j s^{-1} \Big]^{1/2} ds \\
\le & C_j^{1/2} \int_0^1 s^{-1/2}\, ds + \frac{1}{2C_j^{1/2}} \int_0^1 \sum_{k\le j-2} C_{k_j} Q_k'(s) s^{1/2}\, ds \\
\le & 2C_j^{1/2} + \frac{1}{2C_j^{1/2}} \sum_{k\le j-2} C_{k_j} Q_k(1) \\
& - \frac{1}{4C_j^{1/2}} \int_0^1 \sum_{k\le j-2} C_{k_j} Q_k(s) \frac{1}{s^{1/2}}\, ds.
\end{aligned}$$

Since $|Q_k(t)| \le P_k^{1/2}(t) M_k^{1/2}(t)$, by induction we conclude that $\exists \widetilde{C}_j > 0$ such that $M_j(1) \le \tilde{C}_j$. Scaling back, we conclude. ■

As

$$\Gamma_a(x,t;y,s) = \Gamma_{a(x+\cdot,t-\cdot)}(y-x, t-s; 0, 0),$$

we also have the following.

Corollary 15.1 *For each $j \in \mathbb{N}$, there exists $C_j > 0$ such that $\forall s < t$*

$$\int_{\mathbb{R}^N} |x-y|^j\, \Gamma(x,t;y,s)\, dy \le C_j (t-s)^{j/2}.$$

We now have all the elements, to state and prove the Gaussian upper bound on the fundamental solution

Theorem 15.1. *There exists a constant $C > 0$, depending only on the data, such that*

$$\Gamma(x,t;y,s) \le C(t-s)^{-N/2} e^{-|x-y|^2/C(t-s)}.$$

As with Propositions 15.1–15.2, we need a couple of auxiliary results.

Lemma 15.4 *Let*

$$\varphi_j(t) = \int_{\mathbb{R}^N} |x|^{2j}\, \Gamma^2(x,t;0,0)\, dx.$$

Then, there exists a positive constant C, depending only on the data, such that

$$\varphi_j(t) \le C^j j!\, t^{-\frac{N}{2}+j}.$$

Proof. From Lemma 15.1 and Proposition 15.2

$$\varphi_j(t) \le Ct^{-N/2} \int_{\mathbb{R}^N} |x|^{2j}\, \Gamma(x,t;0,0)\, dx \le CC_j t^{-N/2+j}.$$

Hence, for $0 \le j \le \left\lfloor \frac{N}{2} \right\rfloor + 1$, we can find B, depending only on the data, such that

$$\varphi_j(t) \le B^j j!\, t^{-\frac{N}{2}+j}.$$

Now, take $j > \left\lfloor \frac{N}{2} \right\rfloor + 1$. Setting as usual $p(x,t) = \Gamma(x,t;0,0)$, we have

$$\begin{aligned}
\varphi_j'(t) &= 2\int_{\mathbb{R}^N} |x|^{2j}\, p\, \partial_t p\, dx = 2\int_{\mathbb{R}^N} |x|^{2j} p \operatorname{div}(a_{ik}\partial_{x_i} p)\, dx \\
&= -2\int_{\mathbb{R}^N} |x|^{2j} a_{ik}\partial_{x_i} p\, \partial_{x_k} p\, dx - 4j\int_{\mathbb{R}^N} p|x|^{2j-1} a_{ik}\partial_{x_i} p \frac{x_k}{|x|}\, dx \\
&\le \tilde{C}j \int_{\mathbb{R}^N} |x|^{2j-1} p\, |\nabla p|\, dx - 2\lambda \int_{\mathbb{R}^N} |x|^{2j} |\nabla p|^2\, dx \\
&= \tilde{C}j \int_{\mathbb{R}^N} |x|^{j-1}\, p|x|^j\, |\nabla p|\, dx - 2\lambda \int_{\mathbb{R}^N} |x|^{2j} |\nabla p|^2\, dx \\
&\le \tilde{C}j^2 \int_{\mathbb{R}^N} |x|^{2(j-1)}\, p^2\, dx,
\end{aligned}$$

that is

$$\varphi_j'(t) \le \tilde{C}j^2 \varphi_{j-1}(t).$$

Since $j > \left\lfloor \frac{N}{2} \right\rfloor + 1$, we have $\varphi_j(0) = 0$. Hence,

$$\varphi_j(t) \le \tilde{C}j^2 \int_0^t \varphi_{j-1}(s)\, ds.$$

If we assume

$$\varphi_{j-1}(s) \leq C^{j-1}(j-1)!\, s^{-\frac{N}{2}+(j-1)},$$

we obtain

$$\varphi_j(t) \leq \tilde{C} j^2 C^{j-1}(j-1)!\, \frac{t^{-\frac{N}{2}+j}}{-\frac{N}{2}+j} = \tilde{C} j!\, C^{j-1} \frac{j}{j-\frac{N}{2}} t^{-\frac{N}{2}+j}.$$

For $j > \left\lfloor \frac{N}{2} \right\rfloor + 1$, we have $\frac{j}{j-\frac{N}{2}} \leq \frac{\left\lfloor \frac{N}{2} \right\rfloor + 1}{\left\lfloor \frac{N}{2} \right\rfloor + 1 - \frac{N}{2}} = D_N$. Hence, choosing

$$C \geq \max\left\{B, \tilde{C} D_N\right\},$$

we can conclude by induction. ■

Corollary 15.2 *There exists a constant $C > 0$, depending only on the data, such that, for all $j \in \mathbb{N}$*

$$|x|^j \Gamma(x,t;0,0) \leq C^j (j!)^{1/2}\, t^{-N/2+j/2}.$$

Proof. By the reproducing property of the fundamental solution, we have

$$\Gamma(x,t;0,0) = \int_{\mathbb{R}^N} \Gamma\Big(x,t;y,\frac{t}{2}\Big)\, \Gamma\Big(y,\frac{t}{2};0,0\Big)\, dy,$$

and also

$$\begin{aligned} |x|^j \Gamma(x,t;0,0) \leq & 2^j \int_{\mathbb{R}^N} |x-y|^j \Gamma\Big(x,t;y,\frac{t}{2}\Big)\, \Gamma\Big(y,\frac{t}{2};0,0\Big)\, dy \\ & + 2^j \int_{\mathbb{R}^N} |y|^j\, \Gamma\Big(x,t;y,\frac{t}{2}\Big)\, \Gamma\Big(y,\frac{t}{2};0,0\Big)\, dy. \end{aligned}$$

If we now apply the Schwarz inequality, and Lemma 15.4, we conclude. ■

Eventually, we finish the proof of Theorem 15.1.

Proof. As always, we can take $y = 0$, $s = 0$. By Corollary 15.2,

$$\begin{aligned} e^{\frac{\delta|x|^2}{t}}\, \Gamma(x,t;0,0) &= \sum_{j=0}^{\infty} \frac{\delta^j}{j!\, t^j}\, |x|^{2j}\, \Gamma(x,t;0,0) \\ &\leq \sum_{j=0}^{\infty} \frac{\delta^j}{j!} C^{2j} t^{-N/2} [(2j)!]^{1/2}, \end{aligned}$$

whence

$$t^{N/2} e^{\frac{\delta|x|^2}{t}}\, \Gamma(x,t;0,0) \leq \sum_{j=0}^{\infty} \frac{\delta^j C^j ((2j)!)^{1/2}}{j!},$$

where we have relabeled C^2 as C. Now, choosing $\delta = \frac{1}{4C}$ yields

$$t^{N/2} e^{\frac{|x|^2}{4Ct}} \Gamma(x,t;0,0) \leq \sum_{j=0}^{\infty} \frac{((2j)!)^{1/2}}{4^j j!}.$$

We recall the Stirling formula $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$. Hence, for n sufficiently large

$$\sum_{j=n}^{\infty} \frac{((2j)!)^{1/2}}{4^j j!} = \sum_{j=n}^{\infty} \frac{\left(\sqrt{4\pi j}\left(\frac{2j}{e}\right)^{2j}\right)^{1/2}}{4^j \sqrt{2\pi j}\left(\frac{j}{e}\right)^j} = \sum_{j=n}^{\infty} \frac{1}{\pi^{1/4} j^{1/4} 2^j}.$$

Since the series on the right-hand side has finite sum, we conclude that

$$\Gamma(x,t;0,0) \leq \frac{C}{t^{N/2}} e^{-\frac{|x|^2}{4Ct}}.$$

■

15.2 The Gaussian Lower Bound

We now come to the lower bound. Assume for the moment that we have the following.

Proposition 15.3 *There exists a positive constant C, depending only on the data, such that $\forall\, x_o \in \mathbb{R}^N$, $\forall\, x,\, y \in B_{\sqrt{t-s}}(x_o)$, $\forall s < t$, we have*

$$\Gamma(x,t;y,s) \geq \frac{1}{C(t-s)^{N/2}}.$$

Assuming the proposition for the moment, we prove this lower bound, by an argument due to Fabes and Stroock [69].

Theorem 15.2. *There exists a constant $C > 0$, depending only on the data, such that*

$$\Gamma(x,t;y,s) \geq \frac{1}{C(t-s)^{N/2}} \exp\left(-\frac{C|x-y|^2}{t-s}\right).$$

Proof. We assume $y = 0$, $s = 0$. Moreover, by the usual properties of the fundamental solution, we can also take $t = 1$. Hence, it suffices to prove

$$\Gamma(x,1;0,0) \geq \frac{1}{C} \exp\left(-C|x|^2\right).$$

In view of Proposition 15.3, we may also take $|x| > 1$. Given $x \in \mathbb{R}^N$ with $|x| > 1$, let $k \in \mathbb{N}$ be the smallest integer such that $k > 4|x|^2$, and let

$$S = \prod_{l=1}^{k-1} B_{\frac{1}{2\sqrt{k}}}\left(\frac{l}{k}x\right).$$

For $(\xi_1, \xi_2, \ldots, \xi_{k-1}) \in S$, by simple computation we have

$$|\xi_1| < \frac{1}{\sqrt{k}}, \quad \max_{1<l<k} |\xi_l - \xi_{l-1}| < \frac{1}{\sqrt{k}}, \quad |x - \xi_{k-1}| < \frac{1}{\sqrt{k}}.$$

Hence, by Proposition 15.3

$$\begin{aligned}
\Gamma(x,1;0,0) &= \int_{\mathbb{R}^N} \cdots \int_{\mathbb{R}^N} \Gamma\left(x,1;\xi_{k-1},\frac{k-1}{k}\right) \cdot \\
&\quad \cdot \Gamma\left(\xi_{k-1},\frac{k-1}{k};\xi_{k-2},\frac{k-2}{k}\right) \cdots \Gamma\left(\xi_1,\frac{1}{k};0,0\right) d\xi_1 \ldots d\xi_{k-1} \\
&\ge \left(\frac{k^{N/2}}{C}\right)^k |S| \\
&= \left(\frac{k^{N/2}}{C}\right)^k \left[\omega_N \left(\frac{1}{2^{1/2}k^{1/2}}\right)^N\right]^{k-1} \\
&= \left(\frac{\omega_N}{C2^{N/2}}\right)^k \frac{2^{N/2}k^{N/2}}{\omega_N}
\end{aligned}$$

and from here we immediately conclude. ■

Proof of Proposition 15.3 – Without loss of generality, we may assume $s = 0$, and work with $t > 0$. For simplicity, we let $p(x,t,y) = \Gamma(x,t;y,0)$. We want to show that there exists $k > 0$ such that $\forall\, x_o \in \mathbb{R}^N$, $\forall\, x,\, y \in B_{\sqrt{t}}(x_o)$, $\forall\, t > 0$

$$p(x,t,y) \ge kt^{-N/2},$$

where k depends only on the data.

Again, without loss of generality, we can assume $x_o = 0$. In the following we let

$$p_H(x,t,y) = \frac{1}{(4\pi t)^{N/2}} \exp\left(-\frac{|x-y|^2}{4t}\right)$$

and

$$p_o(x,t) = p_H(x,\delta t;0)$$

for any $\delta > \Lambda$. We have

$$\begin{aligned}
\Gamma(x,2t;y,0) &= \int_{\mathbb{R}^N} \Gamma(x,2t;z,t)\, \Gamma(z,t;y,0)\, dz \\
&= \int_{\mathbb{R}^N} \Gamma_{a(\cdot,\cdot+t)}(x,t;z,0)\, \Gamma(z,t;y,0)\, dz \\
&= \int_{\mathbb{R}^N} \Gamma_{a(\cdot,\cdot+t)}(x,t;z,0)\, \Gamma_{a(\cdot,t-\cdot)}(y,t;z,0)\, dz.
\end{aligned}$$

If we let

$$p_+(x,t,z) = \Gamma_{a(\cdot,\cdot+t)}(x,t;z,0), \qquad p_-(y,t,z) = \Gamma_{a(\cdot,t-\cdot)}(y,t;z,0),$$

this yields

$$\begin{aligned}p(x,2t,y) &= \int_{\mathbb{R}^N} p_+(x,t,z)p_-(y,t,z)\,dz\\ &\geq \int_{\mathbb{R}^N} \frac{p_o(z,t)}{\|p_o\|_\infty} p_+(x,t,z)\,p_-(y,t,z)\,dz\\ &= (4\pi\delta t)^{N/2}\int_{\mathbb{R}^N} p_H(z,\delta t,0)\,p_+(x,t,z)\,p_-(y,t,z)\,dz.\end{aligned}$$

By Jensen's inequality

$$\begin{aligned}&\ln\int_{\mathbb{R}^N} p_H(z,\delta t,0)\,p_+(x,t,z)\,p_-(y,t,z)\,dz\\ &\qquad\geq \int_{\mathbb{R}^N} \ln[p_+(x,t,z)\,p_-(y,t,z)]\,p_H(z,\delta t,0)\,dz\\ &\qquad\geq \int_{\mathbb{R}^N} \ln p_+(x,t,z)\,p_H(z,\delta t,0)\,dz + \int_{\mathbb{R}^N} \ln p_-(y,t,z)\,p_H(z,\delta t,0)\,dz.\end{aligned}$$

Hence,

$$\ln p(x,2t,y) \geq \frac{N}{2}\ln(4\pi\delta) + \frac{N}{2}\ln t + \Theta_x^+(t) + \Theta_y^-(t),$$

where we have set $\Theta_z^{\pm}(t) = \int_{\mathbb{R}^N} \ln p_{\pm}(z,t,s)\,p_H(s,\delta t,0)\,ds$. Therefore, if we can show that

$$\forall z \in B_{\sqrt{t}} \quad \Theta_z^{\pm}(t) \geq -\frac{N}{2}\ln t + \gamma \tag{15.13}$$

for some positive γ, depending only on the data, we have finished. In the following we proceed with a general $p(x,t,z)$, irrespective of any time-shift. In order to prove (15.13), we introduce Nash's function

$$G(\tau) = \int_{R^N} p_H(0,\delta t,s)\ln p(z,\tau,s)\,ds$$

for $\frac{t}{2} \leq \tau \leq t$. It is apparent that

$$G(t) = \int_{\mathbb{R}^N} p_H(0,\delta t,s)\ln p(z,t,s)\,ds = \Theta_z(t).$$

If we differentiate, we obtain

$$\begin{aligned}
G'(\tau) &= \int_{\mathbb{R}^N} p_H(0,\delta t,s)\frac{1}{p(z,\tau,s)}\partial_\tau p(z,\tau,s)\,ds\\
&= \int_{\mathbb{R}^N} p_H(0,\delta t,s)\frac{1}{p(z,\tau,s)}\,\mathrm{div}\left(a_{ij}\partial_{s_j}p\right)\,ds\\
&= -\int_{\mathbb{R}^N} a_{ij}\partial_{s_i}p\,\partial_{s_j}\left(\frac{p_H}{p}\right)\,ds\\
&= -\int_{\mathbb{R}^N} a_{ij}\frac{\partial_{s_j}p}{p}\frac{\partial_{s_j}p_H}{p_H}p_H\,ds + \int_{\mathbb{R}^N} a_{ij}\frac{\partial_{s_i}p}{p}\frac{\partial_{s_j}p}{p}p_H\,ds\\
&\geq \int_{\mathbb{R}^N} a_{ij}\frac{\partial_{s_i}p}{p}\frac{\partial_{s_j}p}{p}p_H\,ds - \frac{1}{2}\int_{\mathbb{R}^N} a_{ij}\frac{\partial_{s_ip}}{p}\frac{\partial_{s_j}p}{p}p_H\,ds\\
&\quad -\frac{1}{2}\int_{\mathbb{R}^N} a_{ij}\frac{\partial_{s_i}p_H}{p_H}\frac{\partial_{s_j}p_H}{p_H}p_H\,ds = -\frac{1}{2}J_o + \frac{1}{2}J,
\end{aligned}$$

where we have set

$$J_o = \int_{\mathbb{R}^N} a_{ij}\frac{\partial_{s_i}p_H}{p_H}\frac{\partial_{s_j}p_H}{p_H}p_H\,ds, \quad J = \int_{\mathbb{R}^N} a_{ij}\frac{\partial_{s_i}p}{p}\frac{\partial_{s_j}p}{p}p_H\,ds.$$

If we now let

$$\varphi_o(0,t,s) \overset{\mathrm{def}}{=} \sqrt{p_H(0,t,s)}, \qquad \hat{\varphi}_o(s) \overset{\mathrm{def}}{=} \varphi_o(0,\delta t,s),$$

it is a matter of straightforward computation to check that

$$-\Delta_s\hat{\varphi}_o + \frac{|s|^2}{(4\delta t)^2}\hat{\varphi}_o - \frac{N}{4\delta t}\hat{\varphi}_o = 0, \text{ and } \int_{\mathbb{R}^N}|\nabla\hat{\varphi}_o|^2\,ds \leq \frac{N}{4\delta t}.$$

Hence, we can estimate J_o from above in the following way

$$\begin{aligned}
J_o &\leq \Lambda\int_{\mathbb{R}^N}\frac{|\nabla p_H|^2}{p_H}\,ds = 4\Lambda\int_{\mathbb{R}^N}|\nabla\hat{\varphi}_o|^2\,ds\\
&= \frac{\Lambda}{\delta}\frac{N}{t} = 2\frac{\Lambda}{\delta}\frac{d}{dt}\left(\frac{N}{2}\ln t\right).
\end{aligned}$$

In order to estimate J from below, we rely on the following Poincaré-type inequality

$$\int_{\mathbb{R}^N}\left(f - \overline{\varphi_o^2 f}\right)^2\varphi_o\,ds \leq 2t\int_{\mathbb{R}^N}|\nabla f|^2\varphi_o\,ds$$

where $\overline{\varphi_o^2 f} = \int_{\mathbb{R}^N}\varphi_o^2 f\,ds$. Hence, we have

$$\begin{aligned}
J &= \int_{\mathbb{R}^N} a_{ij}\frac{\partial_{s_i}p}{p}\frac{\partial_{s_j}p}{p}p_H\,ds\\
&= \int_{\mathbb{R}^N} a_{ij}\partial_{s_i}\ln p\,\partial_{s_j}\ln p\,p_H\,ds\\
&\geq \lambda\int_{\mathbb{R}^N}|\nabla\ln p|^2 p_H\,ds = \frac{\lambda}{2\delta t}\int_{\mathbb{R}^N}(\ln p - G(\tau))^2 p_H\,ds.
\end{aligned}$$

We can then conclude that

$$G'(\tau) \geq -\frac{\Lambda}{\delta}\frac{d}{dt}\left(\frac{N}{2}\ln t\right) + \frac{\lambda}{4\delta t}\int_{\mathbb{R}^N} p_H(\ln p - G(\tau))^2\, ds.$$

Setting $\tilde{Q}(t) = \frac{N}{2}\ln t$, it is immediate to check that for $\tau \leq t$, we have

$$\tilde{Q}'(\tau) \geq \tilde{Q}'(t).$$

Taking into account that $\delta \geq \Lambda$, we obtain

$$(G(\tau) + \tilde{Q}(\tau))' \geq \frac{\lambda}{4\delta t}\int_{\mathbb{R}^N} p_H(\ln p - G(\tau))^2\, ds. \tag{15.14}$$

By the upper bound of Theorem 15.1 and thanks to the triangle inequality

$$\forall\, z \in B_{\sqrt{t}},\ \tau \in \left[\frac{t}{2}, t\right],\ \delta > \Lambda, \quad p_H(0,\delta t, s) \geq k\, p(z, \tau s), \tag{15.15}$$

where $k > 0$ depends only on the data. Inequality (15.15) and the Cauchy inequality yield

$$\begin{aligned} k\left[\int_{\mathbb{R}^N} p|\ln p - G(\tau)|\, ds\right]^2 &\leq \left(\int_{\mathbb{R}^N} p\, ds\right) k \int_{\mathbb{R}^N} p|\ln p - G(\tau)|^2\, ds \\ &\leq \int_{\mathbb{R}^N} p_H(\ln p - G(\tau))^2\, ds, \end{aligned}$$

so that

$$\frac{d}{d\tau}\left(G(\tau) + \frac{N}{2}\ln\tau\right) \geq \frac{\lambda k}{4\delta t}\left[\int_{\mathbb{R}^N} p|\ln p - G(\tau)|\, ds\right]^2$$

and also

$$\begin{aligned} \int_{\mathbb{R}^N} p|\ln p - G(\tau)|\, ds &\geq \int_{\mathbb{R}^N} p(\ln p - G(\tau))\, ds \\ &= \int_{\mathbb{R}^N} p\ln p\, ds - G(\tau)\int_{\mathbb{R}^N} p\, ds = -Q(\tau) - G(\tau). \end{aligned}$$

Taking into account (15.5), we have

$$\int_{\mathbb{R}^N} p\,|\ln p - G(\tau)|\ ds = -G(\tau) - \frac{N}{2}\ln\tau - C.$$

We distinguish two alternatives:

a) If $\forall\, \tau \in \left[\frac{t}{2}, t\right]\ \ -G(\tau) - \frac{N}{2}\ln\tau - 2C \geq 0$, then

$$\begin{aligned} \left(G(\tau) + \frac{N}{2}\ln\tau\right)' &\geq \frac{\lambda k}{4\delta t}\left(-G(\tau) - \frac{N}{2}\ln\tau - C\right)^2 \\ &\geq \frac{\lambda k}{16\delta t}\left(-G(\tau) - \frac{N}{2}\ln\tau\right)^2, \end{aligned}$$

whence, after integration over τ in the interval $\left[\frac{t}{2}, t\right]$,

$$G(t) + \frac{N}{2}\ln t \geq \frac{32\delta}{\lambda k}.$$

b) If for some $\tau \in \left[\frac{t}{2}, t\right]$ we have $-G(\tau) - \dfrac{N}{2}\ln \tau - 2C \leq 0$, then by (15.14)

$$G(t) + \frac{N}{2}\ln t \geq G(\tau) + \frac{N}{2}\ln \tau \geq -2C$$

and in any case we have finished. ■

Problems and Complements

3c The Homogeneous Dirichlet Problem

In Section 5 of Chapter 9 the homogeneous Cauchy–Dirichlet problem (4.1) was solved using the Riesz Representation Theorem, which is usually referred to as the Lax–Milgram Theorem in this context. It turns out that in the parabolic case there exists a corresponding result, which plays a comparable role.

Let H be a Hilbert space, endowed with its inner product $(\cdot,\cdot)$ and the norm $|\cdot|$. We identify H with its dual space H'. Moreover, we consider a second Hilbert space V, with norm $\|\cdot\|$. We assume that $V \subset H$, with continuous and dense imbedding, such that

$$V \hookrightarrow H \hookrightarrow V'.$$

Choose $T > 0$ and suppose that for a.e. $t \in [0,T]$ we have a bilinear form $a(t;\cdot,\cdot) : V \times V \to \mathbb{R}$, which satisfies the following properties:

1. $\forall\, u, v \in V$ the application $t \mapsto a(t;\cdot,\cdot)$ is measurable;
2. there exists $M > 0$ such that for a.e. $t \in [0,T]$, $\forall\, u, v \in V$ we have $|a(t;u,v)| \leq M\|u\|\,\|v\|$;
3. there exist α, $C > 0$ such that for a.e. $t \in [0,T]$, $\forall\, v \in V$ we have $a(t;v,v) + C|v|^2 \geq \alpha\|v\|^2$.

We have the following abstract result.

Theorem 3.1c. [174] *Let $f \in L^2(0,T;V')$ and $u_o \in H$. Then there exists a unique function $u \in C([0,T];H) \cap L^2(0,T;V)$ satisfying*

$$-\int_0^T (u(t),\varphi'(t))\,dt + \int_0^T a(t;u(t),\varphi(t))\,dt$$
$$= \int_0^T (f(t),\varphi(t))\,dt + (u_o,\varphi(0))$$

for every $\varphi \in \Phi$*, where*

$$\Phi = \{\varphi :\ \varphi \in C([0,T];V),\ \varphi' \in C([0,T];H),\ \varphi(0)=0\}\,.$$

For the proof, see, for example, page 46 of Lions [174], § 5 of Fujie and Tanabe [88], or Chapter 3, § 1–4 of Lions and Magenes [175].

If we let

$$H = L^2(E),\ V = W_o^{1,2}(E),\ a(t;u,v) = \int_E a_{ij}(x,t)u_{x_j}v_{x_i}\,dx,$$

with a_{ij} as in Section 9.1, and $f, \mathbf{f} \in L^2(E_T)$, Theorem 3.1c ensures the existence of a unique weak solution of the homogeneous Cauchy–Dirichlet problem (3.1).

5c Existence of Solutions of the Homogeneous Dirichlet Problem (3.1) by Galerkin Approximations

It is well-known that given an open set $D \subset \mathbb{R}^{N+1}$ and $(t_o,x_o) \in D$, for a continuous vector field $f : D \to \mathbb{R}^N$, the Cauchy problem

$$\begin{cases} x'(t) = f(t,x(t)) \\ x(t_o) = x_o \end{cases} \tag{5.1c}$$

is equivalent to the integral equation

$$x(t) = x_o + \int_{t_o}^t f(s,x(s))\,ds.$$

Moreover, for f continuous, any solution $x = x(t)$ of the Cauchy problem is of class C^1. It is apparent that (5.1c) makes sense for a more general class of functions f, if x is not required to have a continuous first-order derivative, but, for example, it is assumed to be only absolutely continuous. This remark helps us in extending the notion of solution of an ordinary differential equation.

Take an open set $D \subset \mathbb{R}^{N+1}$ and consider $f : D \to \mathbb{R}^N$. We look for conditions on f, which ensure the existence of an absolutely continuous function $x : I \subset \mathbb{R} \to \mathbb{R}^N$, such that $(t,x(t)) \in D$ for $t \in I$ and

$$x'(t) = f(t,x(t)) \tag{5.2c}$$

for all $t \in I$, except on a set of zero Lebesgue measure. If such a function x and interval I exist, we say that x is a solution of (5.2c). Moreover, a solution x through the point (t_o, x_o) is said to be a solution of the Cauchy problem (5.1c).

In the sequel, we will always assume that our conditions are satisfied except on a set of zero Lebesgue measure, and therefore, we will not repeat it. We have the following.

Theorem 5.1c (Carathéodory [29]). *Let $D \subset \mathbb{R}^{N+1}$ be an open set. Suppose that $f : D \to \mathbb{R}^N$, $f = f(t,x)$ is measurable in t for each fixed x, continuous in x for each t, and such that for each compact set $\mathcal{K} \subset D$, there exists an integrable function $m_{\mathcal{K}} = m_{\mathcal{K}}(t)$ such that*

$$|f(t,x)| \le m_{\mathcal{K}}(t) \quad (t,x) \in \mathcal{K}.$$

Then, for any $(t_o, x_o) \in D$ there exists a solution of (5.1c) *in the sense given before.*

Moreover, if $\varphi = \varphi(t)$ is a solution of (5.1c) *on some interval J, φ can be extended to a maximal interval of existence (a,b), and $\varphi = \varphi(t)$ tends to the boundary of D as $t \to a$ and $t \to b$.*

Finally, if under the previous conditions, for each compact set $\mathcal{K} \subset D$, there exists an integrable function $h = h_{\mathcal{K}}(t)$ such that

$$|f(t,x) - f(t,y)| \le h_{\mathcal{K}}(t)|x - y| \quad (x,t) \in \mathcal{K},\ (t,y) \in \mathcal{K},$$

then for any $(t_o, x_o) \in \mathcal{K}$ there exists a unique solution of (5.1c) *in the sense given before.*

Remark 5.1c It is apparent that for any linear system

$$x'(t) = \mathbf{A}(t)x(t) + b(t),$$

where $\mathbf{A}$ is a $N \times N$ matrix and b is an N vector, whose elements are integrable on any finite interval, all the previous conditions are satisfied, and consequently, the corresponding Cauchy problem has a unique solution.

Proof of Theorem 5.1c – We follow the approach given in Hale [113]. Choose α, $\beta > 0$ such that

$$R \overset{\text{def}}{=} \{(t,x) : \ |t - t_o| \le \alpha,\ |x - x_o| \le \beta\} \subset D,$$

define $I_\alpha = \{t : \ |t - t_o| \le \alpha\}$, let $m(t) = m_R(t)$, and $M(t) \overset{\text{def}}{=} \displaystyle\int_{t_o}^{t} m(s)\,ds$. Pick $\bar\alpha \in (0,\alpha]$, $\bar\beta \in (0,\beta]$ such that $\forall\, t \in I_{\bar\alpha}$, we have $|M(t)| \le \bar\beta$, and let

$$\mathcal{A} = \left\{\varphi \in C(I_{\bar\alpha}; \mathbb{R}^N) : \ \varphi(t_o) = x_o, \quad \text{and} \quad \forall\, t \in I_{\bar\alpha} \quad |\varphi(t) - x_o| \le \bar\beta\right\}.$$

It is straightforward to check that $\mathcal{A}$ is a closed, bounded, convex subset of $C(I_{\bar\alpha}; \mathbb{R}^N)$. For $\varphi \in \mathcal{A}$ and $t \in I_{\bar\alpha}$, define

$$T\varphi(t) = x_o + \int_{t_o}^{t} f(s, \varphi(s))\, ds.$$

Since the fixed points of T in $\mathcal{A}$ coincide with the solutions of (5.1c), if we can prove that such a fixed point exists, we conclude about existence. In order to do so, we rely on the Schauder–Leray Fixed Point Theorem (see Theorem 3.1 of Chapter 14).

First of all, $\forall\, \varphi \in \mathcal{A}$, T is well-defined, since $f(s, \varphi(s))$ is integrable whenever $\varphi \in \mathcal{A}$.

It is clear that $T\varphi(t_o) = x_o$, and $\forall\, t \in I_{\bar{\alpha}}$, $T\varphi(t)$ is continuous at t. Moreover, for all $t \in I_{\bar{\alpha}}$

$$\begin{aligned}|T\varphi(t) - x_o| &= \left|\int_{t_o}^{t} f(s, \varphi(s))\, ds\right| \le \left|\int_{t_o}^{t} |f(s, \varphi(s))|\, ds\right| \\ &\le \left|\int_{t_o}^{t} m_R(s)\, ds\right| = |M(t)| \le \bar{\beta}.\end{aligned}$$

Hence, $T : \mathcal{A} \to \mathcal{A}$.

Let us consider the continuity of T on $\mathcal{A}$. Consider $\{\varphi_n\} \subset \mathcal{A}$ such that $\varphi_n \to \varphi \in \mathcal{A}$ as $n \to \infty$. By the continuity of $f(t, x)$ in x for each fixed t, we have that $\forall\, t \in I_{\bar{\alpha}}$

$$f(t, \varphi_n(t)) \quad \to \quad f(t, \varphi(t)),$$

as $n \to \infty$. Since $|f(t, \varphi_n(t))| \le m(t)$, by the Dominated Convergence Theorem, we conclude that for any $t \in I_{\bar{\alpha}}$

$$\int_{t_o}^{t} f(s, \varphi_n(s))\, ds \quad \to \quad \int_{t_o}^{t} f(s, \varphi(s))\, ds$$

as $n \to \infty$.

Finally, $\forall\, \varphi \in \mathcal{A}$ and for all t, $\tau \in I_{\bar{\alpha}}$

$$\begin{aligned}|T\varphi(t) - T\varphi(\tau)| &= \left|\int_{t_o}^{t} f(s, \varphi(s))\, ds - \int_{t_o}^{\tau} f(s, \varphi(s))\, ds\right| \\ &= \left|\int_{\tau}^{t} f(s, \varphi(s))\, ds\right| \le \left|\int_{\tau}^{t} |f(s, \varphi(s))|\, ds\right| \\ &\le \left|\int_{\tau}^{t} m(s)\, ds\right| = |M(t) - M(\tau)|.\end{aligned}$$

By the continuity of M on $I_{\bar{\alpha}}$, we have that M is also uniformly continuous on $I_{\bar{\alpha}}$; hence, the set $T\mathcal{A}$ is an equi-bounded and equi-continuous set of $C([0, T]; \mathbb{R}^N)$ and by the Ascoli–Arzelà Theorem is relatively compact. We conclude that T is completely continuous. In this way, all the assumptions of the Schauder–Leray Fixed Point Theorem are satisfied, and we finish.

We omit the proof of the second and third parts of the theorem, since they can be obtained by straightforward adaptations of the corresponding classical arguments for a Lipschitz continuous function f. ■.

7c Traces of Functions on $\boldsymbol{\Sigma} \overset{\text{def}}{=} \boldsymbol{\partial E \times (0,T]}$

The statement of Theorem 7.1 is taken from Chapter 4, § 2.2 of Lions and Magenes [176], where a full proof is given, which, however, requires the knowledge of tools from interpolation theory. The same result is also given in Lemma 3.4 of Chapter II of Ladyzhenskaya et al. [151], without any proof, and in Theorem 4.2 of Grisvard [109], whose proof again is based on interpolation theory.

The sharpness of these embedding results is briefly discussed in Weidemaier [273]; the author provides an alternative proof in Weidemaier [272], where he requires the boundary ∂E to be of class C^2.

8c The Inhomogeneous Dirichlet Problem

Besides considering linear equations, one could deal with more general quasilinear equations

$$u_t - \operatorname{div} \mathbf{A}(x,t,u,\nabla u) = B(x,t,u,\nabla u), \tag{8.1c}$$

where $\mathbf{A} : E_T \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$, $B : E_T \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are measurable with respect to (x,t) for all $(u,\xi) \in \mathbb{R}^{N+1}$, continuous with respect to (u,ξ) for a.e. $(x,t) \in E_T$, and they satisfy the structure conditions

$$\begin{cases} \mathbf{A}(x,t,u,\xi)\cdot\xi \ge C_o|\xi|^2 - g_o(x,t), \\ |\mathbf{A}(x,t,u,\xi)| \le C_1|\xi| + g_1(x,t), \\ |B(x,t,u,\xi)| \le C|\xi| + g_2(x,t), \end{cases} \tag{8.2c}$$

with $0 < C_o \le C_1$, $C \ge 0$, and g_o, g_1^2, $g_2 \in L^{q,r}(E_T)$ as in (3.2)–(3.3). Assume that ∂E is of class $C^{1,1}$ and satisfies the segment property. Given $\mathbf{A}$ and B as above, we are interested in the Cauchy–Dirichlet problem

$$\begin{aligned} u_t - \operatorname{div} \mathbf{A}(x,t,u,\nabla u) &= B(x,t,u,\nabla u) && \text{in } E_T, \\ u(x,t) &= \psi && \text{on } \Sigma, \\ u(\cdot,0) &= u_o && \text{in } E, \end{aligned} \tag{8.3c}$$

where $\psi \in L^2(0,T;W^{1,2}(E))$.

A weak solution of the Cauchy–Dirichlet problem (8.3c) is a measurable function $u \in C\big(0,T;L^2(E)\big) \cap L^2\big(0,T;W^{1,2}(E)\big)$ satisfying for all $\tau \in [0,T]$

$$\begin{aligned} &\int_E u(x,\tau)\varphi(x,\tau)dx + \iint_{E_\tau} \big[-u\varphi_t + \mathbf{A}(x,t,u,\nabla u)\cdot\nabla\varphi\big]dxdt \\ &= \iint_{E_\tau} B(x,t,u,\nabla u)\varphi\, dxdt + \int_E u_o\varphi(x,0)dx \end{aligned}$$

for all test functions $\varphi \in W_o^2(E_T)$.

In addition, we take the boundary condition $u = \psi$ to mean that $(u - \psi)(\cdot, t) \in W_o^{1,2}(E)$ for a.e. $t \in (0, T]$.

Here, the boundary condition on Σ is given in a slightly different sense with respect to what we did in Section 8. The solvability of (8.3c) is studied, for example, in Chapter V of Ladyzhenskaya et al. [151].

One can also define *local* weak solutions, as we will do in Chapter 12, and, precisely as in the elliptic context, it turns out that local solutions of (8.3c), whenever they exist, show the same kind of local behavior as local solutions of (8.1).

The situation is much more complicated, in terms of existence, uniqueness, and above all regularity of solutions, if one deals with

$$u_t - \operatorname{div} \mathbf{A}(x, t, u, \nabla u) = 0,$$

where we dropped the lower order terms just for the sake of simplicity, and $\mathbf{A} : E_T \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is measurable with respect to (x, t) for all $(u, \xi) \in \mathbb{R}^{N+1}$, continuous with respect to (u, ξ) for a.e. $(x, t) \in E_T$, and it satisfies the structure conditions

$$\begin{cases} \mathbf{A}(x, t, u, \xi) \cdot \xi \geq C_o |\xi|^p, \\ \ |\mathbf{A}(x, t, u, \xi)| \leq C_1 |\xi|^{p-1}, \end{cases}$$

with $0 < C_o \leq C_1$, and $p > 1$, $p \neq 2$. We refrain from going into further detail here, and refer the interested reader to, for example, DiBenedetto [49] and the references therein.

8.1c Parabolic Quasi-Minima

As we have seen in Section 8.2c of Chapter 9, the notion of Q-minimum can be introduced, in order to provide a unifying framework to some regularity results for elliptic equations, and also for systems.

The same approach can be followed in the parabolic context; it was started by Wieser [280], and it has seen tremendous development in recent years.

Given an open set $E \subseteq \mathbb{R}^N$ with $N \geq 2$ and $T > 0$, consider a function $F : E_T \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, which is measurable in (x, t) for all $(u, \xi) \in \mathbb{R} \times \mathbb{R}^N$, and continuous in (u, ξ) for a.e. $(x, t) \in E_T$, and assume that it satisfies the growth conditions

$$C_o |\xi|^2 - b|u|^\gamma - g(x, t) \leq F(x, t, u, \xi) \leq C_1 |\xi|^2 + b|u|^\gamma + g(x, t),$$

where $g \in L^1(E_T)$, $g \geq 0$, b, $\gamma \in \mathbb{R}_+$, $0 < C_o \leq C_1$.

For $Q \geq 1$ a function $u : E_T \to \mathbb{R}$, $u \in L^2_{\mathrm{loc}}(0, T; W^{1,2}_{\mathrm{loc}}(E)) \cap L^\gamma(E_T)$ is called a *parabolic Q-minimum*, of the functional

$$J[u] = \iint_{E_T} F(x, t, u, \nabla u) \, dxdt,$$

if for every $\varphi \in C_o^\infty(E_T)$ we have

$$-\iint_{\mathcal{K}} u\varphi_t \,dxdt + \iint_{\mathcal{K}} F(x,t,u,\nabla u)\,dxdt \\ \leq Q \iint_{\mathcal{K}} F(x,t,u-\varphi,\nabla u - \nabla\varphi)\,dxdt,$$

where $\mathcal{K} = \operatorname{supp}\varphi$.

Consider the quasi-linear parabolic equation (8.1c) with the structure conditions (8.2c). Then, along the same lines as the elliptic proof, it is not difficult to show that a local weak solution of (8.1c) (see Chapter 12 for more details on *local* solutions), is a parabolic Q-minimum for the functional

$$J[u] = \lambda \iint_{E_T} \left(|\nabla u|^2 + h\right) dxdt,$$

with $\lambda \in (0,1)$ and $h = h(x,t) = (1 + g_o(x,t) + g_1^2(x,t) + g_2^2(x,t))$.

9c The Neumann Problem

It is quite natural to wonder whether the abstract approach discussed in Section 3c of the Complements, in the context of the homogeneous Cauchy–Dirichlet problem, can also be used for the Cauchy–Neumann or mixed problem. In § 4.7.2–4.7.3 of Chapter 3 of Lions and Magenes [175], problems (9.1) and (9.2) respectively are discussed with $\psi = 0$, and it is shown how the framework of Section 3c allows the unique existence of a solution u to be proved. However, the abstract approach requires some care in dealing with the term $\mathbf{f}$ on the right-hand side; indeed, in this case $V' = (W^{1,2}(E))'$, whose characterization is not straightforward (in particular, it is not a space of distributions). We refrain from going into further detail here.

As we did in Section 8c of the Complements, for the Cauchy–Neumann Problem, we can also consider a general quasi-linear operator. Namely, assume that ∂E is of class C^1 and satisfies the segment property. Given $\mathbf{A}$ and B as in (8.2c), we are interested in the problem

$$\begin{aligned} u_t - \operatorname{div}\mathbf{A}(x,t,u,\nabla u) &= B(x,t,u,\nabla u) && \text{in } E_T, \\ \mathbf{A}(x,t,u,\nabla u)\cdot\mathbf{n} &= \psi && \text{on } \Sigma, \\ u(\cdot,0) &= u_o && \text{in } E, \end{aligned} \tag{9.1c}$$

where $\mathbf{n}$ is the outward unit normal to ∂E, and ψ enjoys the same regularity as in (9.2).

By a weak solution of the Cauchy–Neumann Problem (9.1c), under all the previous assumptions, we mean a function $u \in V^2(E_T)$ satisfying the identity

$$-\iint_{E_T} u\varphi_t\,dxdt + \iint_{E_T} \mathbf{A}(x,t,u,\nabla u)\nabla\varphi\,dxdt = \iint_{E_T} B(x,t,u,\nabla u)\varphi\,dxdt$$
$$+\iint_{\Sigma}\psi\varphi\,d\sigma\,dt + \int_E u_o(x)\varphi(x,0)dx$$

for any function $\varphi \in W^2(E_T)$, which vanishes for $t = T$.

10c A Priori $L^\infty(E_T)$ Estimates for Solutions of the Dirichlet Problem (8.1)

For $N \geq 1$, $\epsilon > 0$, the equation

$$u_t - \Delta u = f,$$

where

$$f = -\frac{(2N+1)(T-t)+(2N-3)|x|^2}{[(T-t)+|x|^2]^2},$$

is solved in a cylindrical neighborhood $E_T = B_\epsilon \times (T-\epsilon, T)$ of the point $(0,T)$ by

$$u(x,t) = \ln[(T-t)+|x|^2].$$

One verifies that $f \in L^s(E_T)$ for any $s < \frac{N+2}{2}$, and it is apparent that u is unbounded in E_T.

On the other hand, for $N \geq 3$, if

$$\begin{aligned} f =&\alpha\left[-\ln((T-t)+|x|^2)\right]^{\alpha-1}\frac{(2N+1)(T-t)+(2N-3)|x|^2}{[(T-t)+|x|^2]^2}\\ &-\alpha(\alpha-1)\left[-\ln((T-t)+|x|^2)\right]^{\alpha-2}\frac{4|x|^2}{[(T-t)+|x|^2]^2},\end{aligned}$$

with $\alpha = \frac{N-2}{N+2}$, the solution is

$$u(x,t) = \left[-\ln((T-t)+|x|^2)\right]^\alpha.$$

In such a case one verifies that $f \in L^{\frac{N+2}{2}}(E_T)$, and again u is unbounded in E_T.

10.1. Extend Proposition 10.1 assuming

$$\mathbf{f} \in L^{r_1(p),r_2(p)}(E_T), \quad f \in L^{s_1(p),s_2(p)}(E_T),$$

for proper (r_1, r_2) and (s_1, s_2), where the spaces $L^{p,q}(E_T)$ have been defined in Section 1 (see also Section 12.1 of Chapter 14).

12c A Priori $L^\infty(E_T)$ Estimates for Solutions of the Neumann Problem (9.1)

12.1. Extend Proposition 12.1 assuming

$$\mathbf{f} \in L^{r_1(p),r_2(p)}(E_T), \quad f \in L^{s_1(p),s_2(p)}(E_T),$$

for proper (r_1, r_2) and (s_1, s_2).

15c Gaussian Bounds on the Fundamental Solution

We prove (15.2); it suffices to show that the following hold.

Proposition 15.1c [190] *There exists a positive constant C_N such that for any function $f \in C_0^\infty(\mathbb{R}^N)$ we have*

$$\|f\|_2^{2\left(1+\frac{2}{N}\right)} \le C_N^2 \|\nabla f\|_2^2 \, \|f\|_1^{\frac{4}{N}}.$$

Proof. For $f \in C_0^\infty(\mathbb{R}^N)$, we denote by $\hat{f}$ its Fourier Transform

$$\hat{f}(\xi) = \int_{\mathbb{R}^N} f(x) e^{-i\xi\cdot x}\, dx.$$

It is well-known that

$$\sup_{\mathbb{R}^N} |\hat{f}(\xi)| \le \int_{\mathbb{R}^N} |f|\, dx, \tag{15.1c}$$

$$\|\hat{f}\|_2 = (2\pi)^{N/2} \|f\|_2. \tag{15.2c}$$

Moreover, $|\widehat{\nabla f}| = |\xi||\hat{f}|$. Starting from (15.2c) and relying on (15.1c), we have

$$\begin{aligned}
\|f\|_2^2 &= \frac{1}{(2\pi)^N} \|\hat{f}\|_2^2 \\
&= \frac{1}{(2\pi)^N} \left[\int_{|\xi|\le R} |\hat{f}|^2\, d\xi + \int_{|\xi|>R} |\hat{f}|^2\, d\xi \right] \\
&\le \frac{\omega_N R^N}{(2\pi)^N} \|\hat{f}\|_\infty^2 + \frac{1}{(2\pi)^N R^2} \int_{\mathbb{R}^N} |\xi|^2 |\hat{f}|^2\, d\xi \\
&\le \frac{\omega_N R^N}{(2\pi)^N} \|f\|_1^2 + \frac{1}{(2\pi)^{N/2} R^2} \int_{\mathbb{R}^N} |\nabla f|^2\, dx.
\end{aligned}$$

If we optimize over R, we obtain

$$\|f\|_2^2 \le C\, \|f\|_1^{\frac{4}{N+2}} \, \|\nabla f\|_2^{\frac{2N}{N+2}}$$

with $C = \left[\left(\frac{2}{N}\right)^{\frac{N}{N+2}} + \left(\frac{N}{2}\right)^{\frac{2}{N+2}} \right] \frac{\omega_N^{\frac{2}{N+2}}}{(2\pi)^N}$ and we conclude. ∎

12

PARABOLIC DEGIORGI CLASSES

1 Quasi-Linear Equations and DeGiorgi Classes

A quasi-linear parabolic equation in a set $E_T = E \times (0,T] \subset \mathbb{R}^{N+1}$ is an expression of the form

$$u_t - \operatorname{div} \mathbf{A}(x,t,u,\nabla u) = B(x,t,u,\nabla u) \tag{1.1}$$

where for $u \in C_{\text{loc}}(0,T;L^2_{\text{loc}}(E)) \cap L^2_{\text{loc}}(0,T;W^{1,2}_{\text{loc}}(E))$, the functions

$$E_T \ni (x,t) \to \begin{cases} \mathbf{A}\big(x,t,u(x,t),\nabla u(x,t)\big) \in \mathbb{R}^N \\ B\big(x,t,u(x,t),\nabla u(x,t)\big) \in \mathbb{R} \end{cases}$$

are measurable and satisfy the structure conditions

$$\begin{aligned} \mathbf{A}\big(x,t,u,\xi\big)\cdot\xi &\ge C_o|\xi|^2 - [f(x,t)]^2 \\ |\mathbf{A}\big(x,t,u,\xi\big)| &\le C_1|\xi| + f(x,t) \\ |B\big(x,t,u,\xi\big)| &\le C_2|\xi| + f_o(x,t) \end{aligned} \tag{1.2}$$

for given constants $0 < C_o \le C_1$ and $C_2 > 0$, and given non-negative functions

$$f \in L^{N+2+\varepsilon}(E_T), \qquad f_o \in L^{\frac{N+2+\varepsilon}{2}}(E_T), \qquad \text{for some } \ \varepsilon > 0. \tag{1.3}$$

The Dirichlet and Neumann problems for these equations were introduced in the Complements of Chapter 11, their solvability was established for a class of functions $\mathbf{A}$ and B, and $L^\infty(E_T)$ bounds were derived for suitable data. Here we are interested in the local behavior of these solutions, both at the interior, and at the boundary, either with Dirichlet or with Neumann data.

A function $u \in C_{\text{loc}}(0,T;L^2_{\text{loc}}(E)) \cap L^2_{\text{loc}}(0,T;W^{1,2}_{\text{loc}}(E))$ is a local weak sub(super)-solution of (1.1), if for every compact set $K \subset E$ and every subinterval $[t_1,t_2] \subset (0,T]$ we have

© Springer Nature Switzerland AG 2023

E. DiBenedetto, U. Gianazza, *Partial Differential Equations*,
Cornerstones, https://doi.org/10.1007/978-3-031-46618-2_13

$$\begin{aligned}\int_K u\varphi\,dx\bigg|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_K \big[-u\varphi_t + \mathbf{A}(x,t,u,\nabla u)\cdot D\varphi\big]dx\,dt \\ \le (\ge) \int_{t_1}^{t_2}\int_K B(x,t,u,Du)\varphi\,dx\,dt\end{aligned} \tag{1.4}$$

for all non-negative testing functions

$$\varphi \in W^{1,2}_{\mathrm{loc}}\big(0,T;L^2(K)\big) \cap L^2_{\mathrm{loc}}\big(0,T;W^{1,2}_o(K)\big). \tag{1.5}$$

This guarantees that all the integrals in (1.4) are convergent.

A local weak solution of (1.1) is a function $u \in C_{\mathrm{loc}}(0,T;L^2_{\mathrm{loc}}(E)) \cap L^2_{\mathrm{loc}}(0,T;W^{1,2}_{\mathrm{loc}}(E))$ satisfying (1.4) with the equality sign, for all

$$\varphi \in W^{1,2}_{\mathrm{loc}}\big(0,T;L^2(K)\big) \cap L^2_{\mathrm{loc}}\big(0,T;W^{1,2}_o(K)\big).$$

No further requirements are placed on $\mathbf{A}$ and B other than the structure conditions (1.2). Specific examples of these PDEs are those introduced in the previous chapter. In particular, they include the class of linear parabolic equations with bounded and measurable coefficients (2.2) of Chapter 11. Even though the coefficients are only measurable, nevertheless, local weak solutions of (1.1) are locally Hölder continuous in E_T. This follows from their membership in more general classes of functions called parabolic DeGiorgi classes, which are introduced next.

Let $B_\rho(y) \subset E$ denote a ball of center y and radius ρ; if y is the origin, write $B_\rho(0) = B_\rho$. For $\theta > 0$ let $(y,s) + Q^\pm_\rho(\theta) \subset E_T$ denote the cylinders

$$\begin{aligned}(y,s) + Q^-_\rho(\theta) &= B_\rho(y) \times (s - \theta\rho^2, s],\\ (y,s) + Q^+_\rho(\theta) &= B_\rho(y) \times (s, s + \theta\rho^2];\end{aligned}$$

If $\theta = 1$, write $(y,s) + Q^\pm_\rho(1) = (y,s) + Q^\pm_\rho$.

Consider a piecewise smooth, cutoff function ζ vanishing on $\partial B_\rho(y)$, and such that $0 \le \zeta \le 1$, and let u be a local sub(super)-solution of (1.1). For $k \in \mathbb{R}$, the localized truncations $\pm\zeta^2(u-k)_\pm$ can be taken as test functions φ, modulo a standard Steklov average, as considered in Section 11 of the previous Chapter.

After a translation we may assume $(y,s) = (0,0)$. In (1.4) integrate over $B_\rho \times (-\theta\rho^2, t]$, with $t \in (-\theta\rho^2, 0]$.

$$\begin{aligned}&\pm \iint_{B_\rho\times(-\theta\rho^2,t]} u_\tau (u-k)_\pm \zeta^2 dx\,d\tau \\ &\pm \iint_{B_\rho\times(-\theta\rho^2,t]} \mathbf{A}(x,\tau,u,\nabla u)\nabla[(u-k)_\pm\zeta^2]dx\,d\tau \\ &\le \pm \iint_{B_\rho\times(-\theta\rho^2,t]} B(x,\tau,u,\nabla u)(u-k)_\pm\zeta^2\,dx\,d\tau.\end{aligned}$$

As for the first term

$$
\begin{aligned}
\pm \iint_{B_\rho\times(-\theta\rho^2,t]} u_\tau (u-k)_\pm \zeta^2 dx\, d\tau &= \frac{1}{2}\iint_{B_\rho\times(-\theta\rho^2,t]} [(u-k)_\pm^2]_\tau \zeta^2 dx\, d\tau \\
&= \frac{1}{2}\int_{-\theta\rho^2}^{t} \frac{d}{d\tau}\int_{B_\rho} (u-k)_-^2 \zeta^2 dx\, d\tau - \iint_{B_\rho\times(-\theta\rho^2,t]} (u-k)_\pm^2 \zeta\zeta_\tau dx\, d\tau \\
&\ge \frac{1}{2}\int_{B_\rho} (u-k)_\pm^2 \zeta^2(x,t)dx - \frac{1}{2}\int_{B_\rho} (u-k)_\pm^2 \zeta^2(x,-\theta\rho^2)dx \\
&\quad - \iint_{Q_\rho^-(\theta)} (u-k)_\pm^2 \zeta|\zeta_\tau| dx\, d\tau.
\end{aligned}
$$

The second integral is transformed and estimated as

$$
\begin{aligned}
\pm &\iint_{B_\rho\times(-\theta\rho^2,t]} \mathbf{A}(x,\tau,u,\nabla u)\cdot\nabla[(u-k)_\pm\zeta^2] dx\, d\tau \\
&= \iint_{B_\rho\times(-\theta\rho^2,t]} \pm\mathbf{A}(x,\tau,u,\nabla u)\cdot\nabla(u-k)_\pm\zeta^2 dx\, d\tau \\
&\quad \pm 2\iint_{B_\rho\times(-\theta\rho^2,t]} (u-k)_\pm\zeta\mathbf{A}(x,\tau,u,\nabla u)\cdot\nabla\zeta\, dx\, d\tau \\
&\ge C_o \iint_{B_\rho\times(-\theta\rho^2,t]} |\nabla(u-k)_\pm|^2\zeta^2 dx\, d\tau \\
&\quad - \iint_{Q_\rho^-(\theta)} f^2\chi_{[(u-k)_\pm>0]}\zeta^2 dx\, d\tau \\
&\quad - 2C_1 \iint_{B_\rho\times(-\theta\rho^2,t]} |\nabla(u-k)_\pm|(u-k)_\pm\zeta|\nabla\zeta| dx\, d\tau \\
&\quad - 2\iint_{Q_\rho^-(\theta)} f(u-k)_\pm\zeta|\nabla\zeta| dx\, d\tau.
\end{aligned}
$$

By Young's inequality

$$
\begin{aligned}
2C_1 &\iint_{B_\rho\times(-\theta\rho^2,t]} |\nabla(u-k)_\pm|(u-k)_\pm\zeta|\nabla\zeta| dx\, d\tau \\
&\le \frac{C_o}{4}\iint_{B_\rho\times(-\theta\rho^2,t]} |\nabla(u-k)_\pm|^2\zeta^2 dx\, d\tau \\
&\quad + \gamma(C_o)\iint_{Q_\rho^-(\theta)} (u-k)_\pm^2|\nabla\zeta|^2 dx\, d\tau
\end{aligned}
$$

and

$$
\begin{aligned}
2 &\iint_{Q_\rho^-(\theta)} f(u-k)_\pm\zeta|\nabla\zeta| dx\, d\tau \\
&\le \iint_{Q_\rho^-(\theta)} (u-k)_\pm^2|\nabla\zeta|^2 dx\, d\tau + \iint_{Q_\rho^-(\theta)} f^2\chi_{[(u-k)_\pm>0]}\zeta^2 dx\, d\tau.
\end{aligned}
$$

Combining these terms

$$\begin{aligned}
&\pm\iint_{B_\rho\times(-\theta\rho^2,t]}\mathbf{A}(x,\tau,u,\nabla u)\cdot\nabla[(u-k)_\pm\zeta^2]dx\,d\tau\\
&\quad\geq\frac{3}{4}C_o\iint_{B_\rho\times(-\theta\rho^2,t]}|\nabla(u-k)_\pm|^2\zeta^2dx\,d\tau\\
&\qquad\iint_{Q_\rho^-(\theta)}(u-k)_\pm^2|\nabla\zeta|^2dx\,d\tau-\iint_{Q_\rho^-(\theta)}f^2\chi_{[(u-k)_\pm>0]}\zeta^2dx\,d\tau.
\end{aligned}$$

Finally,

$$\begin{aligned}
&\pm\iint_{B_\rho\times(-\theta\rho^2,t]}B(x,\tau,u,\nabla u)(u-k)_\pm\zeta^2dx\,d\tau\\
&\quad\leq C_2\iint_{B_\rho\times(-\theta\rho^2,t]}|\nabla(u-k)_\pm|(u-k)_\pm\zeta^2dx\,d\tau\\
&\qquad+\iint_{B_\rho\times(-\theta\rho^2,t]}f_o(u-k)_\pm\zeta^2dx\,d\tau\\
&\quad\leq\frac{C_o}{4}\iint_{B_\rho\times(-\theta\rho^2,t]}|\nabla(u-k)_\pm|^2\zeta^2dx\,d\tau\\
&\qquad+\gamma(C_o)C_2^2\iint_{Q_\rho^-(\theta)}(u-k)_\pm^2\zeta^2dx\,d\tau+\iint_{Q_\rho^-(\theta)}f_o(u-k)_\pm\zeta^2dx\,d\tau.
\end{aligned}$$

Combining the previous estimates and recalling that $t\in(-\theta\rho^2,0]$ is arbitrary yields

$$\begin{aligned}
&\frac{1}{2}\operatorname*{ess\,sup}_{-\theta\rho^2<t<0}\int_{B_\rho}(u-k)_\pm^2\zeta^2(x,t)dx-\frac{1}{2}\int_{B_\rho}(u-k)_\pm^2\zeta^2(x,-\theta\rho^2)dx\\
&\qquad+\frac{C_o}{2}\iint_{Q_\rho^-(\theta)}|\nabla[(u-k)_\pm\zeta]|^2dx\,d\tau\\
&\quad\leq\gamma\iint_{Q_\rho^-(\theta)}\left[(u-k)_\pm^2|\nabla\zeta|^2+(u-k)_\pm^2|\zeta_\tau|\right]dx\,d\tau\\
&\qquad+\iint_{Q_\rho^-(\theta)}\left[f^2\chi_{[(u-k)_\pm>0]}+f_o(u-k)_\pm\right.\\
&\qquad\left.+\gamma C_2^2(u-k)_\pm^2\right]\zeta^2dx\,d\tau.
\end{aligned}$$

We point out that the proof traces the dependence of the constant γ on the parameters $\{N,C_o,C_1,C_2\}$. Next, estimate

$$\iint_{Q_\rho^-(\theta)}f^2\chi_{[(u-k)_\pm>0]}dx\,d\tau\leq\|f\|^2_{N+2+\varepsilon;Q_\rho^-(\theta)}|A_{k,\rho}^\pm|^{1-\frac{2}{N+2}+2\delta},$$

where

$$A^{\pm}_{k,\rho} = [(u-k)_{\pm} > 0] \cap Q^-_{\rho}(\theta) \quad \text{ and } \quad \delta = \frac{\varepsilon}{(N+2)(N+2+\varepsilon)}. \tag{1.6}$$

The term involving f_o is estimated by Hölder's inequality and using Proposition 1.1 of Chapter 11. We have

$$\begin{aligned}
\iint_{Q^-_\rho(\theta)} & f_o (u-k)_{\pm}\zeta^2 dx\, d\tau \\
&\le \| f_o \chi_{[(u-k)_\pm>0]}\|_{2\frac{N+2}{N+4};Q^-_\rho(\theta)} \|(u-k)_\pm\zeta]\|_{2\frac{N+2}{N};Q^-_\rho(\theta)} \\
&\le \gamma(N)\|(u-k)_\pm\zeta\|_{V^2(Q^-_\rho(\theta))} \| f_o \chi_{[(u-k)_\pm>0]}\|_{2\frac{N+2}{N+4};Q^-_\rho(\theta)} \\
&\le \min\left\{\frac{C_o}{4};\frac{1}{4}\right\} \|(u-k)_\pm\zeta\|^2_{V^2(Q^-_\rho(\theta))} \\
&\quad + \gamma(N,C_o)\| f_o \chi_{[(u-k)_\pm>0]}\|^2_{2\frac{N+2}{N+4};Q^-_\rho(\theta)} \\
&\le \frac{1}{4} \operatorname*{ess\,sup}_{-\theta\rho^2<t<0} \int_{B_\rho} (u-k)^2_\pm \zeta^2(x,t)dx \\
&\quad + \frac{C_o}{4}\iint_{Q^-_\rho(\theta)} |\nabla[(u-k)_\pm\zeta]|^2 dx\, d\tau \\
&\quad + \gamma(N,C_o)\|f_o\|^2_{\frac{N+2+\hat\delta}{2};Q^-_\rho(\theta)} |A^\pm_{k,\rho}|^{1-\frac{2}{N+2}+2\delta},
\end{aligned}$$

where δ is as in (1.6), and $\hat\delta = \dfrac{\varepsilon(N+2)}{2[2(N+2)+\varepsilon]}$. Combining these estimates yields

$$\begin{aligned}
\operatorname*{ess\,sup}_{-\theta\rho^2<t<0} & \int_{B_\rho} (u-k)^2_\pm \zeta^2(x,t)dx - \int_{B_\rho} (u-k)^2_\pm \zeta^2(x,-\theta\rho^2)dx \\
&+ \iint_{Q^-_\rho(\theta)} |\nabla[(u-k)_\pm\zeta]|^2 dx\, d\tau \\
\le \bar\gamma & \iint_{Q^-_\rho(\theta)} \left[(u-k)^2_\pm |\nabla\zeta|^2 + (u-k)^2_\pm |\zeta_\tau|\right] dx\, d\tau \\
&+ \bar\gamma\, C_2^2 \iint_{Q^-_\rho(\theta)} (u-k)^2_\pm \zeta^2 dx\, d\tau + \gamma^2_* |A^\pm_{k,\rho}|^{1-\frac{2}{N+2}+2\delta},
\end{aligned}$$

where δ is given by (1.6) and

$$\gamma^2_* = \gamma(N,C_o)\Big(\|f\|^2_{N+2+\varepsilon;Q^-_\rho(\theta)} + \|f_o\|^2_{\frac{N+2+\hat\delta}{2};Q^-_\rho(\theta)}\Big). \tag{1.7}$$

If we revert to a general point (y,s) instead of considering $(0,0)$, we have

$$\begin{aligned}
&\operatorname*{ess\,sup}_{s-\theta\rho^2<t<s}\int_{B_\rho(y)}(u-k)_\pm^2\zeta^2(x,t)dx-\int_{B_\rho(y)}(u-k)_\pm^2\zeta^2(x,s-\theta\rho^2)dx\\
&\quad+\iint_{(y,s)+Q_\rho^-(\theta)}|\nabla[(u-k)_\pm\zeta]|^2dx\,d\tau\\
&\le\bar\gamma\iint_{(y,s)+Q_\rho^-(\theta)}\left[(u-k)_\pm^2|\nabla\zeta|^2+(u-k)_\pm^2|\zeta_\tau|\right]dx\,d\tau\\
&\quad+\bar\gamma\,C_2^2\iint_{(y,s)+Q_\rho^-(\theta)}(u-k)_\pm^2\zeta^2dx\,d\tau+\gamma_*^2|A_{k,\rho}^\pm|^{1-\frac{2}{N+2}+2\delta},
\end{aligned}\tag{1.8}$$

with the obvious corresponding meaning for $A_{k,\rho}^\pm$. Analogous estimates hold for “forward” cylinders $(y,s)+Q_\rho^+(\theta)\subset E_T$.

1.1 Parabolic DeGiorgi Classes

Let $E\subset\mathbb{R}^N$ be an open set, let $E_T=E\times(0,T]$, and let $\bar\gamma$, γ_*, C_2, and δ be given positive constants. The parabolic DeGiorgi class $\mathrm{PDG}^+(E_T,\bar\gamma,\gamma_*,C_2,\delta)$ is the collection of all functions $u\in C_{\mathrm{loc}}(0,T;L^2_{\mathrm{loc}}(E))\cap L^2_{\mathrm{loc}}(0,T;W^{1,2}_{\mathrm{loc}}(E))$ such that $(u-k)_+$ satisfy (1.8) for all $k\in\mathbb{R}$, for all piecewise smooth, cutoff function ζ vanishing on $\partial B_\rho(y)$ and such that $0\le\zeta\le1$, and for cylinders $(y,s)+Q_\rho^\pm(\theta)\subset E_T$. Local weak sub-solutions of (1.1) belong to PDG^+, for the constants $\bar\gamma$, γ_*, C_2, and δ identified in (1.6)–(1.7).

The parabolic DeGiorgi class $\mathrm{PDG}^-(E_T,\bar\gamma,\gamma_*,C_2,\delta)$ is defined similarly, with $(u-k)_+$ replaced by $(u-k)_-$. Local weak super-solutions of (1.1) belong to PDG^-. The parabolic DeGiorgi classes $\mathrm{PDG}(E_T,\bar\gamma,\gamma_*,C_2,\delta)$ are the intersection of $\mathrm{PDG}^+\cap\mathrm{PDG}^-$, or equivalently the collection of all functions $u\in C_{\mathrm{loc}}(0,T;L^2_{\mathrm{loc}}(E))\cap L^2_{\mathrm{loc}}(0,T;W^{1,2}_{\mathrm{loc}}(E))$ satisfying (1.8) for all $k\in\mathbb{R}$, for all piecewise smooth, cutoff function ζ vanishing on $\partial B_\rho(y)$ and such that $0\le\zeta\le1$, and for cylinders $(y,s)+Q_\rho^\pm(\theta)\subset E_T$. We refer to these classes as homogeneous if $\gamma_*=0$. In such a case the choice of the parameter δ is immaterial. The set of parameters $\{N,\bar\gamma,C_2\}$ are the homogeneous data of the PDG classes, whereas γ_* and δ are the inhomogeneous parameters. This terminology stems from the structure of (1.8) versus the structure of the quasi-linear elliptic equations in (1.1), and is evidenced by (1.7).

Functions in PDG have remarkable properties, irrespective of their connection with the quasi-linear equations (1.1). In particular, they are locally bounded, and locally Hölder continuous in E_T. Even more striking is that non-negative functions in PDG satisfy the Harnack inequality of Section 5.1 of Chapter 2, which is typical of non-negative caloric functions.

2 Local Boundedness of Functions in the PDG Classes

We say that constants $C,\gamma,\dots$ depend only on the data, and are independent of γ_* and δ, if they can be quantitatively determined a priori only in terms of the homogeneous parameters $\{N, \bar{\gamma}, C_2\}$. The dependence on the inhomogeneous parameters $\{\gamma_*, \delta\}$ will be traced, as a way of identifying those additional properties afforded by inhomogeneous structures.

Theorem 2.1. *Let $u \in PDG^{\pm}$ and $\sigma \in (0,1)$. There exists a constant C depending only on the data such that for every pair of nested cylinders $(y,s)+Q^-_{\sigma\rho}(\theta) \subset (y,s)+Q^-_{\rho}(\theta) \subset E_T$*

$$\operatorname*{ess\,sup}_{(y,s)+Q^-_{\sigma\rho}(\theta)} u_{\pm} \le \max\Bigg\{ \gamma_* \rho^{\delta(N+2)}\, ; \tag{2.1}$$
$$\frac{C\sqrt{\theta}}{(1-\sigma)^{\frac{1}{2\delta}}}\Big(1+\frac{1}{\theta}\Big)^{\frac{1}{4\delta}} \left(\frac{1}{|Q_\rho(\theta)|}\iint_{(y,s)+Q^-_\rho(\theta)} u_{\pm}^2\,dx\,d\tau\right)^{\frac{1}{2}}\Bigg\}.$$

For homogeneous $PDG^{\pm}$ classes, $\gamma_* = 0$ and δ can be taken $\delta = \frac{1}{N+2}$.

Proof. The proof will be given for non-negative $u \in \mathrm{PDG}^+(E_T, \bar{\gamma}, \gamma_*, C_2, \delta)$, the proof for the remaining case being identical. Assume $(y,s) = (0,0)$ and for fixed $\sigma \in (0,1)$ and $n = 0,1,2,\dots$ set

$$\begin{aligned} \rho_n &= \sigma\rho + \frac{1-\sigma}{2^n}\rho, &\qquad t_n &= -\theta\sigma^2\rho^2 - \frac{1-\sigma^2}{2^n}\theta\rho^2,\\ B_n &= B_{\rho_n}, & B_n &= K_n \times (t_n, 0). \end{aligned}$$

This is a family of nested and shrinking cylinders with a common “vertex” at $(0,0)$, and by construction

$$Q_o = B_\rho \times (-\theta\rho^2, 0) \quad \text{and} \quad Q_\infty = B_{\sigma\rho} \times (-\theta\sigma^2\rho^2, 0).$$

Denote by ζ a non-negative, piecewise smooth cutoff function in Q_n that equals one on Q_{n+1}, and has the form $\zeta(x,t) = \zeta_1(x)\zeta_2(t)$, where

$$\zeta_1 = \begin{cases} 1 & \text{in } B_{n+1} \\ 0 & \text{in } \mathbb{R}^N - B_n \end{cases} \qquad |D\zeta_1| \le \frac{2^{n+1}}{(1-\sigma)\rho}$$

$$\zeta_2 = \begin{cases} 0 & \text{for } t \le t_n \\ 1 & \text{for } t \ge t_{n+1} \end{cases} \quad 0 \le \zeta_{2,t} \le \frac{2^{n+1}}{(1-\sigma)\theta\rho^2};$$

introduce the increasing sequence of levels $k_n = k - 2^{-n}k$, where $k > 0$ is to be chosen, stipulate that $\rho \le C_2^{-1}$, and write (1.8) for $(u-k_{n+1})_+\zeta^2$: we obtain

$$\begin{aligned} \sup_{t_n \le \tau \le t} \int_{B_n} [(u-k_{n+1})_+\zeta]^2(x,\tau)dx &+ \iint_{Q_n} |\nabla[(u-k_{n+1})_+\zeta]|^2 dx\,d\tau \\ \le \frac{\gamma 2^{2n}}{(1-\sigma)^2\rho^2}&\left(1+\frac{1}{\theta}\right)\iint_{Q_n}(u-k_{n+1})_+^2 dx\,d\tau \\ + \gamma_*^2 &|A^+_{k_{n+1},\rho_n}|^{1-\frac{2}{N+2}+2\delta} \end{aligned} \tag{2.2}$$

$$\begin{aligned}\leq&\frac{\gamma 2^{2n}}{(1-\sigma)^2\rho^2}\left(1+\frac{1}{\theta}\right)\iint_{Q_n}(u-k_n)_+^2\,dx\,d\tau\\&+\gamma_*^2|A^+_{k_{n+1},\rho_n}|^{1-\frac{2}{N+2}+2\delta},\end{aligned}$$

where γ is a positive constant depending only on the data and that might be different in different contexts.

By Hölder's inequality and the embedding Proposition 1.1 of Chapter 11

$$\begin{aligned}\iint_{Q_{n+1}}(u-k_{n+1})_+^2\,dx\,d\tau\leq&\Big(\iint_{Q_n}[(u-k_{n+1})_+\zeta]^{2\frac{N+2}{N}}dx\,d\tau\Big)^{\frac{N}{N+2}}\\&\times\Big(\iint_{Q_n}\chi_{[(u-k_{n+1})_+>0]}dx\,d\tau\Big)^{\frac{2}{N+2}}\\\leq&\gamma\Big(\iint_{Q_n}|\nabla[(u-k_{n+1})_+\zeta]|^2dx\,d\tau\Big)^{\frac{N}{N+2}}\\&\times\Big(\sup_{t_n\leq\tau\leq 0}\int_{K_n}[(u-k_{n+1})_+\zeta]^2(x,\tau)dx\Big)^{\frac{2}{N+2}}\\&\times\Big(\iint_{Q_n}\chi_{[(u-k_{n+1})_+>0]}dx\,d\tau\Big)^{\frac{2}{N+2}}\\\leq&\gamma\left[\frac{2^{2n}}{(1-\sigma)^2\rho^2}\left(1+\frac{1}{\theta}\right)\iint_{Q_n}(u-k_n)_+^2\,dx\,d\tau\right.\\&\left.+\gamma_*^2|A^+_{k_{n+1},\rho_n}|^{1-\frac{2}{N+2}+2\delta}\right]\times\Big(\iint_{Q_n}\chi_{[(u-k_{n+1})_+>0]}dx\,d\tau\Big)^{\frac{2}{N+2}}.\end{aligned}$$

Since

$$|A^+_{k_{n+1},\rho_n}|=\iint_{Q_n}\chi_{[(u-k_{n+1})_+>0]}dx\,d\tau\leq\frac{2^{2n+1}}{k^2}\iint_{Q_n}(u-k_n)_+^2\,dx\,d\tau,$$

we obtain

$$\begin{aligned}\iint_{Q_{n+1}}(u-k_{n+1})_+^2\,dx\,d\tau\leq\gamma&\left[\frac{2^{2n}}{(1-\sigma)^2\rho^2}\left(1+\frac{1}{\theta}\right)\iint_{Q_n}(u-k_n)_+^2\,dx\,d\tau\right.\\&\left.+\gamma_*^2\left(\frac{2^{2n+1}}{k^2}\iint_{Q_n}(u-k_n)_+^2\,dx\,d\tau\right)^{1-\frac{2}{N+2}+2\delta}\right]\\&\times\Big(\frac{2^{2n+1}}{k^2}\iint_{Q_n}(u-k_n)_+^2\,dx\,d\tau\Big)^{\frac{2}{N+2}}.\end{aligned}$$

Setting

$$Y_n=\frac{\theta}{k^2|Q_n|}\iint_{Q_n}(u-k_n)_+^2\,dx\,d\tau=\frac{\theta}{k^2}\fint\!\!\fint_{Q_n}(u-k_n)_+^2\,dx\,d\tau$$

yields

$$\begin{aligned}Y_{n+1} &\le \frac{\gamma b^n}{(1-\sigma)^2}\Big(1+\frac{1}{\theta}\Big)Y_n^{1+\frac{2}{N+2}} + \frac{\gamma\gamma_*^2 d^n}{k^2}(\rho^{N+2})^{2\delta}Y_n^{1+2\delta}\\ &\le \frac{\gamma b^n}{(1-\sigma)^2}\Big(1+\frac{1}{\theta}\Big)\left(Y_n^{1+\frac{2}{N+2}} + \gamma_*^2\frac{\rho^{2\delta(N+2)}}{k^2}Y_n^{1+2\delta}\right),\end{aligned} \tag{2.3}$$

where

$$b = 2^{2(1+\frac{2}{N+2})}, \qquad d = 2^{2(1+2\delta)}.$$

Stipulate to take k so large that

$$k \ge \gamma_*\rho^{\delta(N+2)}, \qquad k > \sqrt{\theta}\Big(\iint_{Q_\rho^-(\theta)} u_+^2 dx\,d\tau\Big)^{\frac{1}{2}}. \tag{2.4}$$

Then, $Y_n \le 1$ for all n and $Y_n^{\frac{2}{N+2}} \le Y_n^{2\delta}$. With these remarks and stipulations, the previous recursive inequalities take the form

$$Y_{n+1} \le \frac{\gamma b^n}{(1-\sigma)^2}\Big(1+\frac{1}{\theta}\Big)Y_n^{1+2\delta} \qquad \text{for all } \ n = 1, 2, \dots \tag{2.5}$$

From the fast geometric convergence Lemma 15.1 of Chapter 9, it follows that $\{Y_n\} \to 0$ as $n \to \infty$, provided

$$Y_o = \frac{\theta}{k^2}\iint_{Q_\rho^-(\theta)} u_+^2 dx\,d\tau \le b^{-\frac{1}{(2\delta)^2}}\gamma^{-\frac{1}{2\delta}}(1-\sigma)^{\frac{1}{\delta}}\Big(1+\frac{1}{\theta}\Big)^{-\frac{1}{2\delta}}.$$

Therefore, also taking into account (2.4), and choosing

$$k = \max\left\{\gamma_*\rho^{\delta(N+2)}\,;\ \frac{b^{\frac{1}{2(2\delta)^2}}\gamma^{\frac{1}{4\delta}}}{(1-\sigma)^{\frac{1}{2\delta}}}\sqrt{\theta}\Big(1+\frac{1}{\theta}\Big)^{\frac{1}{4\delta}}\left(\iint_{Q_\rho^-(\theta)} u_+^2 dx\,d\tau\right)^{\frac{1}{p}}\right\},$$

one derives

$$Y_\infty = \frac{\theta}{k^2}\iint_{Q_{\sigma\rho}^-(\theta)}(u-k_n)_+^2 dx\,d\tau = 0 \quad\Longrightarrow\quad \operatorname*{ess\,sup}_{Q_{\sigma\rho}^-(\theta)} u_+ \le k.$$

If $\gamma_* = 0$, then (2.3) is already in the form (2.5) with $\delta = \frac{1}{N+2}$. ■

3 Hölder Continuity of Functions in the PDG Classes

For a function $u \in \mathrm{PDG}(E_T, \bar\gamma, \gamma_*, C_2, \delta)$ and $(y,s)+Q_{2\rho}^-(\theta) \subset E_T$ with $\theta > 0$, set

$$\begin{aligned}\mu^+ &= \operatorname*{ess\,sup}_{(y,s)+Q_{2\rho}^-(\theta)} u, \qquad \mu^- = \operatorname*{ess\,inf}_{(y,s)+Q_{2\rho}^-(\theta)} u,\\ \omega(2\rho) &= \mu^+ - \mu^- = \operatorname*{ess\,osc}_{(y,s)+Q_{2\rho}^-(\theta)} u.\end{aligned} \tag{3.1}$$

These quantities are well defined since $u \in L^\infty_{\mathrm{loc}}(E_T)$ by the results of Section 2.

Theorem 3.1. *Let $u \in PDG(E_T, \bar{\gamma}, \gamma_*, C_2, \delta)$. There exist constants $C > 1$ and $\alpha \in (0,1)$ depending only upon the data and independent of u, such that for every pair of cylinders $(y,s) + Q_\rho^-(\theta) \subset (y,s) + Q_R^-(\theta) \subset E_T$*

$$\omega(\rho) \le C \max\left\{\omega(R)\left(\frac{\rho}{R}\right)^\alpha \; ; \, \gamma_* \rho^{\delta(N+2)}\right\}. \tag{3.2}$$

The Hölder continuity is local to E_T, with Hölder exponent $\alpha_o = \min\{\alpha; \delta(N+2)\}$. An upper bound for the Hölder constant is

$$\{\text{Hölder constant}\} \le C \max\{2MR^{-\alpha}; \gamma_*\}, \qquad \text{where} \qquad M = \|u\|_{\infty; (y,s)+Q_R^-(\theta)}.$$

This implies that the local Hölder estimates deteriorate near $\partial_p E_T$. Indeed, fix (x,t), $(y,s) \in E_T$ and let

$$R = \min\{\text{dist}\{(x,t); \partial_p E_T\} \; ; \, \text{dist}\{(y,s); \partial_p E_T\}\},$$

where

$$\text{dist}\{(x,t); \partial_p E_T\} = \inf_{(x_o,t_o)\in\partial_p E_T} \left(|x - x_o| + |t - t_o|^{\frac{1}{2}}\right).$$

If $|x - y| + |t - s|^{\frac{1}{2}} < R$, then (3.2) implies

$$|u(x,t) - u(y,s)| \le C \max\{\omega(R) R^{-\alpha_o}; \gamma_*\} \left(|x - y| + |t - s|^{\frac{1}{2}}\right)^{\alpha_o}.$$

If $|x - y| + |t - s|^{\frac{1}{2}} \ge R$, then

$$|u(x,t) - u(y,s)| \le 2MR^{-\alpha_o} \left(|x - y| + |t - s|^{\frac{1}{2}}\right)^{\alpha_o}.$$

Corollary 3.1 *Let u be a local weak solution of (1.1)–(1.4). Then, for every compact subset $K \subset E_T$, and for every pair $(x,t), (y,s) \in K$*

$$|u(x,t) - u(y,s)| \le C \max\left\{\frac{2M_K}{\text{dist}\{K; \partial_p E_T\}^\alpha}; \gamma_*\right\} \left(|x - y| + |t - s|^{\frac{1}{2}}\right)^{\alpha_o}$$

where $M_K = \operatorname{ess\,sup}_K |u|$ and

$$\text{dist}\{K; \partial_p E_T\} = \inf_{\substack{(x,t)\in K \\ (y,s)\in\partial_p E_T}} \left(|x - y| + |t - s|^{\frac{1}{2}}\right).$$

3.1 On the Proof of Theorem 3.1

Although the parameter δ is fixed, in view of its value in (1.6), which naturally arises from quasi-linear equations, we will assume $\delta \le \frac{1}{N+2}$. The value $\delta = \frac{1}{N+2}$ would occur if $\varepsilon \to \infty$ in the integrability requirements (1.3). For homogeneous PDG classes where $\gamma_* = 0$, although immaterial, we take $\delta = \frac{1}{N+2}$.

In what follows we assume that $u \in \text{PDG}$ is given and ρ is such that $\rho \le C_2^{-1}$. Moreover, according to the different conditions we consider in the auxiliary results, the cylinders $(y,s) + Q_{2\rho}^-(\theta) \subset E_T$ and $(y,s) + Q_{2\rho}^+(\theta) \subset E_T$ are fixed, and $\mu^\pm$ and $\omega(2\rho)$ are defined either as in (3.1), or as

$$\mu^+ = \operatorname*{ess\,sup}_{(y,s)+Q_{2\rho}^+(\theta)} u, \quad \mu^- = \operatorname*{ess\,inf}_{(y,s)+Q_{2\rho}^+(\theta)} u,$$
$$\omega(2\rho) = \mu^+ - \mu^- = \operatorname*{ess\,osc}_{(y,s)+Q_{2\rho}^+(\theta)} u.$$

Some statements are given in $(y,s) + Q_\rho^-(\theta)$, others in $(y,s) + Q_\rho^+(\theta)$, mainly for convenience, and also because they will be needed in Section 10. With an obvious change of notation, they always hold both in backward and forward cylinders. Finally, for simplicity, we denote $\omega(2\rho)$ by ω.

4 Estimating the Values of u by the Measure of the Set Where u is Either Near μ^+ or Near μ^-

Proposition 4.1 *For every $a \in (0,1)$, there exists $\nu \in (0,1)$ depending only on the data, θ and a, but independent of ω, such that if for some $\xi \in (0,1)$*

$$\big|[u > \mu^+ - \xi\omega] \cap (y,s) + Q_\rho^-(\theta)\big| \le \nu |Q_\rho^-(\theta)| \tag*{$(4.1)_+$}$$

then either $\xi\omega \le \gamma_* \rho^{\delta(N+2)}$ or

$$u \le \mu^+ - a\xi\omega \qquad \textit{a.e. in } (y,s) + Q_{\frac{1}{2}\rho}^-(\theta). \tag*{$(4.2)_+$}$$

Similarly, if

$$\big|[u < \mu^- + \xi\omega] \cap (y,s) + Q_\rho^-(\theta)\big| \le \nu |Q_\rho^-(\theta)| \tag*{$(4.1)_-$}$$

then either $\xi\omega \le \gamma_* \rho^{\delta(N+2)}$ or

$$u \ge \mu^- + a\xi\omega \qquad \textit{a.e. in } (y,s) + Q_{\frac{1}{2}\rho}^-(\theta). \tag*{$(4.2)_-$}$$

Proof. We prove only $(4.1)_-$–$(4.2)_-$, the arguments for $(4.1)_+$–$(4.2)_+$ being analogous. Without loss of generality, we may assume $(y,s) = (0,0)$; for $n = 0, 1, \dots,$ set

$$\rho_n = \rho + \frac{\rho}{2^n}, \qquad B_n = B_{\rho_n}, \qquad Q_n = B_n \times (-\theta\rho_n^2, 0]. \tag{4.1}$$

Apply (1.8) over B_n and Q_n to $(u - k_n)_-$, for the levels

$$k_n = \mu_- + \xi_n \omega \qquad \text{where} \qquad \xi_n = a\xi + \frac{1-a}{2^n}\xi. \tag{4.2}$$

The cutoff function ζ is taken of the form $\zeta(x,t)=\zeta_1(x)\zeta_2(t)$, where

$$\begin{aligned}
&\zeta_1=\begin{cases}1 & \text{in } B_{n+1}\\[2ex] 0 & \text{in } \mathbb{R}^N-B_n\end{cases} \qquad\qquad |\nabla\zeta_1|\le\frac{1}{\rho_n-\rho_{n+1}}=\frac{2^{n+1}}{\rho}\\
&\zeta_2=\begin{cases}0 & \text{for } t<-\theta\rho_n^2\\[2ex] 1 & \text{for } t\ge-\theta\rho_{n+1}^2\end{cases} \qquad 0\le\zeta_{2,t}\le\frac{1}{\theta(\rho_n^2-\rho_{n+1}^2)}\le\frac{2^{2(n+1)}}{\theta\rho^2}.
\end{aligned}\tag{4.3}$$

With all these stipulations, inequality (1.8) yields

$$\begin{aligned}
&\operatorname*{ess\,sup}_{-\theta\rho_n^2<t<0}\int_{B_n}(u-k_n)_-^2\zeta^2(x,t)dx+\iint_{Q_n}|\nabla[(u-k_n)_-\zeta]|^2dx\,d\tau\\
&\le\gamma\frac{2^{2n}}{\rho^2}\left(\iint_{Q_n}(u-k_n)_-^2\,dx\,d\tau+\frac{1}{\theta}\iint_{Q_n}(u-k_n)_-^2\,dx\,d\tau\right)\\
&\qquad+\gamma_*^2|A^-_{k_n,\rho_n}|^{1-\frac{2}{N+2}+2\delta}\\
&\le\gamma\frac{2^{2n}(\xi\omega)^2}{\rho^2}\left(1+\frac{1}{\theta}\right)|A^-_{k_n,\rho_n}|+\gamma_*^2|A^-_{k_n,\rho_n}|^{1-\frac{2}{N+2}+2\delta},
\end{aligned}\tag{4.4}$$

where

$$A^-_{k_n,\rho_n}=[(u-k_n)_->0]\cap Q_n(\theta).$$

By the embedding of Proposition 1.1 of Chapter 11

$$\begin{aligned}
\iint_{Q_n}[(u-k_n)_-\zeta]^{2\frac{N+2}{N}}dx\,d\tau&\le\gamma\iint_{Q_n}|\nabla[(u-k_n)_-\zeta]|^2dx\,d\tau\\
&\qquad\times\left(\operatorname*{ess\,sup}_{-\theta\rho_n^2<t<0}\int_{B_n}[(u-k_n)_-\zeta(x,t)]^2dx\right)^{\frac{2}{N}}\\
&\le\gamma\left[\gamma\frac{2^{2n}(\xi\omega)^2}{\rho^2}\left(1+\frac{1}{\theta}\right)|A^-_{k_n,\rho_n}|+\gamma_*^2|A^-_{k_n,\rho_n}|^{1-\frac{2}{N+2}+2\delta}\right]^{\frac{N+2}{N}}\\
&\le\gamma\frac{2^{2\frac{N+2}{N}n}(\xi\omega)^{2\frac{N+2}{N}}}{\rho^{2\frac{N+2}{N}}}\left(1+\frac{1}{\theta}\right)^{\frac{N+2}{N}}|A^-_{k_n,\rho_n}|^{1+\frac{2}{N}}\\
&\qquad+\gamma\,\gamma_*^{2\frac{N+2}{N}}|A^-_{k_n,\rho_n}|^{1+2\delta\frac{N+2}{N}}.
\end{aligned}$$

Estimate below

$$\iint_{Q_n}[(u-k_n)_-\zeta]^{2\frac{N+2}{N}}dx\,d\tau\ge\left(\frac{(1-a)\xi\omega}{2^{n+1}}\right)^{2\frac{N+2}{N}}|A^-_{k_{n+1},\rho_{n+1}}|$$

and set

$$Y_n=\frac{|A^-_{k_n,\rho_n}|}{|Q_n|}.$$

Then,

$$\begin{aligned}
Y_{n+1} \le & \frac{\gamma b^n}{(1-a)^{2\frac{N+2}{N}}}\theta^{\frac{2}{N}}\left(1+\frac{1}{\theta}\right)^{\frac{N+2}{N}} Y_n^{1+\frac{2}{N}} \\
& + \frac{\gamma d^n \theta^{2\delta\frac{N+2}{N}}}{(1-a)^{2\frac{N+2}{N}}}\frac{\gamma_*^{2\frac{N+2}{N}}\rho^{2\delta\frac{(N+2)^2}{N}}}{(\xi\omega)^{2\frac{N+2}{N}}} Y_n^{1+2\delta\frac{N+2}{N}} \\
\le & \frac{\gamma\bar{\theta}b^n}{(1-a)^{2\frac{N+2}{N}}}\left(1+\frac{1}{\theta}\right)^{\frac{N+2}{N}}\left(Y_n^{1+\frac{2}{N}} + \frac{\gamma_*^{2\frac{N+2}{N}}\rho^{2\delta\frac{(N+2)^2}{N}}}{(\xi\omega)^{2\frac{N+2}{N}}} Y_n^{1+2\delta\frac{N+2}{N}}\right),
\end{aligned}$$

where

$$b = 2^{4\frac{N+2}{N}}, \quad d = 2^{2\frac{N+2}{N}}, \quad \bar{\theta} = \max\{\theta^{\frac{2}{N}}; \theta^{2\delta\frac{N+2}{N}}\}. \tag{4.5}$$

From this, one derives

$$Y_{n+1} \le \frac{\gamma\,\bar{\theta}\,b^n}{(1-a)^{(N+2)\frac{2}{N}}}\left(1+\frac{1}{\theta}\right)^{\frac{N+2}{N}} Y_n^{1+2\delta\frac{N+2}{N}}$$

provided $\xi\omega > \gamma_*\rho^{\delta(N+2)}$. Owing to Lemma 15.1 of Chapter 9 these recursive inequalities imply that $\{Y_n\} \to 0$ as $n \to \infty$, provided

$$\begin{aligned}
Y_o = & \frac{\big|[u < \mu^- + \xi\omega] \cap Q_\rho^-(\theta)\big|}{|Q_\rho^-(\theta)|} \\
\le & \frac{(1-a)^{1/\delta}}{(\gamma\bar{\theta})^{\frac{N}{2\delta(N+2)}}\, b^{\frac{N^2}{4\delta^2(N+2)^2}}}\left(\frac{\theta}{\theta+1}\right)^{\frac{1}{2\delta}} \overset{\text{def}}{=} \nu.
\end{aligned} \tag{4.6}$$

Remark 4.1 Formula (4.6) provides a precise dependence of ν on the data, θ, and a. In particular, ν is clearly independent of ξ and ω.

The proof of $(4.2)_+$ is almost identical. One starts from the inequalities (1.8) written for the truncated functions

$$(u - k_n)_+ \qquad \text{with} \qquad k_n = \mu_+ - \xi_n\omega$$

for the same choice of ξ_n as in (4.2). By the definition of μ_+ one estimates

$$(u - k_n)_+ \le \xi\omega.$$

This validates estimates in all analogous to (4.4) with the same functional dependence on $\xi\omega$. The same arguments, with the proper changes in the meaning of the symbols, lead to (4.6) written for ν_+, and conclude the proof. Thus, ν_+ depends on the data, θ, and a the same way as ν_- does. ■

Remark 4.2 In Proposition 4.1 the statement relative to $(4.1)_-$–$(4.2)_-$ is given in terms of μ_- and $\xi\omega$. As a matter of fact, as the proof clearly shows, when dealing with the lower truncations $(u-k)_-$ for *non-negative* functions, all the estimates depend only on $k \ge 0$, without any further assumption on it.

5 Reducing the Measure of the Set Where u is Either Near μ^+ or Near μ^-

Proposition 5.1 *Assume that*

$$\big|[u(\cdot,t) \le \mu^+ - \epsilon\omega] \cap B_\rho(y)\big| \ge \alpha|B_\rho| \tag{5.1$_+$}$$

for some α, $\epsilon \in (0,1)$ *and for all* $t \in (s, s+\theta\rho^2]$ *for some* $\theta > 0$*. Then, for every* $\nu \in (0,1)$ *there exists* $\epsilon_\nu \in (0,1)$ *that can be determined a priori only in terms of the data,* θ*,* ϵ*, and* α*, and independent of* ω*, such that either* $\epsilon_\nu\omega \le \gamma_*\rho^{\delta(N+2)}$ *or*

$$\big|[u > \mu^+ - \epsilon_\nu\omega] \cap (y,s) + Q_\rho^+(\theta)\big| \le \nu\big|Q_\rho^+(\theta)\big|. \tag{5.2$_+$}$$

Similarly, if

$$\big|[u(\cdot,t) \ge \mu^- + \epsilon\omega] \cap B_\rho(y)\big| \ge \alpha|B_\rho| \tag{5.1$_-$}$$

for some α, $\epsilon \in (0,1)$ *and for all* $t \in (s, s+\theta\rho^2]$ *for some* $\theta > 0$*, then for every* $\nu \in (0,1)$ *there exists* $\epsilon_\nu \in (0,1)$ *depending only on the data,* θ*,* ϵ*, and* α*, and independent of* ω*, such that either* $\epsilon_\nu\omega \le \gamma_*\rho^{\delta(N+2)}$ *or*

$$\big|[u < \mu^- + \epsilon_\nu\omega] \cap (y,s) + Q_\rho^+(\theta)\big| \le \nu\big|Q_\rho^+(\theta)\big|. \tag{5.2$_-$}$$

Remark 5.1 An analogous statement holds, if we consider cylinders $(y,s) + Q_\rho^-(\theta)$.

The set $[u < \mu^- + \epsilon_\nu\omega]$ in the cylinder $(y,s) + Q_\rho^+(\theta)$ can be made arbitrarily small, provided ϵ_ν is chosen accordingly. The main tools of the proof are the estimate of the measure of the sets $A_{\mu^- + \frac{1}{2}\omega,\rho}(t)$ for all $t \in (s, s+\theta\rho^2)$, and the discrete isoperimetric inequality of Proposition 5.2 of Chapter 10.

5.1 Proof of Proposition 5.1

Proof. We will establish (5.2)$_-$ starting from (5.1)$_-$. As usual, without loss of generality, we may assume $(y,s) = (0,0)$.

Write down the estimates (1.8) over the cylinder

$$\mathcal{Q}_{2\rho}^+(\theta) \cup \mathcal{Q}_{2\rho}^-(\theta) = (B_{2\rho} \times (-\theta\rho^2, 0]) \cup (B_{2\rho} \times (0, \theta\rho^2]) = B_{2\rho} \times (-\theta\rho^2, \theta\rho^2]$$

for the truncated functions

$$(u - k_j)_- \quad \text{for the levels} \quad k_j = \mu^- + \frac{1}{2^j}\epsilon\omega, \quad \text{for } j = 0, 1, \dots. \tag{5.1}$$

The non-negative, piecewise smooth test function ζ is chosen so that it vanishes outside $B_{2\rho}$ and for $t \le -\theta\rho^2$, and

$$\zeta = 1 \text{ on } Q_\rho^+(\theta), \qquad |\nabla\zeta| \le \frac{1}{\rho}, \qquad \text{and } 0 \le \zeta_t \le \frac{1}{\theta\rho^2}.$$

The first term on the left-hand side of (1.8) is discarded since it is non-negative, and the second vanishes because of our choice of test function. The term involving $|\nabla(u-k_j)_-|$ is minorized by extending the integration over the cylinder $Q_\rho^+(\theta)$, which is the set where $\zeta = 1$. These remarks give the inequalities

$$\begin{aligned}
\iint_{Q_\rho^+(\theta)} &|\nabla(u-k_j)_-|^2\zeta^2 dx\, d\tau \\
&\le \bar{\gamma}\int_{-\theta\rho^2}^{\theta\rho^2}\int_{B_{2\rho}}(u-k_j)_-^2\Big(|\nabla\zeta|^2+\zeta_\tau+\frac{1}{\rho^2}\Big)dx\, d\tau \\
&\quad + \gamma_*^2\big|[(u-k_j)_- > 0]\cup \mathcal{Q}_{2\rho}^+(\theta)\cup\mathcal{Q}_{2\rho}^-(\theta)\big|^{1-\frac{2}{N+2}+2\delta} \qquad (5.2)\\
&\le \gamma\theta\rho^2\left(\frac{\epsilon\omega}{2^j}\right)^2\left(\frac{1}{\rho^2}+\frac{2}{\theta\rho^2}\right)|B_{2\rho}| + \gamma\gamma_*^2\theta^{1-\frac{2}{N+2}+2\delta}\rho^{N+2\delta(N+2)} \\
&\le \gamma\left[\left(\frac{\epsilon\omega}{2^j}\right)^2(\theta+1)|B_\rho| + \gamma_*^2\theta^{1-\frac{2}{N+2}+2\delta}\rho^{N+2\delta(N+2)}\right]
\end{aligned}$$

for a new constant γ depending only on the data $\{N, C_o, C_1\}$.

Apply the discrete isoperimetric inequality of Proposition 5.2 of Chapter 10 with

$$\ell = k_j = \mu^- + \frac{1}{2^j}\epsilon\omega \quad \text{and} \quad k = k_{j+1} = \mu^- + \frac{1}{2^{j+1}}\epsilon\omega \quad \text{for } j = 0, 1, \dots,$$

and take into account $(5.1)_-$, to obtain

$$\frac{\epsilon\omega}{2^{j+1}}\big|A_{k_{j+1},\rho}(t)\big| \le \frac{\gamma}{\alpha}\rho\int_{B_\rho\cap[k_{j+1}<u<k_j]}|\nabla u(\cdot,t)|dx,$$

where $A_{k_{j+1},\rho}(t) = [u(\cdot,t) < k_{j+1}]\cap B_\rho$. Integrate this in dt over $(0,\theta\rho^2)$ and set

$$|A_j| = |[u < k_j]\cap Q_\rho^+(\theta)| = \int_0^{\theta\rho^2}\big|A_{k_j,\rho}(\tau)\big|d\tau.$$

Then, the previous inequality yields

$$\begin{aligned}
\frac{\epsilon\omega}{2^{j+1}}\big|A_{j+1}\big| &\le \frac{\gamma}{\alpha}\rho\iint_{Q_\rho^+(\theta)\cap[k_{j+1}<u<k_j]}|\nabla u|dx\, d\tau \\
&\le \frac{\gamma}{\alpha}\rho\Big(\iint_{Q_\rho^+(\theta)}|\nabla(u-k_j)_-|^2 dx\, d\tau\Big)^{\frac{1}{2}}\big|A_j - A_{j+1}\big|^{\frac{1}{2}} \\
&\le \frac{\gamma}{\alpha}\frac{\epsilon\omega}{2^j}\sqrt{Q_\rho^+(\theta)}\left[\left(1+\frac{1}{\theta}\right)+\frac{\gamma_*^2}{\theta^{\frac{2}{N+2}-2\delta}}\left(\frac{2^j}{\epsilon\omega}\right)^2\rho^{2\delta(N+2)}\right]^{\frac{1}{2}} \\
&\quad \times\big(|A_j| - |A_{j+1}|\big)^{\frac{1}{2}}
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\gamma}{\alpha} \frac{\epsilon\omega}{2^j} \sqrt{Q_\rho^+(\theta)} \left[\frac{\theta+1}{\theta} + \frac{\gamma_*^2}{\theta^{\frac{2}{N+2}-2\delta}} \left(\frac{2^j}{\epsilon\omega} \right)^2 \rho^{2\delta(N+2)} \right]^{\frac{1}{2}} \\
&\quad \times \big(|A_j| - |A_{j+1}| \big)^{\frac{1}{2}} \\
&\leq \frac{\gamma}{\alpha\sqrt{\theta^*}} \frac{\epsilon\omega}{2^j} \sqrt{Q_\rho^+(\theta)} \left[1 + \gamma_*^2 \left(\frac{2^j}{\epsilon\omega} \right)^2 \rho^{2\delta(N+2)} \right]^{\frac{1}{2}} \\
&\quad \times \big(|A_j| - |A_{j+1}| \big)^{\frac{1}{2}},
\end{aligned}
$$

where we have used the estimates (5.2), and we have set

$$
\frac{1}{\theta^*} = \max\left\{ \frac{\theta+1}{\theta}; \frac{1}{\theta^{\frac{2}{N+2}-2\delta}} \right\}. \tag{5.3}
$$

Next, divide by $\dfrac{\epsilon\omega}{2^j}$, and square both sides to obtain the recursive inequalities

$$
|A_{j+1}|^2 \leq \frac{(2\gamma)^2}{\alpha^2\theta^*} |Q_\rho^+(\theta)| \left[1 + \gamma_*^2 \left(\frac{2^j}{\epsilon\omega} \right)^2 \rho^{2\delta(N+2)} \right] \big(|A_j| - |A_{j+1}| \big)
$$

for $j = 0, 1 \ldots$. Stipulate that

$$
\gamma_*^2 \left(\frac{2^{j^*}}{\epsilon\omega} \right)^2 \rho^{2\delta(N+2)} < 1
$$

where j_* is a positive integer to be chosen, and add these inequalities for $j = 0, 1, \ldots, j_* - 1$. Minorize the terms on the left-hand side by their smallest value $|A_{j_*}|^2$ and majorize the right-hand side with the corresponding telescopic series. The indicated estimations yield

$$
\begin{aligned}
j_*|A_{j_*}|^2 \leq \sum_{j=0}^{j_*-1} |A_{j+1}|^2 &\leq \frac{(2\gamma)^2}{\alpha^2\theta^*} |Q_\rho^+(\theta)| \sum_{j=0}^{\infty} \big(|A_j| - |A_{j+1}| \big) \\
&\leq \frac{(2\gamma)^2}{\alpha^2\theta^*} |Q_\rho^+(\theta)|^2.
\end{aligned}
$$

From this

$$
|A_{j_*}| \leq \frac{2\gamma}{\alpha\sqrt{\theta^* j_*}} |Q_\rho^+(\theta)|. \tag{5.4}
$$

Thus, having fixed $\nu \in (0,1)$, one can choose j_* so large that

$$
\frac{\big|[u < \mu^- + \epsilon_\nu\omega] \cap Q_\rho^+(\theta)\big|}{|Q_\rho^+(\theta)|} < \nu, \quad \text{for} \quad \frac{2\gamma}{\alpha\sqrt{\theta^* j_*}} \leq \nu, \quad \text{and} \quad \epsilon_\nu = \frac{\epsilon}{2^{j_*}}. \quad \blacksquare
$$

The proof of Proposition 5.1 shows that it can be rephrased in this equivalent way.

Proposition 5.2 *Assume that*

$$\big|[u(\cdot,t)\le\mu^+-\epsilon\omega]\cap B_\rho(y)\big|\ge\alpha|B_\rho| \tag{5.4$_+$}$$

for some α, $\epsilon\in(0,1)$ *and for all* $t\in(s,s+\theta\rho^2]$ *for some* $\theta>0$*. Then there exists* $\gamma>0$*, depending only on the data, such that for any positive integer* j_* *we have either* $\frac{\epsilon\omega}{2^{j_*}}\le\gamma_*\rho^{\delta(N+2)}$ *or*

$$\frac{\big|[u>\mu^+-\frac{\epsilon\omega}{2^{j_*}}]\cap Q^+_\rho(\theta)\big|}{|Q^+_\rho(\theta)|}\le\frac{2\gamma}{\alpha\sqrt{\theta^* j_*}}. \tag{5.5$_+$}$$

Similarly, if

$$\big|[u(\cdot,t)\ge\mu^-+\epsilon\omega]\cap B_\rho(y)\big|\ge\alpha|B_\rho| \tag{5.4$_-$}$$

for some α, $\epsilon\in(0,1)$ *and for all* $t\in(s,s+\theta\rho^2]$ *for some* $\theta>0$*, then there exists* $\gamma>0$*, depending only on the data, such that for any positive integer* j_* *we have either* $\frac{\epsilon\omega}{2^{j_*}}\le\gamma_*\rho^{\delta(N+2)}$ *or*

$$\frac{\big|[u<\mu^-+\frac{\epsilon\omega}{2^{j_*}}]\cap Q^+_\rho(\theta)\big|}{|Q^+_\rho(\theta)|}\le\frac{2\gamma}{\alpha\sqrt{\theta^* j_*}}. \tag{5.5$_-$}$$

Both in $(5.5)_+$ and in $(5.5)_-$ θ^* is the quantity defined above in (5.3).

Remark 5.2 As in the case of Proposition 5.1, an analogous statement holds, if we consider cylinders $(y,s)+Q^-_\rho(\theta)$.

6 Propagating in Time the Measure-Theoretical Information

Proposition 6.1 *Assume that*

$$\Big|\Big[u(\cdot,s)<\mu^+-\frac{\omega}{4}\Big]\cap B_\rho(y)\Big|\ge\frac{\alpha}{2}|B_\rho|. \tag{6.1$_+$}$$

for some $\alpha\in(0,1]$*. Then, there exist* $\xi\in(0,1)$ *and* $\theta\in(0,1)$*, which depend only on the data, such that either* $\omega\le2\gamma_*\rho^{\delta(N+2)}$ *or*

$$\big|\big[u(\cdot,t)<\mu^+-\xi\omega\big]\cap B_\rho(y)\big|\ge\frac{\alpha}{4}|B_\rho| \tag{6.2$_+$}$$

for all $t\in[s,s+\theta\rho^2]$*. Analogously, if*

$$\Big|\Big[u(\cdot,s)>\mu^-+\frac{\omega}{4}\Big]\cap B_\rho(y)\Big|\ge\frac{\alpha}{2}|B_\rho|, \tag{6.1$_-$}$$

for some $\alpha\in(0,1]$*, then there exist* $\xi\in(0,1)$ *and* $\theta\in(0,1)$*, which depend only on the data, such that either* $\omega\le2\gamma_*\rho^{\delta(N+2)}$ *or*

$$\big|\big[u(\cdot,t)>\mu^-+\xi\omega\big]\cap B_\rho(y)\big|\ge\frac{\alpha}{4}|B_\rho| \tag{6.2$_-$}$$

for all $t\in[s,s+\theta\rho^2]$*.*

Remark 6.1 An analogous statement holds, if we consider cylinders $(y,s) + Q_\rho^-(\theta)$.

In the context of parabolic equations, these kinds of results are usually proven using proper logarithmic estimates (see, for example, DiBenedetto [49], Chapter 2, Section 3). Unfortunately, in the context of PDG classes, we do not know whether such estimates hold or not. Therefore, we give here a different proof, which makes no use of any kind of logarithmic functions.

6.1 Proof of Proposition 6.1

Proof. We will establish $(6.2)_-$ starting from $(6.1)_-$. As usual, without loss of generality, we may assume $(y,s) = (0,0)$.

For simplicity, we set $M = \frac{1}{4}\omega$. We start from (1.8) written over $Q_\rho^+(\theta)$ for $(u-k)_-$ with $k = \mu^- + M$, where θ is to be determined, and we discard the third term on the left-hand side. Moreover, we consider a non-negative, piecewise smooth test function $\zeta = \zeta(x)$ such that

$$\zeta = 1 \text{ in } B_{\sigma\rho}, \quad \zeta = 0 \text{ in } \mathbb{R}^N - B_\rho, \quad |\nabla\zeta| \le \frac{1}{\sigma\rho}.$$

Hence, we conclude that

$$\begin{aligned}
\operatorname*{ess\,sup}_{0<t<\theta\rho^2} \int_{B_\rho} & (u-k)_-^2 \zeta^2(x,t)dx \\
\le & \int_{B_\rho} (u-k)_-^2 \zeta^2(x,0)dx \\
& + \frac{\gamma}{\sigma^2\rho^2} \iint_{Q_\rho^+(\theta)} (u-k)_-^2 \, dx\, d\tau + \gamma_*^2 |A_{k,\rho}^-|^{1-\frac{2}{N+2}+2\delta} \\
\le & M^2 \left(1 - \frac{\alpha}{2}\right) |B_\rho| + \gamma \frac{\theta}{\sigma^2} M^2 |B_\rho| \\
& + \gamma \theta^{1-\frac{2}{N+2}+2\delta} \left(\gamma_*^2 \frac{\rho^{2\delta(N+2)}}{M^2} \right) M^2 |B_\rho|.
\end{aligned}$$

If we stipulate taking

$$\gamma_*^2 \frac{\rho^{2\delta(N+2)}}{M^2} < 1,$$

we have

$$\begin{aligned}
\operatorname*{ess\,sup}_{0<t<\theta\rho^2} \int_{B_\rho} & (u-k)_-^2 \zeta^2(x,t)dx \\
& \le M^2 \left[\left(1 - \frac{\alpha}{2}\right) + \gamma \frac{\theta}{\sigma^2} + \gamma \theta^{1-\frac{2}{N+2}+2\delta} \right] |B_\rho|.
\end{aligned}$$

Let $\ell = \mu^- + \epsilon M$ with $\epsilon \in (0,1)$ to be chosen. Then,

$$\begin{aligned}\int_{B_\rho}(u-k)_-^2\zeta^2(x,t)dx &\ge \int_{B_{(1-\sigma)\rho}\cap[u\le\ell]}(u-k)_-^2(x,t)dx\\ &\ge (1-\epsilon)^2M^2|[u(\cdot,t)\le\ell]\cap B_{(1-\sigma)\rho}|\\ &= (1-\epsilon)^2M^2|A_{\ell,(1-\sigma)\rho}(t)|.\end{aligned}$$

Moreover,

$$\begin{aligned}|A_{\ell,\rho}(t)| &= |A_{\ell,(1-\sigma)\rho}(t)\cup(A_{\ell,\rho}(t)\backslash A_{\ell,(1-\sigma)\rho}(t))|\\ &\le |A_{\ell,(1-\sigma)\rho}(t)|+|B_\rho\backslash B_{(1-\sigma)\rho}|\\ &\le |A_{\ell,(1-\sigma)\rho}(t)|+N\sigma|B_\rho|.\end{aligned}$$

Therefore, we conclude

$$\begin{aligned}(1-\epsilon)^2M^2|A_{\ell,\rho}(t)| \le &M^2\bigg[\Big(1-\frac{\alpha}{2}\Big)+\gamma\frac{\theta}{\sigma^2}\\ &+\gamma\theta^{1-\frac{2}{N+2}+2\delta}+N(1-\epsilon)^2\sigma\bigg]|B_\rho|.\end{aligned}$$

Since we can take $\epsilon<\frac{1}{2}$ without loss of generality, we also have

$$|A_{\ell,\rho}(t)|\le\left[\frac{1-\frac{\alpha}{2}}{(1-\epsilon)^2}+\gamma\frac{\theta}{\sigma^2}+\gamma\theta^{1-\frac{2}{N+2}+2\delta}+N\sigma\right]|B_\rho|.$$

Taking into account that $\theta\in(0,1)$ yields

$$|A_{\ell,\rho}(t)|\le\left[\frac{1-\frac{\alpha}{2}}{(1-\epsilon)^2}+2\gamma\frac{\theta}{\sigma^2}+N\sigma\right]|B_\rho|.$$

Finally, we choose

$$\begin{aligned}&\sigma=\frac{\alpha}{32N},\quad \theta=\frac{\alpha^3}{2\cdot 32^3\gamma N^2},\\ &\frac{1-\frac{\alpha}{2}}{(1-\epsilon)^2}<\frac{1-\frac{\alpha}{4}}{(1-\epsilon)^2}\le 1-\frac{\alpha}{8}\quad\Rightarrow\quad \epsilon\approx\frac{3}{16}\alpha,\end{aligned}$$

and conclude

$$|A_{\ell,\rho}(t)|\le\Big(1-\frac{\alpha}{16}\Big)|B_\rho|$$

for all $t\in[0,\theta\rho^2]$, with θ computed above.

■

Remark 6.2 In Section 6c of the Complements, we present a refined version of Proposition 6.1, based on a clever idea originally introduced in Liao [171].

7 Proof of Theorem 3.1

Without loss of generality, we may assume $(y, s) = (0, 0)$. Assume $\theta = 1$ and consider Q_ρ^-. By the definitions in (3.1), either

$$|[u \le \mu^+ - \tfrac{1}{2}\omega] \cap Q_\rho^-| \ge \frac{1}{2}|Q_\rho^-| \tag{7.1}$$

or

$$|[u \ge \mu^- + \tfrac{1}{2}\omega] \cap Q_\rho^-| \ge \frac{1}{2}|Q_\rho^-|. \tag{7.2}$$

Indeed, if both were false, we would immediately have a contradiction.

Assume the second is in force; in case (7.1) holds, the argument runs exactly in the same way. We can equivalently rewrite (7.2) as

$$|[u < \mu^- + \tfrac{1}{2}\omega] \cap Q_\rho^-| < \frac{1}{2}|Q_\rho^-|.$$

We claim that this implies that there exists $t_o \in [-\rho^2, -\frac{1}{3}\rho^2]$ such that

$$|[u(\cdot, t_o) < \mu^- + \tfrac{1}{2}\omega] \cap B_\rho| \le \frac{3}{4}|B_\rho|$$

Indeed, if not, then

$$|[u < \mu^- + \tfrac{1}{2}\omega] \cap Q_\rho^-| \ge \int_{-\rho^2}^{-\frac{1}{3}\rho^2} |[u(\cdot, \tau) < \mu^- + \tfrac{1}{2}\omega] \cap B_\rho| d\tau > \frac{1}{2}|Q_\rho^-|.$$

We can equivalently conclude that there exists $t_o \in [-\rho^2, \frac{1}{3}\rho^2]$ such that

$$|[u(\cdot, t_o) \ge \mu^- + \tfrac{1}{2}\omega] \cap B_\rho| \ge \frac{1}{4}|B_\rho|.$$

Without loss of generality, we may assume that $t_o = -\rho^2$, so that

$$|[u(\cdot, -\rho^2) \ge \mu^- + \tfrac{1}{4}\omega] \cap B_\rho| \ge \frac{1}{4}|B_\rho|.$$

In such a way we have put ourselves in the worst possible situation, where the measure theoretical information is furthest away from the origin, about which we want to show that the oscillation reduces.

We can then apply Proposition 6.1 with $\alpha = \frac{1}{2}$, and conclude that either

$$\omega \le 2\gamma_* \rho^{\delta(N+2)} \tag{7.3}$$

or

$$|[u(\cdot, t) \ge \mu^- + \xi\omega] \cap B_\rho| \ge \frac{1}{8}|B_\rho| \tag{7.4}$$

for all $t \in [-\rho^2, -\rho^2 + \theta\rho^2]$, with $\theta \in (0, 1)$ as computed in Proposition 6.1.

If (7.3) holds true, we have finished. Otherwise, assuming that (7.4) is in force, fix ν as claimed by Proposition 4.1 in (4.6), for the choices $a = \frac{1}{2}$, with θ as above, and the corresponding $\bar{\theta}$.

We point out that (7.4) is $(5.1)_-$ of Proposition 5.1 with $\alpha = \frac{1}{8}$ and $\epsilon = \xi$; therefore, ν being fixed, determine j_* and hence $\epsilon_\nu = \frac{\xi}{2^{j_*}}$ by the procedure of Proposition 5.1.

Then, by Proposition 4.1, either $\epsilon_\nu \omega \le \gamma_* \rho^{\delta(N+2)}$ and we have finished, or

$$u \ge \mu^- + \frac{1}{2}\epsilon_\nu \omega \quad \text{a.e. in } B_{\frac{\rho}{2}} \times (-\rho^2 + \theta\left(\frac{\rho}{2}\right)^2, -\rho^2 + \theta\rho^2].$$

In particular,

$$u(\cdot, t_1) \ge \mu^- + \frac{1}{2^{j_*+1}}\xi\omega \quad \text{in } B_{\frac{\rho}{2}},$$

where $t_1 = -\rho^2 + \theta\rho^2$. Hence,

$$\begin{aligned} |[u(\cdot,t_1) \ge \mu^- + \frac{1}{2^{j_*+1}}\xi\omega] \cap B_\rho| &\ge |[u(\cdot,t_1) \ge \mu^- + \frac{1}{2^{j_*+1}}] \cap B_{\frac{\rho}{2}}| \\ &= |B_{\frac{\rho}{2}}| = \frac{1}{2^N}|B_\rho|. \end{aligned}$$

We can now work as before, with $\alpha = \frac{1}{2^N}$, $\epsilon = \frac{1}{2^{j_*+1}}\xi$, and conclude that there exist $\tilde{\theta} = \tilde{\theta}(\text{data}, N)$ and $\tilde{j} = \tilde{j}(\text{data}, N)$ such that either

$$\frac{1}{2^{j_*+1+\tilde{j}}}\xi\omega \le \gamma_* \rho^{\delta(N+2)},$$

and we have finished, or for $t_2 = -\rho^2 + \theta\rho^2 + \tilde{\theta}\rho^2$ we have

$$u(\cdot, t_2) \ge \mu^- + \frac{1}{2^{j_*+1+\tilde{j}}}\xi\omega \quad \text{in } B_{\frac{\rho}{2}},$$

which yields that

$$\begin{aligned} |[u(\cdot,t_2) \ge \mu^- + \frac{1}{2^{j_*+1+\tilde{j}}}\xi\omega] \cap B_\rho| &\ge |[u(\cdot,t_2) \ge \mu^- + \frac{1}{2^{j_*+1+\tilde{j}}}] \cap B_{\frac{\rho}{2}}| \\ &= |B_{\frac{\rho}{2}}| = \frac{1}{2^N}|B_\rho|. \end{aligned}$$

The previous step can be further repeated with the same constants as before. Let $n \in \mathbb{N}$ be such that

$$t_{n+1} \overset{\text{def}}{=} -\rho^2 + \theta\rho^2 + n\tilde{\theta}\rho^2 \ge 0.$$

Such a value of n is precisely determined by the previous steps. We conclude that either

$$\frac{1}{2^{j_*+1+n\tilde{j}}}\xi\omega \le \gamma_* \rho^{\delta(N+2)},$$

or

$$u \ge \mu^- + \frac{1}{2^{j_*+1+n\tilde{j}}}\xi\omega \quad \text{in } B_{\frac{\rho}{2}} \times (t_n, 0].$$

For simplicity, we let

$$\tilde{Q}^-_{\frac{1}{2}\rho} = B_{\frac{\rho}{2}} \times (t_n, 0] = B_{\frac{\rho}{2}} \times (-\bar{\theta}\rho^2, 0]$$

with $\bar{\theta} = 1 - \theta - (n-1)\tilde{\theta}$ and we remark that the latter implies

$$-\operatorname*{ess\,inf}_{\tilde{Q}^-_{\frac{1}{2}\rho}} u \le -\operatorname*{ess\,inf}_{Q^-_{2\rho}} u - \tfrac{1}{2^{j_*+1+n\tilde{j}}}\xi \operatorname*{ess\,osc}_{Q^-_{2\rho}} u.$$

Now,

$$\operatorname*{ess\,sup}_{\tilde{Q}^-_{\frac{1}{2}\rho}} u \le \operatorname*{ess\,sup}_{Q^-_{2\rho}} u.$$

Adding these inequalities gives

$$\operatorname*{ess\,osc}_{\tilde{Q}^-_{\frac{1}{2}\rho}} u \le \eta \operatorname*{ess\,sup}_{Q^-_{2\rho}} u, \qquad \text{where } \eta = 1 - \frac{1}{2^{j_*+1+n\tilde{j}}}\xi.$$

Let $Q^-_R \subset E_T$ be fixed and set $\rho_k = 4^{-k}R$. If we let $Q^-_k = B_{\frac{\rho_k}{2}} \times (-\bar{\theta}\rho_k^2, 0]$, the previous remarks imply that

$$\operatorname*{ess\,osc}_{Q^-_k} u \le \max\{\eta \operatorname*{ess\,osc}_{Q^-_{k-1}} u\, ;\, \bar{C}\gamma_* \rho_k^{\delta(N+2)}\} \tag{7.5}$$

for a proper choice of $\bar{C}$ that takes into account all the alternatives. By iteration

$$\operatorname*{ess\,osc}_{Q^-_k} u \le \max\{\eta^k \operatorname*{ess\,osc}_{Q^-_R} u\, ;\, \bar{C}\gamma_* \rho_k^{\delta(N+2)}\}.$$

Compute

$$\rho_k = 4^{-k}R \implies -k = \ln\left(\frac{\rho_k}{R}\right)^{\frac{1}{\ln 4}} \implies \eta^k = \left(\frac{\rho_k}{R}\right)^{\alpha} \quad \text{for } \alpha = -\frac{\ln \eta}{\ln 4}. \qquad \blacksquare$$

Remark 7.1 We have given the proof assuming $\theta = 1$, but a general $\theta > 0$ is also possible, without any substantial change in the previous arguments.

After proving the Harnack inequality below, we will show how such a result also implies the local Hölder continuity of solutions. Moreover, we will give a third proof of the continuity result based on the same set of ideas that are used to show the validity of the Harnack inequality.

8 Boundary Parabolic DeGiorgi Classes: Dirichlet Data

Let ∂E be the finite union of portions of $(N-1)$-dimensional surfaces of class C^1, so that the trace of a function $u \in W^{1,p}(E)$ can be defined on ∂E, except possibly on an $(N-2)$-dimensional subset of ∂E.

Given

$$\begin{cases} g \in L^2\big(0,T;W^{1,2}(E)\big), \\ g \text{ continuous on } \overline{E}_T \text{ with modulus of continuity } \ \omega_g(\cdot), \end{cases} \tag{8.1}$$

we are interested in the boundary behavior of solutions of the Cauchy–Dirichlet problem

$$\begin{cases} u_t - \operatorname{div} \mathbf{A}(x,t,u,\nabla u) = B(x,t,u,\nabla u) \quad \text{ in } \ E_T, \\ u(\cdot,t)\Big|_{\partial E} = g(\cdot,t) \quad \text{ a.e. } \ t \in (0,T], \\ u(\cdot,0) = g(x,0), \end{cases} \tag{8.2}$$

where $\mathbf{A}$ and B satisfy (1.2)–(1.3), and g is as in (8.1).

A weak sub(super)-solution of the Cauchy–Dirichlet problem (8.2) is a measurable function

$$u \in C\big(0,T;L^2(E)\big) \cap L^2\big(0,T;W^{1,2}(E)\big)$$

satisfying

$$\begin{aligned} \int_E u\varphi(x,T)\,dx + \iint_{E_T} \big[-u\varphi_t + \mathbf{A}(x,t,u,\nabla u)\cdot\nabla\varphi \\ - B(x,t,u,\nabla u)\varphi\big]\,dxdt \le (\ge) \int_E g\varphi(x,0)\,dx \end{aligned}$$

for all non-negative test functions $\varphi \in W_o^2(E_T)$.

In addition, we take the boundary condition $u \le g$ $(u \ge g)$ to mean that $(u-g)_+(\cdot,t) \in W_o^{1,2}(E)$ $((u-g)_-(\cdot,t) \in W_o^{1,2}(E))$ for a.e. $t \in (0,T]$. A function u, which is both a weak sub-solution and a weak super-solution, is a weak solution.

The formulation can be rephrased in terms of Steklov averages as in the previous Chapter, namely

$$\int_{E\times\{t\}} \big[u_{h,\tau}\varphi + [\mathbf{A}(x,\tau,u,\nabla u)]_h \cdot \nabla\varphi - [B(x,\tau,u,\nabla u)]_h\varphi\big]\,dx \le (\ge) 0, \tag{8.3}$$

$\forall\, 0 < t < T-h$ and $\forall\, \varphi \in W^{1,2}(E) \cap L^\infty(E)$, $\varphi \ge 0$.

Moreover, the initial datum is taken in the sense of $L^2(E)$, i.e.,

$$(u_h(\cdot,0) - g(\cdot,0))_{+(-)} \ \to \ 0 \ \text{ in } \ L^2(E).$$

Since $g \in C(\partial_p E_T)$, it is natural to ask whether a solution of the Dirichlet problem, whenever it exists, is continuous up the boundary $\partial_p E_T$. The issue can be rephrased by asking what requirements are needed on ∂E for the interior continuity of functions in the PDG classes to extend up to $\partial_p E_T$.

For simplicity, we distinguish the behavior at the lateral boundary and at $t = 0$.

8.1 Lateral Conditions

Assume that ∂E satisfies the property of *positive geometric density*, that is, there exist $\beta \in (0,1)$ and $R > 0$ such that for all $y \in \partial E$, and for all $0 < \rho \leq R$

$$\big|B_\rho(y) \cap (\mathbb{R}^N - E)\big| \geq \beta\big|B_\rho\big|. \tag{8.4}$$

Fix $(x_o,t_o) \in S_T$, and consider the cylinder $(x_o,t_o) + Q_\rho^-$ where ρ, $\theta > 0$ are so small that $t_o - \theta\rho^2 > 0$.

Consider a piecewise smooth, cutoff function ζ vanishing on $\partial B_\rho(x_o)$, and such that $0 \leq \zeta \leq 1$, and let u be a local sub(super)-solution of the Dirichlet problem associated with (1.1) for the given g. Local estimates for u near (x_o,t_o) similar to (1.8) are obtained by taking, in the weak formulation (8.3), the testing functions

$$\varphi_h^\pm = \pm(u_h - k)_\pm \zeta^2,$$

integrating over $[(x_o,t_o) + Q_\rho^-(\theta)]$ and letting $h \to 0$. Such a choice of testing functions is admissible if for a.e. $t \in (t_o - \theta\rho^2, t_o]$,

$$(u(\cdot,t) - k)_\pm \zeta^2(\cdot,t) \in W_o^{1,2}(B_\rho(x_o) \cap E). \tag{8.5}$$

Since $x \to \zeta(x,t)$ vanishes on the boundary of $B_\rho(x_o)$ and not on the boundary of $B_\rho(x_o) \cap E$, condition (8.5) will be verified if for a.e. $t \in (t_o - \theta\rho^2, t_o]$

$$(u-k)_\pm(\cdot,t) = 0 \quad \text{in the sense of the traces on} \quad \partial B_\rho(x_o) \cap E.$$

In view of Lemma 1.1 of Chapter 11, this can be realized for the function $(u-k)_+$ if k is chosen to satisfy

$$k \geq \sup_{[(x_o,t_o)+Q_\rho^-(\theta)]\cap S_T} g. \tag{8.6}$$

Analogously, the functions $-(u-k)_-\zeta^2$ can be taken as testing functions if

$$k \leq \inf_{[(x_o,t_o)+Q_\rho^-(\theta)]\cap S_T} g. \tag{8.7}$$

With these choices of k, we may repeat calculations in all analogous to those of Section 1, with the understanding that the various integrals are now extended over $[(x_o,t_o)+Q_\rho^-(\theta)]\cap E_T$. However, since $\zeta^2(u-k)_\pm(\cdot,t) \in W_o^{1,2}(B_\rho \cap E)$ for almost every $t \in (t_o - \theta\rho^2, t_o]$, we may regard them as elements of $W_o^{1,2}(B_\rho)$ by defining them to be zero outside E. Then, the same calculations lead to the inequalities (1.8), with the same stipulations that the various functions vanish outside E_T and the various integrals are extended over the full cylinder $(x_o,t_o) + Q_\rho^-(\theta)$.

8.2 Initial Conditions

Consider a weak solution of (8.2) that takes the initial datum $g(\cdot,0)$ in the sense that

$$\frac{1}{h}\int_0^h u(\cdot,\tau)\,d\tau \;\to\; g(\cdot,0) \quad \text{in} \quad L^2_{\mathrm{loc}}(E) \quad \text{as} \quad h\to 0. \tag{8.8}$$

We point out that, as far as the initial condition is concerned, u could also be a solution of the Cauchy–Neumann problem we are going to deal with in the next Section. In either case the assumption

$$g(\cdot,0) \quad \text{is continuous in } \; E \; \text{ with modulus of continuity, say } \quad \omega_o(\cdot),$$

in force.

Fix $(x_o,t_o)\in E_T$ and consider the cylinder $(x_o,t_o)+Q^-_\rho(\theta)$, where θ and ρ are such that $t_o-\theta\rho^2=0$. Therefore, $(x_o,t_o)+Q^-_\rho(\theta)$ lies on the bottom of the cylindrical domain E_T. Consider a piecewise smooth, cutoff function ζ vanishing on $\partial B_\rho(x_o)$, and such that $0\le\zeta\le 1$, and in addition ζ is independent of $t\in(0,t_o)$.

Local estimates similar to (1.8) for u near $t=0$ are derived by taking in the weak formulation (8.3) testing functions

$$\varphi_h^\pm = \pm(u_h-k)_\pm\zeta^2,$$

integrating over $(0,t)$, $t\in(0,t_o)$, and letting $h\to 0$. The first term in (8.3) gives

$$\frac{1}{2}\int_{B_\rho(x_o)}(u_h-k)^2_\pm(x,t)\zeta^2(x)\,dx-\frac{1}{2}\int_{B_\rho(x_o)}(u_h-k)^2_\pm(x,0)\zeta^2(x)\,dx.$$

If k is chosen so that $k\ge\sup_{B_\rho(x_o)}g(\cdot,0)$, then in view of (8.7) we have

$$\int_{B_\rho(x_o)}(u_h-k)^2_+(x,0)\zeta^2(x)\,dx\to 0 \;\text{ as }\; h\to 0.$$

Analogous considerations hold for $(u_h-k)^2_-\zeta^2$, provided $k\le\inf_{B_\rho(x_o)}g(\cdot,0)$. Therefore, summarizing, provided we choose

$$\begin{cases} k\ge\displaystyle\sup_{B_\rho(x_o)}g(\cdot,0) \;\text{ for the function }\; (u-k)_+\\[2ex] k\le\displaystyle\inf_{B_\rho(x_o)}g(\cdot,0) \;\text{ for the function }\; (u-k)_-,\end{cases} \tag{8.9}$$

and take θ and ρ such that $t_o-\theta\rho^2=0$, we have

$$\begin{aligned} \operatorname*{ess\,sup}_{t_o-\theta\rho^2<t<t_o} \int_{B_\rho(x_o)} & (u-k)_\pm^2(x,t)\zeta^2(x)dx \\ &+ \iint_{(x_o,t_o)+Q_\rho^-(\theta)} |\nabla[(u-k)_\pm\zeta]|^2 dx\,d\tau \\ \le & \gamma \iint_{(x_o,t_o)+Q_\rho^-(\theta)} (u-k)_\pm^2 |\nabla\zeta|^2\, dx\,d\tau \\ &+ \gamma C_2^2 \iint_{(x_o,t_o)+Q_\rho^-(\theta)} (u-k)_\pm^2 \zeta^2 dx\,d\tau \\ &+ \gamma_*^2 |A_{k,\rho}^\pm|^{1-\frac{2}{N+2}+2\delta}. \end{aligned} \tag{8.10}$$

8.3 Definition of Boundary Parabolic DeGiorgi Classes

Taking into account Section 8.1 and Section 8.2, given $g \in C(\overline{E}_T)$, the boundary parabolic DeGiorgi classes $\mathrm{PDG}_g^\pm = \mathrm{PDG}_g^\pm(E_T, \bar\gamma, \gamma_*, C_2, \delta)$ are the collection of all

$$u \in C\big(0,T; L^2(E)\big) \cap L^2\big(0,T; W^{1,2}(E)\big) \cap L^\infty(E_T),$$

which satisfy these two conditions:

- For all $(x_o,t_o) \in \partial_p E_T$ and all cylinders $(x_o,t_o)+Q_\rho^-(\theta)$ such that $t_o - \theta\rho^2 > 0$ the localized truncations $(u-k)_\pm$ verify (1.8) for all levels k subject to the restrictions (8.6)–(8.7);
- For all $(x_o,t_o) \in E_T$ and all cylinders $(x_o,t_o)+Q_\rho^-(\theta)$ such that $t_o-\theta\rho^2 = 0$ the localized truncations $(u-k)_\pm$ satisfy (8.10) for all levels k subject to the restrictions (8.9).

We further define $\mathrm{PDG}_g = \mathrm{PDG}_g^+ \cap \mathrm{PDG}_g^-$ and refer to these classes as homogeneous if $\gamma_* = 0$.

8.4 Continuity up to $\partial_p E_T$ of Functions in the Boundary PDG Classes (Dirichlet Data)

Theorem 8.1. *Let u be a bounded weak solution of the Cauchy–Dirichlet problem* (8.1)–(8.2). *The boundary ∂E is assumed to satisfy the property of positive geometric density* (8.4). *Then, $u \in C(\overline{E}_T)$, and there exists a continuous, positive, nondecreasing function $s \to \omega(s) : \mathbb{R}_+ \to \mathbb{R}_+$, such that*

$$|u(x_1,t_1) - u(x_2,t_2)| \le \omega\left(|x_1-x_2| + |t_1-t_2|^{\frac12}\right),$$

for every pair of points (x_1,t_1), $(x_2,t_2) \in \overline{E}_T$. In particular, if g is Hölder continuous in S_T with exponent α_{S_T}, and if g is Hölder continuous at $t=0$ in $\overline{E}$ with exponent α_o, then u is Hölder continuous in $\overline{E}_T$, and there exist constants $\gamma > 1$ and $\alpha \in (0,1)$ such that

$$|u(x_1,t_1)-u(x_2,t_2)|\le\gamma\|u\|_{\infty;E_T}\left(|x_1-x_2|+|t_1-t_2|^{\frac{1}{2}}\right)^{\alpha}$$

for every pair of points (x_1,t_1), $(x_2,t_2)\in E_T$. *The constants* γ *and* α *depend only upon the data. Moreover, the constant* α *depends also upon the Hölder exponents* α_{S_T} *and* α_o.

Even though the arguments are fairly similar, nevertheless we will separately discuss the continuity up to the lateral boundary, and the continuity up to $t=0$. In the following, we will always assume that ρ is such that $\rho<C_2^{-1}$.

Let R be the parameter in the condition of positive geometric density (8.4). For $(x_o,t_o)\in S_T$ and for $\theta>0$, $\rho\in(0,R)$ such that $t_o-4\theta\rho^2>0$, consider nested cylinders $(x_o,t_o)+Q_\rho^-(\theta)\subset(x_o,t_o)+Q_{2\rho}^-(\theta)$ and set

$$\begin{gathered}\mu^+=\operatorname*{ess\,sup}_{[(x_o,t_o)+Q_{2\rho}^-(\theta)]\cap E_T}u,\quad \mu^-=\operatorname*{ess\,inf}_{[(x_o,t_o)+Q_{2\rho}^-(\theta)]\cap E_T}u,\\ \omega(2\rho)=\mu^+-\mu^-=\operatorname*{ess\,osc}_{[(x_o,t_o)+Q_{2\rho}^-(\theta)]\cap E_T}u.\end{gathered}\tag{8.11}$$

Moreover, let

$$\begin{gathered}g^+(2\rho)=\operatorname*{ess\,sup}_{[(x_o,t_o)+Q_{2\rho}^-(\theta)]\cap S_T}g,\quad g^-(2\rho)=\operatorname*{ess\,inf}_{[(x_o,t_o)+Q_{2\rho}^-(\theta)]\cap S_T}g,\\ \omega_g(2\rho)=g^+(2\rho)-g^-(2\rho)=\operatorname*{ess\,osc}_{[(x_o,t_o)+Q_{2\rho}^-(\theta)]\cap S_T}g.\end{gathered}$$

As far as the behavior at the lateral boundary is concerned, Theorem 8.1 is a straightforward consequence of the following.

Proposition 8.1 *Let* ∂E *satisfy the positive geometric density condition (8.4), and let* $g\in C(\overline{E}_T)$. *Then, every* $u\in PDG_g$ *is continuous up to* S_T, *and there exist constants* $C>1$ *and* $\alpha\in(0,1)$, *depending only on the data defining the* PDG_g *classes and the parameter* β *in (8.4), and independent of* g *and* u, *such that for all* $(x_o,t_o)\in S_T$ *and all cylinders* $(x_o,t_o)+Q_{2\rho}^-(\theta)\subset(x_o,t_o)+Q_R^-(\theta)$ *with* $\theta>0$ *and* $\rho\in(0,R)$ *such that* $t_o-4\theta\rho^2>0$

$$\omega(\rho)\le C\max\left\{\omega(R)\left(\frac{\rho}{R}\right)^{\alpha};\,\omega_g(2\rho)\,;\,\gamma_*\rho^{\delta(N+2)}\right\}.\tag{8.12}$$

Notice that the boundedness of u is already taken into account in the definition of PDG$_g$ classes.

As usual, we assume $(x_o,t_o)=(0,0)$. The proof of this proposition is almost identical to that of the interior Hölder continuity, except for a few changes, which we outline next. First, Proposition 4.1 and its proof continue to hold, provided the levels $\xi\omega$ satisfy (8.6)–(8.7). Next, Proposition 5.1 and its proof continue to be in force, provided the levels k_j in (5.1) satisfy the restrictions (8.6)–(8.7) for all $j \ge 1$. Finally, Proposition 6.1 and its proof continue to hold, provided the levels ℓ satisfy the restrictions (8.6)–(8.7).

Now either one of the inequalities

$$\mu^+ - \tfrac{1}{4}\omega \ge g^+, \qquad \mu^- + \tfrac{1}{4}\omega \le g^-$$

must be satisfied. Indeed, if both are violated, that is

$$\mu^+ - \tfrac{1}{4}\omega \le g^+ \quad \text{and} \quad -\mu^- - \tfrac{1}{4}\omega \le -g^-,$$

adding these inequalities gives

$$\omega(\rho) \le 2\omega_g(2\rho)$$

and there is nothing to prove. Assuming the second holds, then all levels k_s as defined in (5.1) for $\epsilon = \frac{1}{2}$ satisfy restriction (8.7) and thus are admissible. Moreover, $(u - k_1)_-$ vanishes outside E_T, and therefore

$$\big|[u \le \mu^- + \tfrac{1}{4}\omega] \cap Q_\rho^-\big| \ge \beta |Q_\rho^-|$$

where β is the parameter in the positive geometric density condition (8.1). From this, the procedure of the proof of Theorem 3.1 can be repeated with the understanding that $(u - k_s)_-$ are defined in the full cylinder Q_ρ^- and are zero outside E_T. ■

Let us now consider the behavior at $t = 0$. Fix $(x_o, 0) \in E \times \{0\}$, and $\rho > 0$ so that $B_\rho(x_o) \subset E$. After a translation, without loss of generality, we may assume $x_o = 0$.

The proof of the continuity (or of the Hölder continuity) of u up to $t = 0$ follows from a simple variant of Theorem 3.1 and Proposition 8.1. Here, we briefly sketch how to proceed. Set

$$\mu_o^+ = \operatorname*{ess\,sup}_{B_\rho} g(\cdot, 0), \qquad \mu_o^- = \operatorname*{ess\,inf}_{B_\rho} g(\cdot, 0), \qquad \omega_o(\rho) = \operatorname*{ess\,osc}_{B_\rho} g(\cdot, 0),$$

$$\mu^+ = \operatorname*{ess\,sup}_{Q_\rho^+} u, \qquad \mu^- = \operatorname*{ess\,inf}_{Q_\rho^-} u, \qquad \omega(\rho) = \mu^+ - \mu^- = \operatorname*{ess\,osc}_{Q_\rho^+} u,$$

and consider the two inequalities

$$\mu^+ - \tfrac{1}{4}\omega < \mu_o^+, \qquad \mu^- + \tfrac{1}{4}\omega > \mu_o^-.$$

If both hold, subtracting from one another, we obtain

$$\operatorname*{ess\,osc}_{Q_\rho^+} u \le 2 \operatorname*{ess\,osc}_{B_\rho} g(\cdot, 0)$$

and there is nothing to prove. Let us assume, without loss of generality, that the second one is violated, namely that

$$\mu^- + \tfrac{1}{4}\omega \le \mu_o^- .$$

Then, for all $j \ge 2$, the levels $k = \mu^- + \dfrac{\omega}{2^j}$ satisfy the second of (8.9). Therefore, we may derive estimates for the truncated functions $(u-k)_-$ as in (8.10) with $\theta = 1$, which take the form

$$\begin{aligned}
\operatorname*{ess\,sup}_{0<t<\rho^2} \int_{B_\rho} &(u-k)_-^2(x,t)\zeta^2(x)dx \\
&+ \iint_{Q_\rho^+} |\nabla[(u-k)_-\zeta]|^2 dx\, d\tau \\
\le& \gamma \iint_{Q_\rho^+} (u-k)_-^2 |\nabla\zeta|^2\, dx\, d\tau \\
&+ \gamma C_2^2 \iint_{Q_\rho^+} (u-k)_-^2 \zeta^2 dx\, d\tau + \gamma_*^2 |A_{k,2\rho}^-|^{1-\frac{2}{N+2}+2\delta}.
\end{aligned}$$

They are phrased in terms of cylinders Q_ρ^+, but are obviously equivalent to (8.10).

Since $u(\cdot,0) > \mu^- + \frac{\omega}{4}$ in B_ρ, we can apply Proposition 6.1 and conclude that either $\omega \le 2\gamma_*\rho^{\delta(N+2)}$, or

$$|[u(\cdot,t) > \mu^- + \xi\omega] \cap B_\rho| \ge \frac{1}{4}|B_\rho|, \qquad \forall\, t \in [0, \theta\rho^2],$$

for proper ξ and θ. These two alternatives correspond to (7.3) and (7.4); from here on, we proceed as in the proof of Theorem 3.1 with the obvious adjustments, which are needed in order to deal with cylinders Q_ρ^+, and we conclude. ■

Remark 8.1 The arguments are local in nature and as such they require only local assumptions. For example, the positive geometric density condition (8.4) could be satisfied on only a portion of ∂E, open in the relative topology of ∂E, and for any $t \in (t_o - 4\theta\rho^2, t_o]$, $g(\cdot,t)$ could be continuous only on that portion of ∂E. Then, the boundary continuity of Theorem 8.1 continues to hold only locally, on that portion of S_T. Similar considerations hold for the continuity at $t = 0$.

9 Boundary Parabolic DeGiorgi Classes: Neumann Data

Assume that ∂E is of class $C^{1,\lambda}$, $\lambda \in (0,1)$ so that the outward unit normal, which we denote by $\mathbf{n}$, is everywhere defined on ∂E, and consider the quasi-linear Cauchy–Neumann problem

$$\begin{cases} u_t - \operatorname{div} \mathbf{A}(x,t,u,\nabla u) = B(x,t,u,\nabla u) & \text{in } E_T \\ \mathbf{A}(x,t,u,\nabla u)\cdot \mathbf{n} = \psi(x,t,u) & \text{a.e. } t \in (0,T] \\ u(\cdot,0) = u_o \end{cases} \tag{9.1}$$

where the functions $\mathbf{A}$ and B satisfy the structure conditions (1.2), and u_o is continuous in $\bar{E}$ with modulus of continuity ω_o; moreover, we assume that $\psi(\cdot, t, u(\cdot))$ admits for a.e. $t \in (0,T)$ an extension into E, which we continue to denote by $\psi(\cdot, t, u(\cdot))$, such that

$$\begin{cases} |\psi| \le \psi_o |u| + \psi_1, \\ |\psi_u| \le \psi_o, \\ |\psi_{x_i}| \le \psi_1 \quad \text{for } \ i = 1, 2, \dots, N, \end{cases} \tag{9.2}$$

where ψ_o and ψ_1 are given, non-negative functions satisfying

$$\psi_o,\, \psi_1 \in L^{N+2+\varepsilon}(E_T), \tag{9.3}$$

for some $\varepsilon > 0$. We are interested in the boundary behavior of solutions of this problem.

Weak solutions can be formulated by a straightforward extension of the definition in (9.3) of Chapter 11. However, let us concentrate on *local solutions*.

Let K be an arbitrary compact subset of $\mathbb{R}^N$. A weak sub(super)-solution of (9.1) is a measurable function

$$u \in C(0,T; L^2(E)) \cap L^2(0,T; W^{1,2}(E))$$

satisfying for every compact subset K of $\mathbb{R}^N$, for every subinterval $[t_1, t_2] \subset (0,T]$, and for all non-negative test functions

$$\varphi \in W^{1,2}(0,T; L^2(K)) \cap L^2(0,T; W_o^{1,2}(K)),$$

$$\begin{aligned} &\int_{K\cap E} u\varphi \Big|_{t_1}^{t_2} dx + \iint_{t_1}^{t_2} \int_{K\cap E} [-u\varphi_t + \mathbf{A}(x,t,u,\nabla u)\nabla\varphi \\ &-B(x,t,u,\nabla u)\varphi]\, dxdt \le (\ge) + \int_{t_1}^{t_2} \int_{K\cap\partial E} \psi\varphi \, d\sigma dt, \end{aligned} \tag{9.4}$$

where $d\sigma$ is the surface measure on ∂E.

All terms on the left-hand side are well defined by virtue of the structure conditions (1.2), whereas the boundary integral on the right-hand side is well defined by virtue of (9.2)

We point out that φ vanishes in the sense of traces on ∂K, and not on the boundary of E.

A function that is both a weak sub-solution and a weak super-solution is a weak solution.

The formulation can be rephrased in terms of Steklov averages, as in the previous Section, namely

$$\begin{aligned} \int_{(K\cap E)\times\{t\}} &[u_{h,\tau}\varphi + [\mathbf{A}(x,\tau,u,\nabla u)]_h \cdot \nabla\varphi - [B(x,\tau,u,\nabla u)]_h\varphi]\, dx \\ &\le (\ge) \int_{(K\cap\partial E)\times\{t\}} [\psi(x,\tau,u)]_h \varphi \, d\sigma \end{aligned} \tag{9.5}$$

for all $0 < t < T - h$, and for all $\varphi \in W_o^{1,2}(K)$ with $\varphi \geq 0$.

Moreover, the initial datum is taken in the sense of $L^2(E)$, i.e.,

$$(u_h(\cdot, t) - u_o)_{+(-)} \to 0 \quad \text{in } L^2(E).$$

As we did in the previous Section, also in this case it is quite natural to ask whether a solution of the Neumann problem, provided it exists, is continuous up to the boundary $\partial_p E_T$. In other terms, what do we need to do, for the interior continuity of a function in the PDG classes to be extended up to the boundary in the case of the Neumann Problem?

We concentrate on the behavior at the lateral boundary, since the case of the initial datum was dealt with in the previous Section.

9.1 Lateral Boundary

Fix $(x_o, t_o) \in S_T$, assume without loss of generality that it coincides with the origin; we let K_R denote the N-dimensional cube centered at the origin and wedge $2R$, i.e.,

$$K_R = \{x \in \mathbb{R}^N : \max_{1 \leq i \leq N} |x_i| > R\}.$$

In the following, mainly for simplicity, we will consider these cubes K_R, instead of balls B_R. Cylinders $Q_\rho^\pm(\theta)$ with the corresponding cubic cross-sections will also be used.

We introduce a local change of coordinates by which $\partial E \cap K_R$ for some fixed $R > 0$ coincides with the hyperplane $x_N = 0$, and E lies locally in $\{x_N > 0\}$. Setting

$$K_\rho^+ = K_\rho \cap [x_N > 0] \quad \text{for all } 0 < \rho \leq R$$

and

$$\tilde{Q}_\rho^\pm(\theta) = Q_\rho^\pm(\theta) \cap [x_N > 0]$$

we require that all "concentric" $\frac{1}{2}$-cubes $K_{\sigma\rho}^+ \subset K_\rho^+ \subset K_R^+$ be contained in E.

Without loss of generality, we can assume that (9.5) is written in such a coordinate system.

Let u be a local sub(super)-solution of (9.1) in the sense of (9.4), and in the latter take the test functions $v = \pm(u_h - k)_\pm \zeta^2$, where ζ is the usual cut-off function, integrate over $\tilde{Q}_\rho^-(\theta)$, and let $h \to 0$. Moreover, we assume u to belong to $L^\infty(E_T)$.

All the terms are treated as in the proof of (1.8), except for the boundary integral. We arrive at

$$\begin{aligned}
&\operatorname*{ess\,sup}_{-\theta\rho^2<t<0}\int_{K_\rho^+}(u-k)_\pm^2\zeta^2(x,t)dx-\int_{K_\rho^+}(u-k)_\pm^2\zeta^2(x,-\theta\rho^2)dx\\
&\quad+\frac{C_o}{2}\iint_{\tilde{Q}_\rho^-(\theta)}|\nabla[(u-k)_\pm\zeta]|^2dx\,d\tau\\
&\le\gamma\iint_{\tilde{Q}_\rho^-(\theta)}\left[(u-k)_\pm^2|\nabla\zeta|^2+(u-k)_\pm^2|\zeta_\tau|\right]dx\,d\tau\\
&\quad+\gamma C_2^2\iint_{\tilde{Q}_\rho^-(\theta)}(u-k)_\pm^2\zeta^2dx\,d\tau+\gamma_*^2|A_{k,\rho}^\pm|^{1-\frac{2}{N+2}+2\delta}\\
&\quad+\int_{-\theta\rho^2}^0\int_{\tilde{K}_\rho}\psi(u-k)_\pm\zeta^2\,d\bar{x}dt,
\end{aligned}\tag{9.6}$$

where $\tilde{K}_\rho=K_\rho\cap\partial E$ is the $(N-1)$-dimensional cube, i.e.,

$$\tilde{K}_\rho=\{\bar{x}\equiv(x_1,\dots,x_{N-1})\in\mathbb{R}^{N-1}:\ \max_{1\le i\le N-1}|x_i|<\rho\}.$$

We estimate the boundary integral by transforming it into an interior integral as follows.

$$\begin{aligned}
&\left|\int_{-\theta\rho^2}^0\int_{\tilde{K}_\rho}\psi(u-k)_\pm\zeta^2\,d\bar{x}dt\right|\\
=&\left|\int_{-\theta\rho^2}^0\int_{\tilde{K}_\rho}\left(\int_0^\rho\frac{\partial}{\partial x_N}\left(\psi(x,t,u)(u-k)_\pm\zeta^2\right)dx_N\right)d\bar{x}d\tau\right|\\
\le&\gamma\iint_{\tilde{Q}_\rho^-(\theta)}\left[|\psi_{x_N}|(u-k)_\pm\zeta^2+|\psi||\nabla[(u-k)_\pm\zeta]|\right]dxd\tau\\
&+\gamma\iint_{\tilde{Q}_\rho^-(\theta)}\left[|\psi|(u-k)_\pm\zeta|\nabla\zeta|+|\psi_u||\nabla(u-k)_\pm|(u-k)_\pm\zeta^2\right]dxd\tau
\end{aligned}$$

By virtue of (9.2)–(9.3), we have

$$\iint_{\tilde{Q}_\rho^-(\theta)}|\psi_{x_N}|(u-k)_\pm\zeta^2\,dxd\tau\le\|u\|_{\infty;E_T}\iint_{\tilde{Q}_\rho^-(\theta)}\psi_1\chi_{[(u-k)_\pm>0]}\,dxd\tau.$$

Moreover,

$$\begin{aligned}
&\iint_{\tilde{Q}_\rho^-(\theta)}|\psi|\left[|\nabla[(u-k)_\pm\zeta]|+|(u-k)_\pm\zeta|\nabla\zeta|\right]dxd\tau\\
&\quad\le\frac{C_o}{8}\iint_{\tilde{Q}_\rho^-(\theta)}|\nabla[(u-k)_\pm\zeta]|^2\,dxd\tau+\iint_{\tilde{Q}_\rho^-(\theta)}|\psi|^2\chi_{[(u-k)_\pm>0]}\,dxd\tau\\
&\qquad+\gamma\iint_{\tilde{Q}_\rho^-(\theta)}(u-k)_\pm^2|\nabla\zeta|^2\,dxd\tau+\gamma\iint_{\tilde{Q}_\rho^-(\theta)}\zeta^2|\psi|^2\chi_{[(u-k)_\pm>0]}\,dxd\tau.
\end{aligned}$$

By (9.2) it is apparent that

$$\begin{aligned}
\iint_{\tilde{Q}_\rho^-(\theta)} &|\psi|^2\chi_{[(u-k)_\pm>0]}\,dxd\tau \\
&\le \iint_{\tilde{Q}_\rho^-(\theta)} u^2\psi_o^2\chi_{[(u-k)_\pm>0]}\,dxd\tau + \iint_{\tilde{Q}_\rho^-(\theta)} \psi_1^2\chi_{[(u-k)_\pm>0]}\,dxd\tau \\
&\le \gamma(\text{data}, \|u\|_{\infty;E_T}) \iint_{\tilde{Q}_\rho^-(\theta)} (\psi_o^2+\psi_1^2)\chi_{[(u-k)_\pm>0]}\,dxd\tau \\
&\le \gamma(\text{data}, \|u\|_{\infty;E_T}) \|\psi_o+\psi_1\|_{N+2+\varepsilon;\tilde{Q}_\rho^-(\theta)} |A_{k,\rho}^\pm|^{1-\frac{2}{N+2}+2\delta},
\end{aligned}$$

with $\delta = \frac{\varepsilon}{(N+2)(N+2+\varepsilon)}$. Finally,

$$\begin{aligned}
\iint_{\tilde{Q}_\rho^-(\theta)} &|\psi_u||\nabla(u-k)_\pm|(u-k)_\pm\zeta^2\,dxd\tau \\
&\le \iint_{\tilde{Q}_\rho^-(\theta)} |\psi_u| \left\{|\nabla[(u-k)_\pm\zeta]|(u-k)_\pm\zeta + (u-k)_\pm^2\zeta|\nabla\zeta|\right\} dxd\tau \\
&\le \frac{C_o}{8} \iint_{\tilde{Q}_\rho^-(\theta)} |\nabla[(u-k)_\pm\zeta]|^2\,dxd\tau + \gamma \iint_{\tilde{Q}_\rho^-(\theta)} |\psi_u|^2(u-k)_\pm^2\zeta^2\,dxd\tau \\
&\quad + \gamma \int_{\tilde{Q}_\rho^-(\theta)} (u-k)_\pm^2|\nabla\zeta|^2\,dxd\tau + \gamma \int_{\tilde{Q}_\rho^-(\theta)} (u-k)_\pm^2\zeta^2\,dxd\tau,
\end{aligned}$$

and

$$\begin{aligned}
\iint_{\tilde{Q}_\rho^-(\theta)} &|\psi_u|^2(u-k)_\pm^2\zeta^2\,dxd\tau \\
&\le \gamma(\text{data}, \|u\|_{\infty;E_T}) \iint_{\tilde{Q}_\rho^-(\theta)} \zeta^2\psi_o^2\chi_{[(u-k)_\pm>0]}\,dxd\tau \\
&\le \gamma(\text{data}, \|u\|_{\infty;E_T}) \|\psi_o\|_{N+2+\varepsilon;\tilde{Q}_\rho^-(\theta)}^2 |A_{k,\rho}^\pm|^{1-\frac{2}{N+2}+2\delta}.
\end{aligned}$$

Collecting all the terms yields

$$\begin{aligned}
\operatorname*{ess\,sup}_{-\theta\rho^2<t<0} &\int_{K_\rho^+} (u-k)_\pm^2\zeta^2(x,t)dx - \int_{K_\rho^+} (u-k)_\pm^2\zeta^2(x,-\theta\rho^2)dx \\
&+ \iint_{\tilde{Q}_\rho^-(\theta)} |\nabla[(u-k)_\pm\zeta]|^2 dx\,d\tau \\
\le \gamma &\iint_{\tilde{Q}_\rho^-(\theta)} \left[(u-k)_\pm^2|\nabla\zeta|^2 + (u-k)_\pm^2|\zeta_\tau|\right] dx\,d\tau \\
&+ \gamma C_2^2 \iint_{\tilde{Q}_\rho^-(\theta)} (u-k)_\pm^2\zeta^2 dx\,d\tau + \gamma_{**}^2 |A_{k,\rho}^\pm|^{1-\frac{2}{N+2}+2\delta},
\end{aligned}$$

where the value of δ has been given before, and

$$\gamma_{**}^2 = \gamma_*^2 + \gamma(\text{data}, \|u\|_{\infty;E_T}) \|\psi_o+\psi_1\|_{N+2+\varepsilon;\tilde{Q}_\rho^-(\theta)}^2.$$

9.2 Definition of Boundary Parabolic DeGiorgi Classes

Given ψ as in (9.2)–(9.3), and taking into account that $u \in L^\infty(E_T)$, the boundary parabolic DeGiorgi classes

$$\mathrm{PDG}_\psi^\pm = \mathrm{PDG}_\psi^\pm(E_T, \bar\gamma, \gamma_{**}, C_2, \delta)$$

are the collection of all

$$u \in C\big(0,T;L^2(E)\big) \cap L^2\big(0,T;W^{1,2}(E)\big) \cap L^\infty(E_T)$$

such that for all $(x_o,t_o) \in S_T$, all $\frac{1}{2}$-cubes $K_\rho^+(x_o) \subset K_R^+(x_o)$ for $\rho < R$, and all cylinders $(x_o,t_o) + \tilde Q_\rho^-(\theta)$ such that $t_o - \theta\rho^2 > 0$, the localized truncations $(u-k)_\pm$ satisfy

$$\begin{aligned}
&\operatorname*{ess\,sup}_{t_o-\theta\rho^2<t<t_o} \int_{K_\rho^+(x_o)} (u-k)_\pm^2 \zeta^2(x,t)dx \\
&\quad - \int_{K_\rho^+(x_o)} (u-k)_\pm^2 \zeta^2(x,-\theta\rho^2)dx \\
&\quad + \iint_{(x_o,t_o)+\tilde Q_\rho^-(\theta)} |\nabla[(u-k)_\pm\zeta]|^2 dx\,d\tau \qquad\qquad (9.7)\\
&\le \bar\gamma \iint_{(x_o,t_o)+\tilde Q_\rho^-(\theta)} \big[(u-k)_\pm^2|\nabla\zeta|^2 + (u-k)_\pm^2|\zeta_\tau|\big]\,dx\,d\tau \\
&\quad + \bar\gamma C_2^2 \iint_{(x_o,t_o)+\tilde Q_\rho^-(\theta)} (u-k)_\pm^2\zeta^2 dx\,d\tau + \gamma_{**}^2 |A_{k,\rho}^\pm|^{1-\frac{2}{N+2}+2\delta}.
\end{aligned}$$

We further define $\mathrm{PDG}_\psi = \mathrm{PDG}_\psi^+ \cap \mathrm{PDG}_\psi^-$ and refer to these classes as homogeneous if $\gamma_{**} = 0$.

9.3 Continuity up to S_T of Functions in the Boundary PDG Classes (Neumann Data)

Having fixed $(x_o,t_o) \in S_T$, let

$$K_R(x_o) = \{x \in \mathbb{R}^N : \max_{1\le i\le N} |x_i - x_{i,o}| < R\},$$

assume after a flattening of ∂E about x_o that ∂E coincides with the hyperplane $x_N = 0$ within the cube $K_R(x_o)$ for some $R > 0$. For $\theta > 0$ assume that $t_o - 4\theta\rho^2 > 0$, consider the "concentric" cylinders $(x_o,t_o) + \tilde Q_\rho^-(\theta) \subset (x_o,t_o) + \tilde Q_{2\rho}^-(\theta) \subset E_T$ and set

$$\begin{aligned}
&\mu^+ = \operatorname*{ess\,sup}_{(x_o,t_o)+\tilde Q_{2\rho}^-(\theta)} u, \qquad \mu^- = \operatorname*{ess\,inf}_{(x_o,t_o)+\tilde Q_{2\rho}^-(\theta)} u, \\
&\omega(2\rho) = \mu^+ - \mu^- = \operatorname*{ess\,osc}_{(x_o,t_o)+\tilde Q_{2\rho}^-(\theta)} u.
\end{aligned} \qquad (9.8)$$

Theorem 9.1. *Let ∂E be of class $C^{1,\lambda}$, with $\lambda \in (0,1)$. Then, every $u \in PDG_\psi$ is Hölder continuous up to S_T, and there exist constants $C > 1$ and $\alpha \in (0,1)$, depending only on the data defining the PDG_ψ classes and the $C^{1,\lambda}$ structure of ∂E, such that for all $(x_o,t_o) \in S_T$ and all "concentric" cylinders $(x_o,t_o) + \tilde{Q}_\rho^-(\theta) \subset (x_o,t_o) + \tilde{Q}_{2\rho}^-(\theta) \subset E_T$*

$$\omega(\rho) \le C \max\left\{\omega(R)\left(\frac{\rho}{R}\right)^\alpha ; \gamma_{**}\rho^{N\delta}\right\}. \tag{9.9}$$

The proof of this theorem is almost identical to that of the interior Hölder continuity, the only difference being that we are working with "concentric" $\frac{1}{2}$-cubes and corresponding cylinders, instead of full balls and related cylinders. Proposition 4.1 and its proof continue to hold. Since $(u-k)_\pm\zeta(\cdot,t)$ do not vanish on ∂K_ρ^+, the parabolic embedding deduced from Theorem 2.1 of Chapter 9 is used, instead of the usual multiplicative embedding. Next, Proposition 5.1 relies on the discrete isoperimetric inequality of Proposition 5.2 of Chapter 10, which holds for convex domains, and thus in particular for $\frac{1}{2}$-cubes. The rest of the proof is identical to the indicated change in the use of the embedding inequalities. ∎

Remark 9.1 The regularity of ψ enters only in the requirements (9.2)–(9.3) through the constant γ_{**}.

Remark 9.2 The arguments are local in nature, and as such they require only local assumptions.

Taking into account Theorem 9.1, we conclude.

Corollary 9.1 *Let ∂E be of class $C^{1,\lambda}$, with $\lambda \in (0,1)$. A bounded, weak solution u of the Neumann problem* (9.1) *for a datum ψ satisfying* (9.2)–(9.3)*, is Hölder continuous in $\bar{E} \times (0,T)$. Analogous local statements are in force, if the assumptions on ∂E and ψ hold on portions of S_T.*

Remark 9.3 As already remarked above, the continuity of u can be claimed up to $t = 0$, provided that u_o in (9.1) is continuous in $\bar{E}$ with modulus of continuity ω_o.

10 The Harnack Inequality

We have already seen the Harnack inequality in the context of non-negative solutions of the heat equation (Section 13 of Chapter 5). After Pini's and Hadamard's results, it was shown to hold for non-negative solutions first of linear parabolic equations with bounded and measurable coefficients, and then of quasi-linear parabolic equations of the type of (1.1). It is quite remarkable that it continues to hold for non-negative functions in the PDG classes.

Theorem 10.1. [48, 187, 189, 262] *Let $u \ge 0$ be an element of the parabolic DeGiorgi class $PDG(E_T, \bar{\gamma}, \gamma_*, C_2, \delta)$. There exist positive constants c_* and θ_* that can be quantitatively determined a priori in terms of only the parameters N, γ and independent of u, γ_*, and δ such that for every radius $\rho < C_2^{-1}$, provided that $B_{4\rho}(x_o) \times (t_o - \theta_*(4\rho)^2, t_o + \theta_*(4\rho)^2) \subset E_T$, either $u(x_o, t_o) \le c_*^{-1}\gamma_*\rho^{\delta(N+2)}$ or*

$$c_*\, u(x_o, t_o) \le \inf_{B_\rho(x_o)} u(\cdot, t_o + \theta_*\rho^2). \tag{10.1}$$

The first proof of Theorem 10.1 that relies only on the structure of the classes is in DiBenedetto [48]. This is the proof presented here, in view of its relative flexibility, with one fundamental difference: in the spirit of what we did in the elliptic context in Chapter 10, we do not rely on the Hölder continuity of solution.

10.1 Proof of Theorem 10.1. Preliminaries

Fix $(x_o, t_o) \in E_T$, assume that $u(x_o, t_o) > 0$, and construct the cylinders $(x_o, t_o) + Q^\pm_{8\rho} \subset E_T$. The change of variables

$$x \to \frac{x - x_o}{\rho} \qquad t \to \frac{t - t_o}{\rho^2}$$

maps these cylinders into $Q^\pm$, where

$$Q^+ = B_8 \times (0, 8^2], \qquad Q^- = B_8 \times (-8^2, 0].$$

Denoting again by (x, t) the transformed variables, the rescaled function

$$w(x,t) = \frac{1}{u(x_o, t_o)} u\Big(x_o + \rho x, t_o + t\rho^2\Big)$$

satisfies $w(0,0) = 1$ and is a bounded, non-negative, element of the PDG classes relative to the cylinders $Q^\pm$, with the same parameters as the original PDG classes, except that γ_* is now replaced by

$$\Gamma_* = (2\rho)^{\delta(N+2)} \frac{\gamma_*}{u(x_o, t_o)}. \tag{10.2}$$

In particular, the truncations $(w - k)_\pm$ satisfy

$$\begin{aligned}
&\operatorname*{ess\,sup}_{s-\theta r^2 < t < s} \int_{B_r(y)} (w-k)_\pm^2 \zeta^2(x,t)dx - \int_{B_r(y)} (w-k)_\pm^2 \zeta^2(x, s - \theta r^2)dx \\
&\quad + \iint_{(y,s)+Q_r^-(\theta)} |\nabla[(w-k)_\pm \zeta]|^2 dx\, d\tau \\
&\le \bar{\gamma} \iint_{(y,s)+Q_r^-(\theta)} \big[(w-k)_\pm^2 |\nabla\zeta|^2 + (w-k)_\pm^2 |\zeta_\tau|\big]\, dx\, d\tau \\
&\quad + \bar{\gamma} C_2^2 \iint_{(y,s)+Q_r^-(\theta)} (w-k)_\pm^2 \zeta^2 dx\, d\tau + \Gamma_*^2 |A^\pm_{k,r}|^{1 - \frac{2}{N+2} + 2\delta},
\end{aligned} \tag{10.3}$$

for all $(y,s)+Q_r^-(\theta)\subset Q^\pm$ and for all $k>0$. Analogous estimates hold for "forward" cylinders $(y,s)+Q_r^+(\theta)\subset Q^\pm$. By these transformations, proving (10.1) reduces to finding positive constant γ_o, γ_1, γ_2 that can be determined a priori only in terms of the parameters of the original PDG classes, such that either $u(x_o,t_o)<\gamma_2\gamma_*\rho^{\delta(N+2)}$ or

$$\inf_{B_1} w(\cdot,\gamma_1)\geq\gamma_o. \tag{10.4}$$

10.2 Proof of Theorem 10.1. Expansion of Positivity

Let w be a non-negative element of $\mathrm{PDG}^-(Q^\pm,\bar\gamma,\gamma_*,C_2,\delta)$.

Proposition 10.1 *Assume that for some $(y,s)\in Q^\pm$ and some $r\in(0,C_2^{-1})$*

$$|[w(\cdot,s)\geq M]\cap B_r(y)|\geq\alpha\big|B_r(y)\big| \tag{10.5}$$

for some $M>0$ and some $\alpha\in(0,1)$. Then there exist constants η and $\theta\in(0,1)$ depending only on the data $\{N,C_o,C_1\}$, and α, such that either $\eta M\leq\Gamma_* r^{\delta(N+2)}$ or

$$w\geq\eta M\qquad a.e.\ in\ \ B_{2r}(y)\times\big(s+\tfrac{1}{2}\theta r^2\,,\,s+\theta r^2\big]. \tag{10.6}$$

Proof. Without loss of generality, we can assume $(y,s)=(0,0)$. We prove the statement in Q^+, but it can be easily extended to other frameworks, with minor adjustments. Since w is non-negative, we can directly assume that

$$\mu^-=\inf_{Q_{2r}^+} w=0.$$

The assumption (10.5) can then be read as assumption $(6.1)_-$ in Proposition 6.1. If we repeat its proof, we can conclude that there exist θ and ϵ in $(0,1)$, depending only on the data $\{N,C_o,C_1\}$, and α, and independent of M, such that either

$$M\leq\Gamma^* r^{\delta(N+2)}$$

or

$$|[w(\cdot,t)>\epsilon M]\cap B_r|\geq\tfrac{1}{2}\alpha|B_r|\quad\text{ for all }\ t\in(0,\theta r^2]. \tag{10.7}$$

The latter conclusion implies that

$$|[w(\cdot,t)>\epsilon M]\cap B_{4r}|>\tfrac{1}{2}\alpha 4^{-N}|B_{4r}|,\quad\text{ for all }\ t\in(0,\theta r^2]. \tag{10.8}$$

This represents $(5.1)_-$ with $\mu^-=0$ and $\epsilon\omega$ substituted by ϵM. Apply Proposition 4.1 over the cylinder $B_{4r}\times(0,\theta r^2]$ with $\mu_-=0$, $\xi\omega=\epsilon_\nu M$, and $a=\frac{1}{2}$, where ϵ_ν is the quantity claimed in Proposition 5.1. Choose ν from (4.1) and observe that the number ν is independent of $\epsilon_\nu M$. It only depends on the data $\{N,C_o,C_1\}$ and θ, which itself has been determined and fixed in terms of the data $\{N,C_o,C_1\}$ and α. Such a ν being fixed a priori only in terms of the data, choose $j_* \in \mathbb{N}$ as in Proposition 5.1. Then, Proposition 4.1 implies that

$$w(x,t) > \tfrac{1}{2}\epsilon_\nu M \quad \text{a.e. in } B_{2r} \times \left(\tfrac{1}{2}\theta r^2, \theta r^2\right].$$

Thus, the conclusion holds with $\eta = \frac{1}{2}\epsilon_\nu$. ■

Remark 10.1 Proposition 4.1 is a "shrinking" proposition, in that information on a cylinder Q_r^-, yields information on a smaller cylinder $Q_{\frac{1}{2}r}^-$. Proposition 10.1 is an "expanding" proposition, in the sense that information on a ball $B_r(y)$ at time s yields information on a larger ball $B_{2r}(y)$ for all times in the interval $(s + \frac{1}{2}\theta r^2, s + \theta r^2]$. Moreover, a measure-theoretical information is converted into a pointwise information. This "expansion of positivity" is at the heart of the Harnack inequality (10.1).

10.3 Proof of Theorem 10.1

In the light of Theorem 3.1, we may assume that w is continuous. The only way we are using this information, is to give unique meaning to pointwise values of w.

For $\tau \in [0,1)$, introduce the family of nested cylinders $\{Q_\tau^-\}$ with the same "vertex" at $(0,0)$, and the families of non-negative numbers $\{M_\tau\}$ and $\{N_\tau\}$, defined by

$$Q_\tau^- = K_\tau \times (-\tau^2, 0], \qquad M_\tau = \sup_{Q_\tau^-} w, \qquad N_\tau = (1-\tau)^{-\beta},$$

where $\beta > 1$ is to be chosen. The two functions $[0,1) \ni \tau \to M_\tau, N_\tau$ are increasing, and $M_o = N_o = 1$ since $w(0,0) = 1$. Moreover, $N_\tau \to \infty$ as $\tau \to 1$, whereas M_τ is bounded, since w is locally bounded. Therefore, the equation $M_\tau = N_\tau$ has roots and we let τ_* denote the largest one. By the continuity of w, there exists $(y,s) \in \bar{Q}_{\tau_*}$ such that

$$w(y,s) = M_{\tau_*} = N_{\tau_*} = (1-\tau_*)^{-\beta} \overset{\text{def}}{=} M. \tag{10.9}$$

Moreover,

$$(y,s) + Q_r^- \subset Q_{\frac{1+\tau_*}{2}}^- \subset Q_1, \qquad \text{where } r \overset{\text{def}}{=} \tfrac{1}{2}(1-\tau_*). \tag{10.10}$$

Therefore, by the definition of M_τ and N_τ

$$\sup_{(y,s)+Q_r^-} w \le \sup_{Q_{\frac{1+\tau_*}{2}}^-} w \le 2^\beta (1-\tau_*)^{-\beta} \overset{\text{def}}{=} M_*.$$

The parameter τ_*, and hence the upper bound M_*, is only known qualitatively, and β has to be chosen. The arguments below have the role of eliminating the qualitative knowledge of τ_* by a quantitative choice of β.

10.3.1 Local Largeness of w Near (y, s)

The largeness of w at (y, s) as expressed by (10.9), propagates to a full space-time neighborhood nearby (y, s). To render this quantitative, set

$$\xi = 1 - \frac{1}{2^{\beta+1}}, \qquad a = \frac{1 - \dfrac{3}{2}\dfrac{1}{2^{\beta+1}}}{1 - \dfrac{1}{2^{\beta+1}}}.$$

Lemma 10.1 *Either $\Gamma_* \ge 1$, or*

$$|[w > \tfrac{1}{2}M] \cap [(y, s) + Q_r^-]| > \nu |Q_r^-|, \tag{10.11}$$

where

$$\nu = \Big(\frac{1-a}{\gamma(\text{data})}\Big)^{N+2} \frac{1}{2^{\frac{N+2}{2}}}.$$

Proof. Assume that $\Gamma_* < 1$. If (10.11) is violated, apply Proposition 4.1 over the cylinder

$$(y, s) + Q_r^- = B_r(y) \times (s - r^2, s]$$

in the form $(4.1)_+$–$(4.2)_+$, for the choices $\mu^+ = \omega = M_*$ and $\theta = 1$, to conclude that

$$w(y, s) \le M_*(1 - a\xi) = \tfrac{3}{4}(1 - \tau_*)^{-\beta},$$

contradicting (10.9). ■

Remark 10.2 The indicated expressions of ξ, a, and ν imply that $\nu(\beta)$ depends on the data and β, but is independent of τ_*. Such a constant will be made quantitative whence β is chosen, dependent only on the data. We continue to denote by ν such a constant, keeping in mind its dependence on β.

Corollary 10.1 *Either $\Gamma_* \ge 1$, or there exists a time level*

$$s - r^2 \le \bar{s} \le s$$

such that

$$|[w(\cdot, \bar{s}) > \tfrac{1}{2}M] \cap K_r(y)| > \nu |K_r|. \tag{10.12}$$

10.3.2 Expanding the Positivity of w

Starting from (10.12) apply the expansion of positivity of Proposition 10.1 to w with $\frac{1}{2}M$ and r given by (10.9)–(10.10) and $\alpha = \nu$. Then, taking into account the expression (10.2) of Γ_*, either

$$u(x_o, t_o) \le \bar{\gamma}_* \, \gamma_* \rho^{\delta(N+2)} \frac{r^{\delta(N+2)}}{M} \tag*{$(10.13)_*$}$$

or

$$w(\cdot,t) \geq \eta_* M \qquad \text{in } \ B_{2r}(y) \tag{10.14$_*$}$$

for all t in the range

$$\bar{s} + \tfrac{1}{2}\theta_* r^2 \leq t \leq \bar{s} + \theta_* r^2 \ = \ s_*. \tag{10.15$_*$}$$

Remark 10.3 The constants $\{\bar{\gamma}_*, \theta_*, \eta_*\}$ in (10.13)$_*$–(10.15)$_*$ depend on the data $\{N, C_o, C_1\}$ and β, through the constant $\nu(\beta)$ in (10.12). However, they are independent of the constant Γ_* in (10.2). These constants are also independent of M and r. The parameter β is still to be chosen.

The expansion of positivity implies in particular

$$|[w(\cdot, s_*) > \eta_* M] \cap B_{2r}(y)| = |B_{2r}|. \tag{10.16}$$

Therefore, the expansion of positivity of Proposition 10.1 can be applied again, starting at the time level s_*, with M replaced by $(\eta_* M)$, $\rho = 2r$, and $\alpha = 1$. It gives that either

$$u(x_o, t_o) \leq \bar{\gamma}\gamma_* \rho^{\delta(N+2)} \frac{(2r)^{\delta(N+2)}}{\eta_* M} \tag{10.13$_1$}$$

or

$$w(\cdot,t) \geq \eta(\eta_* M) \qquad \text{in } \ B_{4r}(y) \tag{10.14$_1$}$$

for all t in the range

$$s_* + \tfrac{1}{2}\theta(2r)^2 \leq t \leq s_* + \theta(2r)^2 \ = \ s_1. \tag{10.15$_1$}$$

Remark 10.4 The constants $\{\bar{\gamma}, \theta, \eta\}$ in (10.13)$_1$–(10.15)$_1$ are different from the set of constants $\{\bar{\gamma}_*, \theta_*, \eta_*\}$ in (10.13)$_*$–(10.15)$_*$. They depend on the data $\{N, C_o, C_1\}$ but they are no longer dependent on β. By the expansion of positivity of Proposition 10.1 these parameters depend only on $\{N, C_o, C_1\}$, and the measure-theoretical lower bound α. Such a measure-theoretical lower bound in the current context is $\alpha = 1$, as provided by (10.16). The parameter β is still to be chosen.

Starting from (10.14)$_1$, the expansion of positivity can now be applied again with M replaced by $\eta(\eta_* M)$, and ρ replaced by $4r$, and $\alpha = 1$ to yield that either

$$u(x_o, t_o) \leq \bar{\gamma}\gamma_* \rho^{\delta(N+2)} \frac{(4r)^{\delta(N+2)}}{\eta(\eta_* M)} \tag{10.13$_2$}$$

or

$$w(\cdot,t) \geq \eta^2(\eta_* M) \qquad \text{in } \ B_{8r}(y) \tag{10.14$_2$}$$

for all t in the range

$$s_1 + \tfrac{1}{2}\theta(4r)^2 \leq t \leq s_1 + \theta(4r)^2 \ = \ s_2 \tag{10.15$_2$}$$

for the same set of parameters $\{\bar{\gamma},\theta,\eta\}$ as in $(10.13)_1$–$(10.15)_1$. These parameters depend on $\{N,C_o,C_1\}$ but they are independent of β.

The process can be iterated to yield that either

$$u(x_o,t_o)\le\bar{\gamma}\gamma_*\rho^{\delta(N+2)}\frac{(2^nr)^{\delta(N+2)}}{\eta^{n-1}(\eta_*M)} \tag{10.13$_n$}$$

or

$$w(\cdot,t)\ge\eta^n(\eta_*M)\quad\text{ in }\ B_{2^{n+1}r}(y) \tag{10.14$_n$}$$

for all t in the range

$$s_{n-1}+\tfrac{1}{2}\theta(2^nr)^2\le t\le s_{n-1}+\theta(2^nr)^2\ =\ s_n. \tag{10.15$_n$}$$

10.3.3 Proof of Theorem 10.1 Concluded

Without loss of generality we may assume that $(1-\tau_*)$ is a negative, integral power of 2. Then, choosing n so that $2^{n+1}r=2$, the ball $B_2(y)$ covers the ball B_1 centered at $x=0$, and

$$w(\cdot,t)\ge\eta^n(\eta_*M)\quad\text{ in }\ B_1,$$

for all t in the interval $(10.15)_n$. For the indicated choice of n, and the values of M and r given by (10.9)–(10.10)

$$\begin{aligned}\eta^n(\eta_*M)&=\eta^n\frac{\eta_*(\beta)}{(1-\tau_*)^\beta}=\eta^n\frac{2^{-\beta}\eta_*(\beta)}{r^\beta}\\&=(2^\beta\eta)^n\frac{\eta_*(\beta)}{(2^{n+1}r)^\beta}=(2^\beta\eta)^n\gamma_o,\end{aligned}$$

where

$$\gamma_o=2^{-\beta}\eta_*(\beta).$$

To remove the qualitative knowledge of τ_* and hence n, choose β from $2^\beta\eta=1$. Notice that such a choice is possible, since by Remark 10.4 the parameter η is independent of β. This makes γ_o quantitative. The time level s_n is computed from

$$s_n=s_*+\theta r^2\sum_{j=1}^n2^{2j}.$$

Therefore, the range of t for which $(10.14)_n$ holds, can be estimated as

$$s_*+\tfrac{1}{2}\theta(2^nr)^2\le t\le s_*+2\theta(2^nr)^2.$$

From the previous choices one estimates

$$s_*+\bar{\gamma}_1\le t\le s_*+4\bar{\gamma}_1\quad\text{ where }\ \bar{\gamma}_1=\frac{\theta}{2}.$$

By choosing η_* even smaller if necessary, we may ensure that $\bar{\gamma}_1 \geq 1$ so that $s_* + \bar{\gamma}_1 \geq 0$ and hence

$$\gamma_1 = 3\bar{\gamma}_1 \tag{10.17}$$

is included in the times for which $(10.14)_n$ holds.

From Remark 10.4 it follows that b and η do not depend on η_*, and hence the assumption of possibly taking η_* smaller is justified.

Finally, from the indicated choices of n and β the alternatives $(10.13)_*$–$(10.13)_n$ can be rewritten as $u(x_o, t_o) \leq \gamma_2 \gamma_* \rho^{\delta(N+2)}$ for $\gamma_2 = \frac{\bar{\gamma}\,\bar{\gamma}_*}{\gamma_o}$. ∎

10.4 The Mean Value Harnack Inequality

The Harnack inequality of Theorem 10.1 can be given an equivalent formulation, which we refer to as the *mean value form* of the Harnack inequality for non-negative functions in PDG classes.

Corollary 10.2 *Let $u \geq 0$ be an element of the parabolic DeGiorgi class $PDG(E_T, \bar{\gamma}, \gamma_*, C_2, \delta)$. There exist positive constants c_* and θ_* that can be quantitatively determined a priori only in terms of the parameters N, γ and independent of u, γ_*, and δ, such that for every radius $\rho < C_2^{-1}$, provided that $B_{4\rho}(x_o) \times (t_o - \theta_*(4\rho)^2, t_o + \theta_*(4\rho)^2) \subset E_T$, either $u(x_o, t_o) \leq c_*^{-1}\gamma_*\rho^{\delta(N+2)}$ or*

$$c_* \sup_{B_\rho(x_o)} u(\cdot, t_o - \theta_*\rho^2) \leq u(x_o, t_o) \leq c_*^{-1} \inf_{B_\rho(x_o)} u(\cdot, t_o + \theta_*\rho^2). \tag{10.18}$$

Remark 10.5 The terminology is suggested by the mean value property of harmonic functions. As shown in Chapter 2, Section 5, the latter implies that the value $u(x_o)$ at one point x_o of a non-negative harmonic function u controls its maximum and minimum in a ball centered at x_o.

Proof. Fix $(x_o, t_o) \in E_T$, and assume $u(x_o, t_o) > 0$. Seek those values of $t < t_o$, if any, for which

$$u(x_o, t) = 2c_*^{-1} u(x_o, t_o). \tag{10.19}$$

If such a t does not exist

$$u(x_o, t) < 2c_*^{-1} u(x_o, t_o) \qquad \text{for all } \ t \in (t_o - (4\rho)^2, t_o). \tag{10.20}$$

We establish by contradiction that this in turn implies

$$\sup_{B_\rho(x_o)} u(\cdot, t_o - \theta_*\rho^2) \leq 2c_*^{-1} u(x_o, t_o). \tag{10.21}$$

Indeed, if not, by continuity there exists $x_* \in B_\rho(x_o)$ such that

$$u(x_*, t_o - \theta_*\rho^2) = 2c_*^{-1} u(x_o, t_o).$$

Apply the Harnack inequality in (10.1) with (x_o, t_o) replaced by $(x_*, t_o - \theta_*\rho^2)$, to get

$$u(x_*, t_o - \theta_*\rho^2) \le c_*^{-1} \inf_{B_\rho(x_*)} u(\cdot, t_o - \theta_*\rho^2 + \theta_*\rho^2) = c_*^{-1} \inf_{B_\rho(x_*)} u(\cdot, t_o). \quad (10.22)$$

Now, $x_o \in K_\rho(x_*)$ and therefore,

$$2c_*^{-1}u(x_o, t_o) = u(x_*, t_o - \theta_*\rho^2) \le c_*^{-1}u(x_o, t_o).$$

The contradiction establishes (10.21).

10.4.1 There Exists $t < t_o$ Satisfying (3.1)

Let $\tau < t_o$ be the first time for which (10.19) holds. For such a time

$$t_o - \tau > \theta_*\rho^2. \quad (10.23)$$

Indeed, if such an inequality were violated, applying the Harnack inequality in (10.1) with (x_o, t_o) replaced by (x_o, τ) would give

$$2c_*^{-1}u(x_o, t_o) = u(x_o, \tau) \le c_*^{-1}u(x_o, t_o).$$

Set

$$s = t_o - \theta_*\rho^2.$$

From the definitions, the continuity of u and (10.5), we have

$$\tau < s < t_o \qquad \text{and} \qquad u(x_o, s) \le 2c_*^{-1}u(x_o, t_o).$$

We claim that

$$u(y, s) < 2c_*^{-1}u(x_o, t_o) \quad \text{for all } y \in B_\rho(x_o). \quad (10.24)$$

Proceeding by contradiction, let $y \in B_\rho(x_o)$ be such that

$$u(y, s) = 2c_*^{-1}u(x_o, t_o).$$

Apply the Harnack inequality in (10.1) with (x_o, t_o) replaced by (y, s) to obtain

$$u(y, s) \le c_*^{-1} \inf_{B_\rho(y)} u(\cdot, s + \theta_*\rho^2).$$

Using the definition of s, since $y \in B_\rho(x_o)$

$$2c_*^{-1}u(x_o, t_o) = u(y, s) \le c_*^{-1} \inf_{B_\rho(y)} u(\cdot, t_o) \le c_*^{-1}u(x_o, t_o).$$

The contradiction implies that (10.24) holds true. Summarizing the results of these alternatives, either (10.21) holds or (10.24) is in force. The proof is now concluded by using the arbitrariness of ρ and by properly redefining c_*^{-1}. ■

In Moser [187] the Harnack inequality is given a third equivalent statement. Fix $(x_o,t_o)\in E_T$ and $\rho>0$ and construct the cylinders

$$\begin{aligned}(x_o,t_o)+Q_\rho^+&=B_\rho(x_o)\times(t_o,t_o+\rho^2]\\(x_o,t_o)+Q_\rho^-&=B_\rho(x_o)\times(t_o-\rho^2,t_o].\end{aligned}$$

Assume that ρ is so small that $(x_o,t_o)+Q_{4\rho}^\pm\subset E_T$. Fix $\sigma\in(0,1)$ and inside $(x_o,t_o)+Q_\rho^\pm$ construct the two subcylinders

$$\begin{aligned}Q_{\sigma\rho}^+&=B_{\sigma\rho}(x_o)\times(t_o+(\sigma\rho)^2,t_o+\rho^2]\\Q_{\sigma\rho}^-&=B_{\sigma\rho}(x_o)\times(t_o-\rho^2,t_o-(\sigma\rho)^2].\end{aligned}$$

Assume for simplicity that u is a member of a homogeneous PDG class, and also that $C_2=0$. Then there exists a constant $\gamma(\sigma)$ depending only on the data $\{N,C_o,C_1\}$ and σ, and independent of (x_o,t_o) and ρ, such that

$$\sup_{Q_{\sigma\rho}^-}u\le\gamma(\sigma)\inf_{Q_{\sigma\rho}^+}u.\tag{10.25}$$

The two cylinders $Q_{\sigma\rho}^\pm$ are separated along the time axis by a distance $2(\sigma\rho)^2$, and the constant $\gamma(\sigma)\to\infty$ as $\sigma\to0$. However, $\gamma(\sigma)$ is stable as $\sigma\to1$. Other than the indicated separation between $Q_{\sigma\rho}^+$ and $Q_{\sigma\rho}^-$, there is great freedom in choosing these cylinders. For example, one could take $\sigma\approx1$, keep $Q_{\sigma\rho}^+$ fixed, and choose $Q_{\sigma\rho}^-$ with its top "vertex" at (x_o,t_o). This would keep it separated by a distance $(\sigma\rho)^2$ from $Q_{\sigma\rho}^+$. By possibly modifying the form of the constant γ, this would imply

$$u(x_o,t_o)\le\gamma\inf_{B_\rho(x_o)}u(\cdot,t_o+\rho^2),$$

which is precisely (10.1). Likewise, one could take $\sigma\approx1$, keep $Q_{\sigma\rho}^-$ fixed, and choose $Q_{\sigma\rho}^+$ with its bottom "vertex" at (x_o,t_o). This would keep it separated by a distance $(\sigma\rho)^2$ from $Q_{\sigma\rho}^-$. By possibly modifying the form of the constant γ, this would imply

$$\gamma^{-1}\sup_{B_\rho(x_o)}u(\cdot,t_o-\rho^2)\le u(x_o,t_o),$$

which is just the left-hand side of (10.18). Hence, (10.25) implies (10.18), but the opposite is also true (see Section 11c in the Complements).

11 The Harnack Inequality Implies the Hölder Continuity

The Hölder continuity of a function u in the PDG classes in the form (3.2) has been established in Theorem 3.1. However, as we have already shown in the elliptic framework in Chapter 10, in the parabolic context the Harnack inequality can also be used to prove the Hölder continuity [187].

Let $\mu^{\pm}$ and $\omega(2\rho)$ be defined as in (3.1), with θ the quantity denoted with θ_* in Theorem 10.1. Again, for simplicity we assume that $(y, s) = (0, 0)$. Applying Theorem 10.1 to the two non-negative functions $w^+ = \mu^+ - u$ and $w^- = u - \mu^-$ gives either

$$\begin{aligned} \operatorname*{ess\,sup}_{Q_\rho^-} w^+ &= \mu^+ - \operatorname*{ess\,inf}_{Q_\rho^-} u \le c_*^{-1}\gamma_*\rho^{\delta(N+2)} \\ \operatorname*{ess\,sup}_{Q_\rho^-} w^- &= \operatorname*{ess\,sup}_{Q_\rho^-} u - \mu^- \le c_*^{-1}\gamma_*\rho^{\delta(N+2)} \end{aligned} \tag{11.1}$$

or

$$\begin{aligned} c_*(\mu^+ - \operatorname*{ess\,inf}_{Q_\rho^-} u) &\le \mu^+ - \operatorname*{ess\,sup}_{Q_\rho^-} u \\ c_*(\operatorname*{ess\,sup}_{Q_\rho^-} u - \mu^-) &\le \operatorname*{ess\,inf}_{Q_\rho^-} u - \mu^-. \end{aligned} \tag{11.2}$$

If either one of (11.1) holds, then

$$\omega(\rho) \le \omega(2\rho) \le c_*^{-1}\gamma_*\rho^{\delta(N+2)}. \tag{11.3}$$

Otherwise, both inequalities in (11.2) are in force. Adding them gives

$$c_*\omega(2\rho) + c_*\omega(\rho) \le \omega(2\rho) - \omega(\rho),$$

whence

$$\omega(\rho) \le \eta\omega(2\rho), \qquad \text{where} \qquad \eta = \frac{1 - c_*}{1 + c_*}. \tag{11.4}$$

The alternatives (11.3)–(11.4) yield recursive inequalities of the same form as (7.5), from which the Hölder continuity follows.

12 A Consequence of the Harnack Inequality

Consider the functions

$$\Gamma_\nu(x, t; y, s) = \frac{k\rho^\nu}{[(t - s) + \rho^2]^{\frac{\nu}{2}}} \exp\left[-\frac{|x - y|^2}{4[(t - s) + \rho^2]}\right]$$

pointwise in $\mathbb{R}^N \times [t > s]$. For $\nu = N$ the latter are exact solutions of the heat equation and one verifies that for $\nu > N$ they are sub-solutions.

Given the general quasi-linear structure of (1.1)–(1.2), the functions Γ_ν are not sub-solutions of these equations in any sense. Moreover, no comparison principle holds for functions in PDG. Nevertheless, the “fundamental sub-solutions” Γ_ν drive, in a sense made precise by Proposition 12.1 below, the structural behavior of non-negative functions in these homogeneous classes.

The content of Propositions 12.1 is that these functions are, locally, bounded below by one Γ_ν for some $\nu > N$, and thus they do not decay in space, faster than these “sub-potentials.”

Proposition 12.1 [187] *Let* $E \subset \mathbb{R}^N$ *and* $E' \subset E$ *a convex subdomain, such that* $\operatorname{dist}(E', \partial E) = d > 0$. *Let* $u > 0$ *be a continuous element of the homogeneous parabolic DeGiorgi class* $PDG(E_T, \bar{\gamma}, 0, 0, \delta)$. *There exists a positive constant* $\gamma > 1$, *depending only on the data* $\{N, C_o, C_1\}$, *such that for all* $x, y \in E'$, $0 < d^2 \leq s < t \leq T$ *we have*

$$\ln \frac{u(y,s)}{u(x,t)} \leq \gamma \left[\frac{|x-y|^2}{t-s} + \frac{t-s}{d^2} + 1 \right]. \tag{12.1}$$

Moreover, if $E \equiv \mathbb{R}^N$ *and* $0 < s < t < T$, *we have*

$$\ln \frac{u(y,s)}{u(x,t)} \leq \gamma \left[\frac{|x-y|^2}{t-s} + \ln \frac{t}{s} + 1 \right]. \tag{12.2}$$

Remark 12.1 The continuity of u, although a given fact, is assumed only in order to give a unique meaning to the pointwise values of u.

Proof. First of all, we point out that the proof of Theorem 10.1 shows that, once the constant θ_* has been determined, there is no need to have further room above, and it is enough to assume that $t_o + \theta_* \rho^2 < T$. Hence, it suffices to assume that $B_{4\rho} \times [t_o - \rho^2, t_o + \theta_* \rho^2) \subset E_T$.

Without loss of generality, we can assume that $(y, s) = (0, 0) \in E'$ and that $u \in \mathrm{PDG}(E \times (-d^2, T), \bar{\gamma}, 0, 0, \delta)$ is positive in $E \times (-d^2, T - d^2)$. The Harnack inequality can be rephrased, saying that

$$u(x,t) \geq c_* u(0,0)$$

for

$$|x|^2 \leq \frac{t}{\theta_*} \leq \rho^2. \tag{12.3}$$

We take $(x, t) \in E' \times (0, T - d^2)$ and proceed to estimate $u(0,0)/u(x,t)$ from above.

We connect $(0, 0)$ and (x, t) with a straight line and let

$$t_j = \frac{j}{n} t, \qquad x_j = \frac{j}{n} x, \quad j = 0, 1, \ldots, n.$$

By the convexity of E', $\forall j$ we have that $x_j \in E'$. We have to determine n: we choose $4\rho = d$, so that $u > 0$ in

$$|x - x_j| < 4\rho, \qquad -\rho^2 < t - t_j < \frac{t}{n} < \theta_* \rho^2$$

if $n > \dfrac{t}{\theta_* \rho^2}$. To satisfy (12.3), we need

$$|x_{j+1} - x_j|^2 = \frac{|x|^2}{n^2} \leq \frac{t}{n\theta_*} < \rho^2 \quad \Rightarrow \quad n \geq |x|^2 \frac{\theta_*}{t},$$

so that

$$n \ge \max\left\{\theta_* \frac{|x|^2}{t}; \frac{t}{\theta_* \rho^2}\right\}.$$

Under these circumstances, we have

$$\frac{u(x_j, t_j)}{u(x_{j+1}, t_{j+1})} \le c_*^{-1}, \qquad j = 0, 1, \dots, n-1.$$

Multiplying, we have

$$\begin{aligned}
\frac{u(0,0)}{u(x,t)} &\le c_*^{-n} \\
\ln \frac{u(0,0)}{u(x,t)} &\le n \ln \frac{1}{c_*} \\
\ln \frac{u(0,0)}{u(x,t)} &\le \gamma \left[\theta_* \frac{|x|^2}{t} + \frac{t}{\theta_* \rho^2}\right] \\
\ln \frac{u(0,0)}{u(x,t)} &\le \gamma \left[\frac{|x|^2}{t} + \frac{t}{d^2} + 1\right]
\end{aligned}$$

for a constant $\gamma > 1$, which depends only on the data.

In order to prove (12.2), we still assume $y = 0$, and we distinguish two possibilities, as far as t and s are concerned.

If $s < t \le 4s$, choosing $s = d^2$ we obtain

$$\ln \frac{u(0,s)}{u(x,t)} \le 4\gamma \left[\frac{|x|^2}{t-s} + 1\right]$$

and we are done.

Otherwise, we have $t > 4s$. In such a case, choose $n \in \mathbb{N}$, such that $2^{n+1}s < t \le 2^{n+2}s$, and let $\tau = 2^n s$. Since $2\tau < t \le 4\tau$ and $t - \tau > \frac{t}{2} > \frac{t-s}{2}$, working as in the first alternative, we obtain

$$\ln \frac{u(0,\tau)}{u(x,t)} \le 4\gamma \left[\frac{|x|^2}{t-\tau} + 1\right] \le 8\gamma \left[\frac{|x|^2}{t-s} + 1\right]. \tag{12.4}$$

To estimate $u(0,s)/u(0,\tau)$, we introduce

$$t_j = 2^j s, \qquad u_j = u(0, 2^j s), \quad j = 0, 1, \dots, n.$$

From (12.1) with $d^2 = t_{j-1}$ we have $\ln(u_{j-1}/u_j) \le 2\gamma$, so that

$$\ln \frac{u(0,s)}{u(0,\tau)} = \ln \frac{u_o}{u_n} \le 2\gamma n \le 2\gamma \log_2 \frac{t}{s}.$$

Adding this last inequality to (12.4) yields

$$\ln \frac{u(0,s)}{u(x,t)} \le 8\gamma \left[\frac{|x|^2}{t-s} + \log_2 \frac{t}{s} + 1\right]$$

and we are done. ■

13 A More Straightforward Proof of the Hölder Continuity

We conclude the chapter by giving a more streamlined proof of the Hölder continuity of functions in PDG classes, based on the ideas developed for the proof of the Harnack inequality. For simplicity, we deal only with homogeneous classes, but just minor adjustments are needed in order to cover the general case.

Take $(x_o,t_o)\in E_T$, assume that $\rho<C_2^{-1}$ is such that $(x_o,t_o)+Q_{2\rho}^-\subset E_T$, and let

$$\mu^+=\sup_{(x_o,t_o)+Q_{2\rho}^-}u,\quad \mu^-=\inf_{(x_o,t_o)+Q_{2\rho}^-}u,\quad \omega=\mu^+-\mu^-.$$

Moreover, let ν be the quantity in (4.6) with $a=\frac{1}{2}$ and $\theta=1$. We have two alternatives, namely either

$$\big|[u>\mu^+-\frac{1}{2}\omega]\cap(x_o,t_o)+Q_\rho^-\big|\le\nu|Q_\rho^-|,$$

or

$$\big|[u>\mu^+-\frac{1}{2}\omega]\cap(x_o,t_o)+Q_\rho^-\big|>\nu|Q_\rho^-|.$$

If the first alternative holds, by Proposition 4.1 we have

$$u\le\mu^+-\frac{1}{2}\omega\qquad\text{a.e. in}\ \ (x_o,t_o)+Q_{\frac{1}{2}\rho}^-$$

and also

$$\begin{aligned}\operatorname*{ess\,osc}_{(x_o,t_o)+Q_{\frac{1}{2}\rho}^-}u=\operatorname*{ess\,sup}_{(x_o,t_o)+Q_{\frac{1}{2}\rho}^-}u-\operatorname*{ess\,inf}_{(x_o,t_o)+Q_{\frac{1}{2}\rho}^-}u&\le\mu^+-\frac{1}{2}\omega-\mu^-\\ =\frac{1}{2}\operatorname*{ess\,osc}_{(x_o,t_o)+Q_{2\rho}^-}u.&\end{aligned}$$

If the second alternative is satisfied, there exists $s\in[-\rho^2,-\frac{\nu}{2}\rho^2]$ such that

$$\big|[u(\cdot,s)>\mu^+-\frac{1}{2}\omega]\cap B_\rho(x_o)\big|>\frac{\nu}{2}|B_\rho(x_o)|.$$

Indeed, if not, we would have

$$\begin{aligned}\big|[u>\mu^+-\frac{1}{2}\omega]\cap(x_o,t_o)+Q_\rho^-\big|=&\int_{-\rho^2}^{-\frac{\nu}{2}\rho^2}\big|[u(\cdot,\tau)>\mu^+-\frac{1}{2}\omega]\cap B_\rho(x_o)\big|d\tau\\ &+\int_{-\frac{\nu}{2}\rho^2}^{0}\big|[u(\cdot,\tau)>\mu^+-\frac{1}{2}\omega]\cap B_\rho(x_o)\big|d\tau\\ \le&\nu||Q_\rho^-|.\end{aligned}$$

Since

$$\mu^+ - \frac{1}{2}\omega = \mu^- + \frac{1}{2}\omega,$$

we have

$$\Big|[u(\cdot, s) > \mu^- + \frac{1}{2}\omega] \cap B_\rho(x_o)\Big| > \frac{\nu}{2}|B_\rho(x_o)|.$$

If we set $w = u - \mu^-$, we have (10.5) with $M = \frac{1}{2}\omega$. By a (possibly repeated) application of Proposition 10.1, we conclude that there exists a $\bar{\eta} \in (0, 1)$, which depends only on the data $\{N, C_o, C_1\}$, such that

$$u \geq \mu^- + \bar{\eta}\omega \quad \text{ a.e. in } \ (x_o, t_o) + Q^-_{\frac{1}{2}\rho},$$

and also

$$\begin{aligned} \operatorname*{ess\,osc}_{(x_o,t_o)+Q^-_{\frac{1}{2}\rho}} u &= \operatorname*{ess\,sup}_{(x_o,t_o)+Q^-_{\frac{1}{2}\rho}} u - \operatorname*{ess\,inf}_{(x_o,t_o)+Q^-_{\frac{1}{2}\rho}} u \leq \mu^+ - \mu^- - \bar{\eta}\omega \\ &= (1 - \bar{\eta}) \operatorname*{ess\,osc}_{(x_o,t_o)+Q^-_{2\rho}} u. \end{aligned}$$

Combining the two alternatives, we conclude that

$$\omega(\frac{1}{2}\rho) \leq \max\left\{\frac{1}{2}; (1 - \bar{\eta})\right\} \omega(2\rho)$$

and from here on, we conclude as in Section 7.

Problems and Complements

2c Local Boundedness of Functions in the PDG Classes

The results of Theorem 2.1 can be slightly extended and at the same time generalized to cover a wider situation. We limit ourselves to a qualitative statement, even though it could be phrased in a quantitative fashion, with a rather limited extra effort.

For $N \geq 2$ consider the quasi-linear parabolic equation

$$u_t - \operatorname{div} \mathbf{A}(x, t, u, \nabla u) = B(x, t, u, \nabla u) \quad \text{ weakly in } \ E_T \tag{2.1c}$$

where the functions

$$E_T \ni (x,t) \to \begin{cases} \mathbf{A}\big(x,t,u(x,t),\nabla u(x,t)\big) \in \mathbb{R}^N \\ B\big(x,t,u(x,t),\nabla u(x,t)\big) \in \mathbb{R} \end{cases}$$

are measurable and satisfy the structure conditions

$$\begin{aligned} \mathbf{A}\big(x,t,u,\nabla u\big)\cdot\nabla u &\geq C_o|\nabla u|^2 - f_o \\ |\mathbf{A}\big(x,t,u,\nabla u\big)| &\leq C_1|\nabla u| + f_1 \\ |B\big(x,t,u,\nabla u\big)| &\leq C_2|\nabla u| + f_2 \end{aligned} \tag{2.2c}$$

for given constants $0 < C_o \leq C_1$ and $C_2 > 0$, and given non-negative functions

$$\begin{aligned} &f_1 \in L^2_{\mathrm{loc}}(E_T), \quad f_o \in L^\mu_{\mathrm{loc}}(E_T) \ \text{ with } \ \mu > 1, \\ &f_1,\, f_2 \in L^s_{\mathrm{loc}}(E_T),\ \ s > \frac{2(N+2)}{N+4}. \end{aligned} \tag{2.3c}$$

Theorem 2.1c. *Let u be a local, weak solution of* (2.1c). *Under conditions* (2.2c)–(2.3c) *we have*

- *If both $s > \frac{N+2}{2}$ and $\mu > \frac{N+2}{2}$, then $u \in L^\infty_{\mathrm{loc}}(E_T)$.*
- *If both $s = \frac{N+2}{2}$ and $\mu = \frac{N+2}{2}$, then $u \in L^q_{\mathrm{loc}}(E_T)$ for all $q < +\infty$.*
- *If both $s < \frac{N+2}{2}$ and $\mu < \frac{N+2}{2}$, then $u \in L^q_{\mathrm{loc}}(E_T)$ for all $q < q_*$ with*

$$q_* = \min\left\{\frac{\frac{N+2}{2}}{1-\left(1-\frac{1}{s}\right)\left(1+\frac{2}{N}\right)};\, \frac{\frac{2N+4}{N}}{1-\left(1-\frac{1}{\mu}\right)\left(1+\frac{2}{N}\right)}\right\}.$$

The result was proved in [194] for a larger class of quasi-linear parabolic equations. Conditions (2.3c) are rather natural, as the same conditions were imposed in Ladyzhenskaya et al. [151], Chapter 3, Sections 8–9, in order to have the corresponding properties for the solutions of a large class of linear equations.

The assumption $f_1 \in L^2_{\mathrm{loc}}(E_T)$ may seem out of place, but its purpose is to guarantee that the terms of the form $\mathbf{A}\big(x,t,u,\nabla u\big)\cdot\nabla u$ are locally integrable.

3c Hölder Continuity of Solutions of Linear Parabolic Equations with Bounded and Measurable Coefficients

As discussed in Section 15 of Chapter 11, the local Hölder continuity of locally bounded solutions of linear parabolic equations in divergence form with bounded and measurable coefficients was first proved in 1958 by Nash [190].

Since his proof relies in a fundamental way on deep estimates of the fundamental solution (again, see Section 15 mentioned above), it has an intrinsic linear feature, and it does not seem possible to extend it to the general quasi-linear setting considered in this Chapter.

6c Propagating in Time the Measure-Theoretical Information

We give a slightly stronger version of Proposition 6.1, where the measure-theoretical information is spread not just for a short time, but up to the top of the original cylinder. It is taken from Liao [171].

Proposition 6.1c *Assume that*

$$\Big|\Big[u(\cdot,s)<\mu^+-\frac{\omega}{4}\Big]\cap B_\rho(y)\Big|\ge\frac{\alpha}{2}|B_\rho|. \tag*{(6.1c)$_+$}$$

for some $\alpha\in(0,1]$. Then, there exist $\xi\in(0,1)$ and $C_\xi>1$, which depend only on the data, such that either $\omega\le\gamma_* C_\xi\rho^{\delta(N+2)}$ or

$$\big|\big[u(\cdot,t)<\mu^+-\xi\omega\big]\cap B_\rho(y)\big|\ge\frac{\alpha}{4}|B_\rho| \tag*{(6.2c)$_+$}$$

for all $t\in[s,s+\rho^2]$. Analogously, if

$$\Big|\Big[u(\cdot,s)>\mu^-+\frac{\omega}{4}\Big]\cap B_\rho(y)\Big|\ge\frac{\alpha}{2}|B_\rho|, \tag*{(6.1c)$_-$}$$

for some $\alpha\in(0,1]$, then there exist $\xi\in(0,1)$ and $C_\xi>1$, which depend only on the data, such that either $\omega\le\gamma_* C_\xi\rho^{\delta(N+2)}$ or

$$\big|\big[u(\cdot,t)>\mu^-+\xi\omega\big]\cap B_\rho(y)\big|\ge\frac{\alpha}{4}|B_\rho| \tag*{(6.2c)$_-$}$$

for all $t\in[s,s+\rho^2]$.

Remark 6.1c An analogous statement holds, if we consider cylinders $(y,s)+Q_\rho^-(\theta)$.

6.1c Proof of Proposition 6.1c

Proof. We will establish (6.2c)$_-$ starting from (6.1c)$_-$. As usual, without loss of generality, we may assume $(y,s)=(0,0)$.

For simplicity, we set $M=\frac{1}{4}\omega$. We start from (1.7) written over $Q_\rho^+(\theta)$ for $(u-k)_-$, where θ is to be determined and $k=\mu^-+M$, we discard the third term on the left-hand side.

Moreover, we consider a non-negative, piecewise smooth, test function $\zeta=\zeta(x)$ such that

$$\zeta=1 \text{ in } B_{\sigma\rho},\quad \zeta=0 \text{ in } \mathbb{R}^N-B_\rho,\quad |\nabla\zeta|\le\frac{1}{\sigma\rho}.$$

Hence, we conclude that

$$\begin{aligned}
\operatorname*{ess\,sup}_{0<t<\theta\rho^2}\int_{B_\rho}&(u-k)_-^2\zeta^2(x,t)dx\\
&\le\int_{B_\rho}(u-k)_-^2\zeta^2(x,0)dx\\
&\quad+\frac{\gamma}{\sigma^2\rho^2}\iint_{Q_\rho^+(\theta)}(u-k)_-^2\,dx\,d\tau+\gamma_*^2|A_{k,\rho}^-|^{1-\frac{2}{N+2}+2\delta}\\
&\le M^2\left(1-\frac{\alpha}{2}\right)|B_\rho|+\gamma\frac{\theta}{\sigma^2}M^2\frac{|[u<k]\cap Q_\rho^+(\theta)|}{|Q_\rho^+(\theta)|}|B_\rho|\\
&\quad+\gamma\theta^{1-\frac{2}{N+2}+2\delta}\left(\gamma_*^2\frac{\rho^{2\delta(N+2)}}{M^2}\right)M^2|B_\rho|.
\end{aligned}$$

If we stipulate to take

$$\gamma_*^2\frac{\rho^{2\delta(N+2)}}{M^2}<1,$$

we have

$$\begin{aligned}
\operatorname*{ess\,sup}_{0<t<\theta\rho^2}&\int_{B_\rho}(u-k)_-^2\zeta^2(x,t)dx\\
&\le M^2\left[\left(1-\frac{\alpha}{2}\right)+\gamma\frac{\theta}{\sigma^2}\frac{|[u<k]\cap Q_\rho^+(\theta)|}{|Q_\rho^+(\theta)|}+\gamma\theta^{1-\frac{2}{N+2}+2\delta}\right]|B_\rho|.
\end{aligned}$$

Let $\ell=\mu^-+\epsilon M$ with $\epsilon\in(0,1)$ to be chosen. Then,

$$\begin{aligned}
\int_{B_\rho}(u-k)_-^2\zeta^2(x,t)dx&\ge\int_{B_{(1-\sigma)\rho}\cap[u\le\ell]}(u-k)_-^2(x,t)dx\\
&\ge(1-\epsilon)^2M^2|[u\le\ell]\cap B_{(1-\sigma)\rho}|.
\end{aligned}$$

Moreover,

$$\begin{aligned}
|A_{\ell,\rho}(t)|&=|A_{\ell,(1-\sigma)\rho}(t)\cup(A_{\ell,\rho}(t)\backslash A_{\ell,(1-\sigma)\rho}(t))|\\
&\le|A_{\ell,(1-\sigma)\rho}(t)|+|B_\rho\backslash B_{(1-\sigma)\rho}|\\
&\le|A_{\ell,(1-\sigma)\rho}(t)|+N\sigma|B_\rho|.
\end{aligned}$$

Therefore, we conclude

$$\begin{aligned}
(1-\epsilon)^2M^2|A_{\ell,\rho}(t)|\le M^2\Bigg[&\left(1-\frac{\alpha}{2}\right)+\gamma\frac{\theta}{\sigma^2}\frac{|[u<k]\cap Q_\rho^+(\theta)|}{|Q_\rho^+(\theta)|}\\
&+\gamma\theta^{1-\frac{2}{N+2}+2\delta}+N(1-\epsilon)^2\sigma\Bigg]|B_\rho|.
\end{aligned}$$

Since we can take $\epsilon<\frac{1}{2}$ without loss of generality, we also have

$$|A_{\ell,\rho}(t)|\le\left[\frac{1-\frac{\alpha}{2}}{(1-\epsilon)^2}+\gamma\frac{\theta}{\sigma^2}\frac{|[u<k]\cap Q_\rho^+(\theta)|}{|Q_\rho^+(\theta)|}+\gamma\theta^{1-\frac{2}{N+2}+2\delta}+N\sigma\right]|B_\rho|.$$

Choosing

$$\sigma = \theta^{\frac{1}{3}} \left(\frac{|[u < k] \cap Q_\rho^+(\theta)|}{|Q_\rho^+(\theta)|} \right)^{\frac{1}{3}},$$

taking into account that $\theta \in (0, 1)$, and relabelling $2\gamma + N$ with γ yields

$$|A_{\ell,\rho}(t)| \le \left[\frac{1 - \frac{\alpha}{2}}{(1-\epsilon)^2} + \gamma \theta^{\frac{1}{3}} \left(\frac{|[u < k] \cap Q_\rho^+(\theta)|}{|Q_\rho^+(\theta)|} \right)^{\frac{1}{3}} \right] |B_\rho|.$$

Finally, we choose

$$\begin{aligned} \gamma\theta^{\frac{1}{3}} &= \frac{\alpha}{16}, \\ \frac{1-\frac{\alpha}{2}}{(1-\epsilon)^2} &< \frac{1-\frac{\alpha}{4}}{(1-\epsilon)^2} \le 1 - \frac{\alpha}{8} \quad \Rightarrow \quad \epsilon \approx \frac{3}{16}\alpha, \end{aligned}$$

and conclude

$$|A_{\ell,\rho}(t)| \le \left(1 - \frac{\alpha}{16}\right) |B_\rho|$$

for all $t \in [0, \theta\rho^2]$, with θ computed above. We now use this estimate and we refine it.

Let

$$M_1 = \epsilon M, \quad k_1 = \mu^- + \frac{M_1}{2^{j_1}}, \quad \ell_1 = \mu^- + \frac{M_1}{2^{j_1+n_1}},$$

where j_1 and n_1 are to be chosen. If we repeat the previous computations with the new quantities, we have

$$\begin{aligned} |A_{\ell_1,\rho}(t)| \le & \Bigg[\frac{1-\frac{\alpha}{2}}{(1-2^{-n_1})^2} + \gamma \frac{\theta}{\sigma^2} \frac{|[u < k_1] \cap Q_\rho^+(\theta)|}{|Q_\rho^+(\theta)|} \\ & + \gamma \theta^{1-\frac{2}{N+2}+2\delta} \left(\gamma_* \frac{2^{j_1} \rho^{\delta(N+2)}}{M_1} \right)^2 + N\sigma \Bigg] |B_\rho|. \end{aligned}$$

We now rely on Proposition 5.2 and we assume that

$$\gamma_* \frac{2^{j_1} \rho^{\delta(N+2)}}{M_1} \le \frac{1}{j_1^{\frac{1}{12}}};$$

since in (5.3) we have $\theta^* = \theta$ because $\theta \in (0, 1)$, with the same choice for σ as above, we obtain

$$\begin{aligned} |A_{\ell_1,\rho}(t)| \le & \Bigg[\frac{1-\frac{\alpha}{2}}{(1-2^{-n_1})^2} + \gamma \theta^{\frac{1}{3}} \left(\frac{1}{\alpha\sqrt{\theta j_1}} \right)^{\frac{1}{3}} \\ & + \gamma \theta^{1-\frac{2}{N+2}+2\delta} \left(\frac{1}{\sqrt{j_1}} \right)^{\frac{1}{3}} + N \left(\frac{1}{\alpha\sqrt{\theta j_1}} \right)^{\frac{1}{3}} \Bigg] |B_\rho| \end{aligned}$$

$$\leq \left[\frac{1-\frac{\alpha}{2}}{(1-2^{-n_1})^2} + \gamma\theta^{\frac{1}{3}} \left(\frac{1}{\alpha\sqrt{\theta j_1}} \right)^{\frac{1}{3}} \right] |B_\rho|,$$

where in the last inequality we have also relabeled γ as before. We choose j_1 and n_1 such that

$$\gamma\theta^{\frac{1}{3}} \left(\frac{1}{\alpha\sqrt{\theta j_1}} \right)^{\frac{1}{3}} \leq \theta\frac{\alpha}{8}$$
$$\frac{1-\frac{\alpha}{2}}{(1-2^{-n_1})^2} \leq 1 - \frac{\alpha}{2} + \theta\frac{\alpha}{8}.$$

As a result

$$|A_{\ell_1,\rho}(t)| \leq \left(1 - \frac{\alpha}{2} + \theta\frac{\alpha}{4} \right) |B_\rho|$$

for all $t \in [s_o, s_1] \overset{\text{def}}{=} [0, \theta\rho^2]$.

We now proceed by induction. Either at every step $m = 1, \dots, i-1$

$$\gamma_* \frac{2^{j_m}\rho^{\delta(N+2)}}{M_m} \leq \frac{1}{j_m^{\frac{1}{12}}},$$

and in such a case the sequences $\{M_i\}$, $\{n_i\}$ and $\{j_i\}$ have been chosen, and

$$|A_{\ell_{i-1},\rho}(t)| \leq \left(1 - \frac{\alpha}{2} + (i-1)\theta\frac{\alpha}{4} \right) |B_\rho| \quad \forall\, t \in [s_{i-1}, s_i]$$

where

$$\ell_{i-1} = \mu^- + \frac{M_{i-1}}{2^{n_{i-1}+j_{i-1}}} \overset{\text{def}}{=} \mu^- + \hat{M}_{i-1}.$$

Otherwise, the process stops and ξ and C_ξ are chosen accordingly. Suppose we have reached step $i-1$. If we now set

$$\ell_{i-1}^\epsilon = \mu^- + \epsilon\hat{M}_{i-1}, \quad s_{i+1} = s_i + \theta\rho^2, \quad Q_i^+ = B_\rho \times (s_i, s_{i+1}],$$

using the measure-theoretical information at times s_i, we can work exactly as above and conclude that

$$|A_{\ell_{i-1}^\epsilon,\rho}(t)| \leq \left[\frac{1-\frac{\alpha}{2}+(i-1)\theta\frac{\alpha}{4}}{(1-\epsilon)^2} + \gamma\theta^{\frac{1}{3}} \left(\frac{|[u<k] \cap Q_i^+|}{|Q_i^+|} \right)^{\frac{1}{3}} \right] |B_\rho|.$$

If we assume $(i-1)\theta < 1$, we may choose ϵ and θ as above, and conclude that

$$|A_{\ell_{i-1}^\epsilon,\rho}(t)| \leq \left(1 - \frac{\alpha}{16} \right) |B_\rho| \qquad \forall\, t \in [s_i, s_{i+1}].$$

Now we let

$$M_i = \epsilon\hat{M}_{i-1}, \quad k_1 = \mu^- + \frac{M_i}{2^{j_i}}, \quad \ell_i = \mu^- + \frac{M_i}{2^{j_i+n_i}},$$

where j_i and n_i are still to be chosen. Using Proposition 5.2 as above and stipulating that

$$\gamma_* \frac{2^{j_i} \rho^{\delta(N+2)}}{M_i} \le \frac{1}{j_i^{\frac{1}{12}}},$$

yields

$$|A_{\ell_i,\rho}(t) \le \left[\frac{1 - \frac{\alpha}{2} + (i-1)\theta \frac{\alpha}{4}}{(1-2^{-n_i})^2} + \gamma \theta^{\frac{1}{3}} \left(\frac{1}{\alpha \sqrt{\theta j_i}} \right)^{\frac{1}{3}} \right] |B_\rho| \qquad \forall\, t \in [s_i, s_{i+1}].$$

Finally, we choose j_i and n_i such that

$$\begin{aligned} \gamma \theta^{\frac{1}{3}} \left(\frac{1}{\alpha \sqrt{\theta j_i}} \right)^{\frac{1}{3}} &\le \theta \frac{\alpha}{8} \\ \frac{1 - \frac{\alpha}{2} + (i-1)\theta \frac{\alpha}{4}}{(1-2^{-n_i})^2} &\le 1 - \frac{\alpha}{2} + (i-1)\theta \frac{\alpha}{4} + \theta \frac{\alpha}{8}. \end{aligned}$$

As a result

$$|A_{\ell_i,\rho}(t)| \le \left(1 - \frac{\alpha}{2} + i\theta \frac{\alpha}{4} \right) |B_\rho|$$

for all $t \in [s_i, s_{i+1}]$. The induction finishes when

$$\text{either} \quad \gamma_* \frac{2^{j_i} \rho^{\delta(N+2)}}{M_i} \ge \frac{1}{j_i^{\frac{1}{12}}}, \quad \text{or} \quad i\theta \ge 1.$$

Hence, we have that either

$$\left| \left[u(\cdot,t) > \mu^- + \xi\omega \right] \cap B_\rho(y) \right| \ge \frac{\alpha}{4} |B_\rho|$$

for all $t \in [s, s+\rho^2]$, with $\xi\omega = \frac{M_i}{2^{n_i + j_i}}$, or

$$\omega \le \gamma_* C_\xi \rho^{\delta(N+2)},$$

where $C_\xi > 1$ takes into account at which step m, if any, the stipulation $\gamma_* \dfrac{2^{j_i} \rho^{\delta(N+2)}}{M_i} \le \dfrac{1}{j_i^{\frac{1}{12}}}$ is violated. ■

Remark 6.2c The proof shows that we can take a general $\theta\rho^2$ for the height of the cylinder, with any $\theta > 0$ and not just $\theta = 1$.

7c Proof of Theorem 3.1

Relying on Proposition 6.1c, the proof of Theorem 3.1 is revised accordingly. In particular, it is concluded in one stroke. The first part is as in Section 7.

Without loss of generality, we may assume $(y,s)=(0,0)$. We consider Q_ρ^-, that is, we take $\theta=1$. By the definitions (3.1), either

$$|[u \le \mu^+ - \tfrac{1}{2}\omega] \cap Q_\rho^-| \ge \frac{1}{2}|Q_\rho^-| \tag{7.1c}$$

or

$$|[u \ge \mu^- + \tfrac{1}{2}\omega] \cap Q_\rho^-| \ge \frac{1}{2}|Q_\rho^-|. \tag{7.2c}$$

Assume that the second is in force; if (7.1c) holds, the argument runs exactly in the same way. We can equivalently rewrite (7.2c) as

$$|[u < \mu^- + \tfrac{1}{2}\omega] \cap Q_\rho^-| < \frac{1}{2}|Q_\rho^-|.$$

We claim that this implies that there exists $t_o \in [-\rho^2, -\frac{1}{3}\rho^2]$ such that

$$|[u(\cdot,t_o) < \mu^- + \tfrac{1}{2}\omega] \cap B_\rho| \le \frac{3}{4}|B_\rho|.$$

Indeed, if not, then

$$|[u < \mu^- + \tfrac{1}{2}\omega] \cap Q_\rho^-| \ge \int_{-\rho^2}^{-\frac{1}{3}\rho^2} |[u(\cdot,\tau) < \mu^- + \tfrac{1}{2}\omega] \cap B_\rho| d\tau > \frac{1}{2}|Q_\rho^-|.$$

We can equivalently conclude that there exists $t_o \in [-\rho^2, \frac{1}{3}\rho^2]$ such that

$$|[u(\cdot,t_o) \ge \mu^- + \tfrac{1}{2}\omega] \cap B_\rho| \ge \frac{1}{4}|B_\rho|.$$

Without loss of generality, we can assume that $t_o = -\rho^2$, so that

$$|[u(\cdot,-\rho^2) \ge \mu^- + \tfrac{1}{4}\omega] \cap B_\rho| \ge \frac{1}{4}|B_\rho|.$$

We can then apply Proposition 6.1c with $\alpha = \frac{1}{2}$, and conclude that either

$$\omega \le \gamma_* C_\xi \rho^{\delta(N+2)} \tag{7.3c}$$

or

$$|[u(\cdot,t) \ge \mu^- + \xi\omega] \cap B_\rho| \ge \frac{1}{8}|B_\rho| \tag{7.4c}$$

for all $t \in [-\rho^2, 0]$.

If (7.3c) holds true, we have finished. Otherwise, assuming that (7.4c) is in force, fix the number ν as the one claimed by Proposition 4.1 in (4.6), for the choices $a=\frac{1}{2}$, $\theta=\bar{\theta}=1$.

We point out that (7.4c) is $(5.1)_-$ of Proposition 5.1 with $\alpha=\frac{1}{8}$ and $\epsilon=\xi$; therefore, ν being fixed, determine j_* and hence $\epsilon_\nu = \frac{\xi}{2^{j_*}}$ by the procedure of Proposition 5.1.

Then, by Proposition 4.1, either $\epsilon_\nu \omega \le \gamma \rho^{\delta(N+2)}$, or $(4.2)_+$ holds. The latter implies

$$-\operatorname*{ess\,inf}_{Q^-_{\frac{1}{2}\rho}} u \le -\operatorname*{ess\,inf}_{Q^-_{2\rho}} - \tfrac{1}{2}\epsilon_\nu \operatorname*{ess\,osc}_{Q^-_{2\rho}} u.$$

Now,

$$\operatorname*{ess\,sup}_{Q^-_{\frac{1}{2}\rho}} u \le \operatorname*{ess\,sup}_{Q^-_{2\rho}} u.$$

Adding these inequalities gives

$$\omega(\tfrac{1}{2}\rho) \le \eta\omega(2\rho), \qquad \text{where } \ \eta = 1 - \frac{1}{2}\epsilon_\nu.$$

Let $Q^-_R \subset E_T$ be fixed and set $\rho_n = 4^{-n}R$. The previous remarks imply that

$$\omega(\rho_{n+1}) \le \max\{\eta\omega(\rho_n)\,;\, \bar{C}\gamma_* \rho_n^{\delta(N+2)}\} \tag{7.5c}$$

for a proper choice of $\bar{C}$ that takes into account all the alternatives, and by iteration

$$\omega(\rho_{n+1}) \le \max\{\eta^n\omega(R)\,;\, \bar{C}\gamma_* \rho_n^{\delta(N+2)}\}.$$

Compute

$$\rho_n = 4^{-n}R \implies -n = \ln\Big(\frac{\rho_n}{R}\Big)^{\frac{1}{\ln 4}} \implies \eta^n = \Big(\frac{\rho_n}{R}\Big)^\alpha \ \text{ for } \ \alpha = -\frac{\ln\eta}{\ln 4}. \qquad \blacksquare$$

Remark 7.1c We have given the proof assuming $\theta = 1$, but a general $\theta > 0$ is also possible, without any substantial change in the previous arguments.

11c The Harnack Inequality

The first to prove a Harnack inequality for linear parabolic equations in divergence form with bounded and measurable coefficient was Jürgen Moser [187], expanding upon ideas he had previously developed for elliptic equations [186]. The proof of Lemma 4 in Moser [187] contained a faulty argument, which was later corrected [188]. An easier proof was given later [189].

As we have shown in Section 11, the Harnack inequality can be used to prove that weak solutions are locally Hölder continuous in E_T. Even though Moser's method is quite different from the one we used here in Section 10, nevertheless, the linearity assumed in (1.1) is immaterial to the proof, and one might then expect, as in the elliptic case, an extension of these results to quasi-linear equations of the type

$$u_t - \operatorname{div}\mathbf{A}(x,t,u,\nabla u) = B(x,t,u,\nabla u) \quad \text{ in } \ E_T,$$

where the structure conditions are as in (1.2) of Chapter 10, that is, with a growth of order p for any $p > 1$. Surprisingly, however, Moser's proof could be extended only for the case $p = 2$, i.e., for equations whose principal part has a linear growth with respect to ∇u. This appears in the work of Aronson and Serrin [12] and Trudinger [262].

Also, the approach based on parabolic DeGiorgi classes cannot be simply extended, and the problem remained open for quite a number of years, until a partial solution was first given in the late 1990s by the first author of this monograph. For a very interesting historical perspective, we refer to the survey paper by Kassmann [134], whereas the interested reader can look at DiBenedetto et al. [55] for a proof of the Harnack inequality for general operators with growth of order $p > 2$.

11.1. Prove that (10.18) implies (10.25).

13

PARABOLIC EQUATIONS IN NONDIVERGENCE FORM

1 Introductory Material

The aim of this chapter is to present some of the known estimates for solutions of parabolic PDEs in nondivergence form with only bounded and measurable coefficients.

The time derivative of a function u will be equivalently denoted with u_t, $\partial_t u$, and $\frac{\partial u}{\partial t}$, whereas for the space derivatives of u we equivalently write Du, u_{x_i}, $\frac{\partial u}{\partial x_i}$.

1.1 Introduction

Let E be a bounded open set in $\mathbb{R}^N$, $N \ge 1$ with smooth boundary ∂E and for $0 < T < \infty$ let $E_T \equiv E \times (0, T]$. The following types of equations are considered in this chapter.

1.1.1 Linear Equations

$$u_t - \mathcal{L}_o(u) = 0 \ \text{ in } \ E_T, \qquad \mathcal{L}_o = a_{ij}(x,t) u_{x_i x_j}, \tag{1.1}$$

where the summation convention is adopted. The basic assumptions on the coefficients are:

> The functions $(x,t) \to a_{ij}(x,t)$, $i,j = 1, \dots, N$ are only bounded and measurable, defined in E_T.
>
> The matrix $(a_{ij}(x,t))$ is symmetric and positive definite uniformly in E_T. Equivalently, if $\lambda(x,t) \le \Lambda(x,t)$ are respectively the minimum and the maximum eigenvalues of $a_{ij}(x,t)$ as $(x,t) \in E_T$, there exists $\lambda_o \le \Lambda_o$ such that
>
> $$0 < \lambda_o \le \lambda(x,t) \le \Lambda(x,t) \le \Lambda_o \quad \forall\,(x,t) \in E_T. \qquad (1.2)_{(i)}$$
>
> In turn, this can be formulated as
>
> $$\lambda_o |\boldsymbol{\xi}|^2 \le a_{ij}(x,t)\xi_i \xi_j \le \lambda_o |\boldsymbol{\xi}|^2, \quad \forall \boldsymbol{\xi} \in \mathbb{R}^N. \qquad (1.2)_{(ii)}$$

© Springer Nature Switzerland AG 2023
E. DiBenedetto, U. Gianazza, *Partial Differential Equations*,
Cornerstones, https://doi.org/10.1007/978-3-031-46618-2_14

We assume that $u \in C^{2,1}(E_T)$ is a solution of (1.1), where the space $C^{2,1}(E_T)$ has been defined in Section 1 of Chapter 11.

Remark 1.1 There is no loss of generality in assuming $(a_{ij}(x,t))$ symmetric, since $\mathcal{L}_o$ can always be transformed into a new operator $\mathcal{L}_o^*$ with symmetric coefficients

$$a_{ij}^*(x,t) = \frac{1}{2}(a_{ij}(x,t) + a_{ji}(x,t)).$$

This follows from the fact that $x \to u(x,t) \in C^2(E)$, and therefore, $u_{x_i x_j} = u_{x_j x_i}$.

Remark 1.2 Condition $(1.2)_{(ii)}$ implies that

$$\lambda_o \le a_{ij}(x,t) \le \Lambda_o \qquad \forall\, (x,t) \in E_T,\ i,j = 1,\dots,N.$$

The equivalence of $(1.2)_{(i)}$ and $(1.2)_{(ii)}$ can be proved using the following result, which we also state here for future reference.

Lemma 1.1 *If $(a_{ij}(x,t))$ is a $N \times N$ symmetric matrix, then there exists a unitary matrix $\mathbb{X} = \mathbb{X}(x,t)$ that diagonalizes $(a_{ij}(x,t))$, i.e.,*

$$\mathbb{X}^{-1}(x,t)(a_{ij}(x,t))\mathbb{X}(x,t) = \begin{pmatrix} \lambda_1(x,t) & 0 & 0 & \dots & 0 \\ 0 & \lambda_2(x,t) & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \lambda_N(x,t) \end{pmatrix},$$

where $\lambda_1(x,t)$, ..., $\lambda_N(x,t)$ are the eigenvalues of $(a_{ij}(x,t))$, and

$$\mathbb{X}^{-1} = \mathbb{X}^t, \quad (\mathbb{X}^t \textit{ being the transpose of } \mathbb{X})$$
$$\sum_{i=1}^{N} x_{ij}^2 = 1, \quad \forall\, j = 1,2,\dots,N, \quad \sum_{j=1}^{N} x_{ij}^2 = 1, \quad \forall\, i = 1,2,\dots,N.$$

For the proof, see Section 1.1.1c in the Complements.

For future reference, we also note the following.

Remark 1.3 Let $(D^2u) = (u_{x_i x_j})$ be the $N \times N$ symmetric Hessian matrix of the second-order space derivatives of $u \in C^{2,1}(E_T)$. Let $\mathbb{A}$ be the matrix $(a_{ij}(x,t))$. Then,

$$a_{ij}(x,t)u_{x_i x_j} = \operatorname{tr}(\mathbb{A} \cdot (D^2u)).$$

Nonhomogeneous variations of (1.1) are

$$u_t - a_{ij}(x,t)u_{x_i x_j} - b(x,t,u,Du) = 0 \quad \text{in } E_T, \tag{1.3}$$

where Du denotes the gradient of u with respect to the space variables only, and b is a given measurable function from $E_T \times \mathbb{R} \times \mathbb{R}^N$ into $\mathbb{R}$, which will be assumed to satisfy

$$|b(x,t,u,Du)| \le \varphi_o(x,t) + \varphi_1(x,t)|u|^\sigma + \varphi_2(x,t)|Du|^\theta.$$

Here, $(x,t) \to \varphi_i(x,t)$, $i = 0,1,2$ are given, non-negative, measurable functions defined in E_T, and σ, θ are given positive numbers.

The precise regularity of the φ_i and the order of growth of b with respect to u and $|Du|$, i.e., the numbers σ and θ, will be specified later, depending on the estimates we will be seeking.

The operator in (1.3) will be denoted by $u_t - \mathcal{L}(u)$, where $\mathcal{L}(u) = \mathcal{L}_o(u) + b(x,t,u,Du)$. Thus, $\mathcal{L}_o(u)$ in (1.1) is the principal part of $\mathcal{L}(u)$.

1.1.2 Quasi-linear Equations

We consider

$$u_t - Q_o(u) = 0 \text{ in } E_T, \qquad Q_o = A_{ij}(x,t,u,Du)u_{x_ix_j}. \tag{1.4}$$

Here, A_{ij}, $i,j = 1,2,\dots,N$ are given measurable functions from $E_T \times \mathbb{R} \times \mathbb{R}^N$ into $\mathbb{R}$.

As before, we assume that $u \in C^{2,1}(E_T)$ is a solution of (1.4), and with an abuse of notation, we denote by (x,t,u,Du) points in $E_T \times \mathbb{R} \times \mathbb{R}^N$.

Moreover, for the $N \times N$ matrix $(A_{ij}(x,t,u,Du))$, we assume that it is symmetric, positive definite uniformly in $E_T \times \mathbb{R} \times \mathbb{R}^N$ and there exist numbers $0 < \lambda_o \le \Lambda_o$ such that, for some given $\alpha \in \mathbb{R}$

$$\lambda_o(1+|Du|^\alpha)|\boldsymbol{\xi}|^2 \le A_{ij}(x,t,u,Du)\xi_i\xi_j \le \Lambda_o(1+|Du|^\alpha)|\boldsymbol{\xi}|^2.$$

Obviously, $\mathcal{L}_o(u)$ is a special case of $Q_o(u)$. A nonhomogeneous variation of (1.4) is

$$u_t - A_{ij}(x,t,u,Du)u_{x_ix_j} - B(x,t,u,Du) = 0.$$

We set

$$Q(u) = A_{ij}(x,t,u,Du)u_{x_ix_j} + B(x,t,u,Du),$$

and observe that $Q_o(u)$ is the principal part of $Q(u)$.

No smoothness is assumed on the functions A_{ij}, i, $j = 1,2,\dots,N$, and on $B : E_T \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ the basic assumption is

$$\iint_{E_T} \frac{|B(x,t,u,Du)|^{N+1}}{\det[A_{ij}(x,t,u,Du)]}\,dxdt \le C$$

for a given constant C.

1.1.3 Fully Nonlinear Equations

Let $\mathbb{R}^{N\times N}$ denote the space of all real, symmetric, $N \times N$ matrices. Such a space has dimension $\frac{N(N+1)}{2}$.

Let K be the set $E_T \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^{N\times N}$ and let F be a real-valued function from K into $\mathbb{R}$. Consider the evolution equation

$$u_t - F(x,t,u,Du,D^2u) = 0 \quad \text{in } E_T. \tag{1.5}$$

We assume that $u \in C^{2,1}(E_T)$ is a solution of (1.5) in E_T, and with an abuse of notation we denote with (x,t,u,Du,D^2u) points in K. We also assume that $D^2u \to F(x,t,u,Du,D^2u)$ is a.e. differentiable in $\mathbb{R}^{N\times N}$.

Equation (1.5) is parabolic in K in the sense that the matrix given by $(F_{u_{x_ix_j}}(x,t,u,Du,D^2u))$ is non-negative definite in K, i.e., $\forall\,\boldsymbol{\xi} \in \mathbb{R}^N$

$$\lambda(x,t,u,Du,D^2u)|\boldsymbol{\xi}|^2 \le F_{u_{x_ix_j}}\xi_i\xi_j \le \Lambda(x,t,u,Du,D^2u)|\boldsymbol{\xi}|^2 \tag{1.6}$$

for measurable functions λ, $\Lambda : K \to \mathbb{R}$ satisfying $0 \le \lambda \le \Lambda < \infty$ in K.

If there exist $0 < \lambda_o \le \Lambda_o < \infty$ such that

$$0 < \lambda_o \le \lambda(x,t,u,Du,D^2u) \le \Lambda(x,t,u,Du,D^2u) \le \Lambda_o < \infty \tag{1.7}$$

uniformly in K, then (1.5) is uniformly parabolic in K.

The notion of parabolicity or uniform parabolicity is local in K and could be formulated in terms of subsets K' of K. If (1.6) (or (1.7) respectively) hold for all points in $K' \subset K$, we say that (1.5) is parabolic (or, uniformly parabolic respectively) in K'.

Given the a.e. differentiability of F in $\mathbb{R}^{N\times N}$, equation (1.5) could be written in a way that resembles the quasi-linear equations $u_t - Q(u) = 0$, except that now the coefficients A_{ij} also depend on D^2u. Indeed,

$$\begin{aligned} F(x,t,u,Du,D^2u) - F(x,t,u,Du,0) &= \int_0^1 \frac{d}{ds}F(x,t,u,Du,sD^2u)\,ds \\ &= \left[\int_0^1 F_{u_{x_ix_j}}(x,t,u,Du,sD^2u)\,ds\right] u_{x_ix_j}. \end{aligned}$$

Hence, we have

$$\begin{aligned} &u_t - F(x,t,u,Du,D^2u) \\ &\quad = u_t - A_{ij}(x,t,u,Du,D^2u)u_{x_ix_j} - B(x,t,u,Du), \end{aligned}$$

where

$$\begin{aligned} A_{ij}(x,t,u,Du,D^2u) &= \int_0^1 F_{u_{x_ix_j}}(x,t,u,Du,sD^2u)\,ds, \\ B(x,t,u,Du) &= F(x,t,u,Du,0). \end{aligned}$$

Next, we discuss in some detail the fully nonlinear notion, by looking at specific examples.

1.2 The Pucci Equation

Let $\alpha \in (0, \frac{1}{N})$ and denote by $\mathcal{L}_\alpha$ the class of all linear elliptic operators of the type

$$L = \sum_{i,j=1}^{N} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j},$$

where

$$\begin{cases} \alpha|\boldsymbol{\xi}|^2 \le a_{ij}(x)\xi_i\xi_j, & \forall\, \boldsymbol{\xi} \in \mathbb{R}^N \\ \operatorname{tr}(a_{ij}(x)) = 1. \end{cases} \tag{1.8}$$

For $u \in C^{2,1}(E_T)$ consider the quantities

$$M(u(x,t)) = \sup_{L \in \mathcal{L}_\alpha} Lu,$$

$$m(u(x,t)) = \inf_{L \in \mathcal{L}_\alpha} Lu,$$

and the associated parabolic equations

$$\begin{cases} u_t - M(u(x,t)) = 0 & \text{in } E_T, \\ u_t - m(u(x,t)) = 0 & \text{in } E_T. \end{cases}$$

These are called *extremal operators*, and are of the form (1.5) with F a.e. differentiable in K. This will follow from the pointwise representation of $M(u)$ and $m(u)$ in terms of the eigenvalues of the Hessian matrix (D^2u). In fact, we have the following result.

Lemma 1.2 *For any $u \in C^{2,1}(E_T)$, we have*

$$M(u(x,t)) = \alpha\Delta u + (1 - N\alpha)C_N(u), \tag{1.9}$$

$$m(u(x,t)) = \alpha\Delta u + (1 - N\alpha)C_1(u), \tag{1.10}$$

where $C_1(u)$ and $C_N(u)$ are respectively the smallest and the largest eigenvalues of (D^2u).

Proof. We prove (1.9), since the proof for (1.10) is analogous. Let $u \in C^2(E)$ and let $\mathbb{X}$ be the unitary $N \times N$ matrix that diagonalizes the symmetric matrix (D^2u), i.e.,

$$\mathbb{X}^{-1}(D^2u)\mathbb{X} = \begin{pmatrix} C_1(u) & \dots & 0 \\ 0 & \ddots & 0 \\ 0 & \dots & C_N(u) \end{pmatrix},$$

and

$$\mathbb{X}^{-1} = \mathbb{X}^t, \quad \sum_{j=1}^{N} x_{ij}^2 = 1, \; i = 1, 2, \dots, N.$$

Let $L \in \mathcal{L}_\alpha$ and let $(a_{ij}(x))$ be the associated matrix of the coefficients. Then,

$$\begin{aligned}\operatorname{tr}[(a_{ij}(x)) \cdot (u_{x_i x_j})] &= a_{ij}(x) u_{x_i x_j} \\ &= \text{ sum of the eigenvalues of } [(a_{ij}(x)) \cdot (u_{x_i x_j})] \\ &= \text{ sum of the eigenvalues of } [\mathbb{X}^{-1} \cdot (a_{ij}(x)) \cdot (u_{x_i x_j}) \cdot \mathbb{X}] \\ &= \operatorname{tr}[\mathbb{X}^{-1} \cdot (a_{ij}(x)) \cdot (u_{x_i x_j}) \cdot \mathbb{X}].\end{aligned}$$

Therefore,

$$\begin{aligned}a_{ij}(x) u_{x_i x_j} &= \operatorname{tr}\{[\mathbb{X}^{-1} \cdot (a_{ij}(x)) \cdot \mathbb{X}] \cdot [\mathbb{X}^{-1} \cdot (D^2 u) \cdot \mathbb{X}]\} \\ &= \operatorname{tr}\left\{ [\mathbb{X}^{-1} \cdot (a_{ij}(x)) \cdot \mathbb{X}] \cdot \begin{pmatrix} C_1(u) & \dots & 0 \\ 0 & \ddots & 0 \\ 0 & \dots & C_N(u) \end{pmatrix} \right\} \\ &= \sum_{i=1}^N d_{ii} C_i(u),\end{aligned}$$

where d_{ii} are the diagonal elements of $[\mathbb{X}^{-1} \cdot (a_{ij}(x)) \cdot \mathbb{X}]$, and have the following properties

$$(a) \ \sum_{i=1}^N d_{ii} = 1, \qquad (b) \ d_{ii} \geq \alpha, \ \ i = 1, 2, \dots, N.$$

Property (a) follows from

$$\sum_{i=1}^N d_{ii} = \operatorname{tr}[\mathbb{X}^{-1} \cdot (a_{ij}(x)) \cdot \mathbb{X}] = \operatorname{tr}(a_{ij}(x)) = 1.$$

Property (b) follows from (1.8). Indeed,

$$d_{ii} = \sum_{l,k=1}^N a_{lk} x_{ki} x_{li} \geq \alpha \left(\sum_{k=1}^N x_{ki}^2 \right) = \alpha,$$

where we have used that $\sum_{k=1}^N x_{ki}^2 = 1$. Combining these remarks, we have

$$\begin{aligned}(Lu)(x) &= (a_{ij} u_{x_i x_j})(x) \\ &= \sum_{i=1}^N d_{ii}(x) C_i(u) \\ &= \alpha \sum_{i=1}^N C_i(u) + \sum_{i=1}^N (d_{ii} - \alpha) C_i(u) \\ &= \alpha \Delta u + \sum_{i=1}^N (d_{ii} - \alpha) C_i(u),\end{aligned}$$

where we have taken into account that $\sum_{i=1}^{N} C_i(u) = \operatorname{tr}(D^2u) = \Delta u$. Since $d_{ii} - \alpha > 0 \;\forall\, i = 1, 2, \ldots, N$, we have

$$(Lu)(x) \le \alpha\Delta u + \sum_{i=1}^{N}(d_{ii} - \alpha)C_N(u) = \alpha\Delta u + (1 - N\alpha)C_N(u). \tag{1.11}$$

Inequality (1.11) holds for every $L \in \mathcal{L}_\alpha$, and therefore,

$$M[u(x)] \le \alpha\Delta u + (1 - N\alpha)C_N(u) \quad \forall\, u \in C^2(E).$$

Now we show that the operator on the right-hand side of (1.11) belongs to $\mathcal{L}_\alpha$. In order to prove this, we have to find a matrix $(\tilde{a}_{ij}(x))$ such that $\sum_{i=1}^{N} \tilde{a}_{ii}(x) = 1$ for all $x \in E$, and the eigenvalues of $(\tilde{a}_{ij})$ are larger than α. Let $u \in C^2(E)$ be fixed, construct the matrix $x \to \mathbb{X}(x)$ that diagonalizes (D^2u), i.e.,

$$(D^2u) = \mathbb{X} \begin{pmatrix} C_1(u) & \ldots & 0 \\ 0 & \ddots & 0 \\ 0 & \ldots & C_N(u) \end{pmatrix} \mathbb{X}^{-1}$$

and let

$$(\tilde{a}_{ij}(x)) = \mathbb{X}(x) \begin{pmatrix} \alpha & \ldots & 0 \\ 0 & \alpha & \ldots \\ & \vdots & \\ 0 & \ldots & 1 - (N-1)\alpha \end{pmatrix} \mathbb{X}^{-1}(x).$$

Since $x \to \mathbb{X}(x)$ is unitary, we have

$$\operatorname{tr}(\tilde{a}_{ij}(x)) = 1, \qquad \text{and} \qquad \tilde{a}_{ij}(x)\xi_i\xi_j \ge \alpha|\boldsymbol{\xi}|^2, \;\; \forall\, \boldsymbol{\xi} \in \mathbb{R}^N.$$

Therefore, the operator $(\tilde{L}v)(x) = \operatorname{tr}[(\tilde{a}_{ij}(x))(D^2v)]$ belongs to $\mathcal{L}_\alpha$, $v \in C^2(E)$. For $v = u$ we have

$$\begin{aligned} M[u(x)] &\le \alpha\Delta u + (1 - N\alpha)C_N(u) \\ &= \begin{pmatrix} \alpha & \ldots & 0 \\ 0 & \alpha & \ldots \\ & \vdots & \\ 0 & \ldots & 1 - (N-1)\alpha \end{pmatrix} \begin{pmatrix} C_1(u) & \ldots & 0 \\ 0 & \ddots & 0 \\ 0 & \ldots & C_N(u) \end{pmatrix} \\ &= \operatorname{tr} \begin{pmatrix} \alpha & \ldots & 0 \\ 0 & \alpha & \ldots \\ & \vdots & \\ 0 & \ldots & 1 - (N-1)\alpha \end{pmatrix} \begin{pmatrix} C_1(u) & \ldots & 0 \\ 0 & \ddots & 0 \\ 0 & \ldots & C_N(u) \end{pmatrix} \end{aligned}$$

$$= \operatorname{tr}\left\{ \mathbb{X}(x) \begin{pmatrix} \alpha \dots & & 0 \\ 0 \ \ \alpha & & \dots \\ \vdots & & \\ 0 \ \dots & & 1-(N-1)\alpha \end{pmatrix} \mathbb{X}^{-1}(x)(D^2u) \right\}$$

$$= \operatorname{tr}(\tilde{a}_{ij}(x))(D^2u) = \sum_{i,j=1}^{N} \tilde{a}_{ij}(x) u_{x_i x_j} = (\tilde{L}u)(x) \le M[u(x)],$$

whence (1.9) follows. ■

1.3 The Bellman–Dirichlet Equation

Let A be an arbitrary set of indices, and consider the family $\mathcal{F}$ of linear elliptic operators

$$L_\nu[u] = a_{ij}^\nu(x,t) u_{x_i x_j} + b_i^\nu(x,t) u_{x_i} + c^\nu(x,t) u, \quad \nu \in A,$$

where $(x,t) \to a_{ij}^\nu(x,t)$, $b_i^\nu(x,t)$, $c^\nu(x,t)$ are real-valued, measurable functions defined in E, $i,j = 1,2,\dots,N$, $\nu \in A$.

Let $u \in C^{2,1}(E_T)$ and $(x,t) \to f^\nu(x,t)$, $\nu \in A$ be a family of real-valued, measurable functions defined in E_T. Consider the quantity

$$F[u](x,t) = \inf_{\nu \in A} [L_\nu[u](x,t) + f^\nu(x,t)],$$

and the associated parabolic equation

$$u_t - F[u] = 0 \quad \text{in } E_T. \tag{1.12}$$

Equation (1.12) is a parabolic version of the Bellman–Dirichlet equation $F[u] = 0$.

The assumptions on the matrices $(a_{ij}^\nu(x,t))$ are

$$\lambda(x,t)|\boldsymbol{\xi}|^2 \le a_{ij}^\nu(x,t)\xi_i\xi_j \le \Lambda(x,t)|\boldsymbol{\xi}|^2, \quad \forall \boldsymbol{\xi} \in \mathbb{R}^N, \tag{1.13}$$

where $(x,t) \to \lambda(x,t)$, $\Lambda(x,t)$ are given non-negative functions in E_T, such that $0 \le \lambda(x,t) \le \Lambda(x,t)$ for any $(x,t) \in E_T$.

Besides the presence of the lower order terms in the definition of $L_\nu[u]$, $\nu \in A$, the difference between the Bellman–Dirichlet equations and the Pucci equation is that no uniform lower bound on the eigenvalues of $(a_{ij}^\nu(x,t))$, $\nu \in A$, nor conditions on the trace (see (1.8)) are imposed.

For each $\nu \in A$, the function $y^\nu : \mathbb{R}^{N\times N} \to \mathbb{R}$ given by

$$y^\nu(D^2u) = a_{ij}^\nu u_{x_i x_j} + b_i^\nu u_{x_i} + c^\nu u + f^\nu$$

is affine in $\mathbb{R}^{N\times N}$, and therefore, the function G defined by

$$G(D^2u) = \inf_{\nu \in A} y^\nu(D^2u)$$

is a.e. differentiable in $\mathbb{R}^{N\times N}$.

Let (η) be a $N\times N$, nontrivial, positive semi-definite matrix in $\mathbb{R}^{N\times N}$. Then,

$$\begin{aligned} G(D^2u+\eta)-G(D^2u) &= \inf_{\nu\in A} y^\nu(D^2u+\eta) - \inf_{\nu\in A} y^\nu(D^2u) \\ &\geq \inf_{\nu\in A}(a_{ij}^\nu(x,t)(u_{x_ix_j}+\eta_{ij}) - a_{ij}^\nu(x,t)u_{x_ix_j}) \\ &\geq \inf_{\nu\in A} a_{ij}^\nu(x,t)\eta_{ij} = \inf_{\nu\in A}\operatorname{tr}[(a_{ij}^\nu(x,t))\cdot(\eta)]. \end{aligned}$$

Suppose now that (η) is of the form $(\eta_{ij})=(\xi_i\xi_j)\epsilon^2$, where $\epsilon\boldsymbol{\xi}\in\mathbb{R}^N$. Then, for a.e. $(D^2u)\in\mathbb{R}^{N\times N}$

$$G(D^2u+\eta)-G(D^2u)\geq \epsilon^2|\boldsymbol{\xi}|^2\lambda(x,t),$$

and

$$G_{u_{x_ix_j}}(D^2u+\theta(\epsilon)\epsilon^2(\xi_i\xi_j))\epsilon^2\xi_i\xi_j \geq \epsilon^2|\boldsymbol{\xi}|^2\lambda(x,t),$$

where $\theta(\epsilon)\in(0,1)$ and $\theta(\epsilon)\searrow 0$ as $\epsilon\to 0$. Letting $\epsilon\to 0$ for a.e. $(D^2u)\in\mathbb{R}^{N\times N}$ we have

$$F_{u_{x_ix_j}}(x,t,u,Du,D^2u)\xi_i\xi_j \geq \lambda(x,t)|\boldsymbol{\xi}|^2, \quad \forall\boldsymbol{\xi}\in\mathbb{R}^N.$$

Likewise,

$$G(D^2u+\eta)-G(D^2u)\leq \sup_{\nu\in A}(a_{ij}^\nu(x,t)\epsilon^2\xi_i\xi_j)\leq \Lambda(x,t)\epsilon^2|\boldsymbol{\xi}|^2, \quad \forall\boldsymbol{\xi}\in\mathbb{R}^N.$$

Therefore, $\forall\boldsymbol{\xi}\in\mathbb{R}^N$

$$\lambda(x,t)|\boldsymbol{\xi}|^2\leq F_{u_{x_ix_j}}(x,t,u,Du,D^2u)\xi_i\xi_j\leq \Lambda(x,t)|\boldsymbol{\xi}|^2.$$

This proves that (1.12) is parabolic. These remarks suggest a way of generalizing the concept of ellipticity to nondifferentiable F.

1.4 Remarks on the Concept of Ellipticity

Definition 1.1. *We say that* $F[u]=F(x,t,u,Du,D^2u)$ *is* increasing *in* $\mathbb{R}^{N\times N}$ *if*

$$F(\cdot,D^2u+\eta)>F(\cdot,D^2u), \qquad (D^2u)\in\mathbb{R}^{N\times N}$$

for every nontrivial, positive semi-definite $N\times N$ *matrix* $\eta\in\mathbb{R}^{N\times N}$ *(in particular, symmetric).*

We have the following.

Lemma 1.3 *Suppose* $F[u]=F(x,t,u,Du,D^2u)$ *is elliptic in the sense that* $D^2u\to F(x,t,u,Du,D^2u)$ *is a.e. differentiable in* $\mathbb{R}^{N\times N}$ *and* (1.6) *holds. Then,* $D^2u\to F(\cdot,D^2u)$ *is increasing in* $\mathbb{R}^{N\times N}$.

Proof. If $(D^2u) \in \mathbb{R}^{N\times N}$ is a point where F is differentiable, there exists $\epsilon > 0$ sufficiently small, such that

$$F(\cdot, D^2u + \eta) - F(\cdot, D^2u) = F_{u_{x_i x_j}}(\cdot, D^2u)\eta_{ij} + o(\epsilon)$$

for every matrix (η) such that $|\eta| \le \epsilon$. Let (η) be symmetric, positive semi-definite and nontrivial in $\mathbb{R}^{N\times N}$, and let $\mathbb{Y}$ be the unitary matrix that diagonalizes η, i.e.,

$$\mathbb{Y}^t \cdot (\eta) \cdot \mathbb{Y} = \begin{pmatrix} \eta_1 & \dots & 0 \\ 0 & \ddots & 0 \\ 0 & \dots & \eta_N \end{pmatrix}, \qquad \eta_i \ge 0.$$

If (η) is nontrivial, there exists at least one index $1 \le j \le N$, such that for the corresponding eigenvalue $\eta_j > 0$. Then,

$$\begin{aligned} F(\cdot, D^2u + \eta) - F(\cdot, D^2u) &= \operatorname{tr}\Big\{[\mathbb{Y}^t \cdot (F_{u_{x_i x_j}}) \cdot \mathbb{Y}] \cdot [\mathbb{Y}^t \cdot (\eta) \cdot \mathbb{Y}]\Big\} + o(\epsilon) \\ &= g_{ii}\eta_i + o(\epsilon), \end{aligned}$$

where g_{ii} are the diagonal elements of $[\mathbb{Y}^t \cdot (F_{u_{x_i x_j}}) \cdot \mathbb{Y}]$, and the elements $F_{u_{x_i x_i}}$ are larger than or equal to the smallest eigenvalue of $(F_{u_{x_i x_j}})$. Taking into account that $\mathbb{Y}$ is unitary, it easily follows that for ϵ small enough

$$F(\cdot, D^2u + \eta) - F(D^2u) > 0$$

for any nontrivial, positive semi-definite $(\eta) \in \mathbb{R}^{N\times N}$. ∎

If $D^2u \to F(x,t,u,Du,D^2u)$ is not differentiable in $\mathbb{R}^{N\times N}$, then Definition 1.1 can be taken as the definition of ellipticity.

1.5 Equations of Mini-Max Type

Let A, B be arbitrary set of indices and consider the family $\mathcal{F}$ of linear elliptic operators

$$L_{\alpha,\beta}[u] = a_{ij}^{\alpha,\beta}(x,t)u_{x_i x_j} + b_i^{\alpha,\beta}(x,t)u_{x_i} + c^{\alpha,\beta}u,$$

where $(x,t) \to a_{ij}^{\alpha,\beta}(x,t)$, $b_i^{\alpha,\beta}(x,t)$, $c^{\alpha,\beta}(x,t)$ are real-valued, measurable functions defined in E_T for i, $j = 1, 2, \dots, N$, $(\alpha,\beta) \in A \times B$.

Let $u \in C^{2,1}(E_T)$ and $(x,t) \to f^{\alpha,\beta}(x,t)$, $(\alpha,\beta) \in A \times B$ be a family of real-valued, bounded functions defined in E_T. Consider the quantity

$$F[u](x,t) = \inf_{\beta\in B}\sup_{\alpha\in A}[L_{\alpha,\beta}[u](x,t) + f^{\alpha,\beta}(x,t)] \tag{1.14}$$

and the associated evolution equation

$$u_t - F[u] = 0 \qquad \text{in } E_T. \tag{1.15}$$

If we assume that the matrices $(a_{ij}^{\alpha,\beta}(x,t)) \in \mathbb{R}^{N\times N}$ and satisfy (1.13) for any $(\alpha,\beta) \in A \times B$, then (1.14) is elliptic, and (1.15) is parabolic. This can be shown as in Section 1.3.

One of the reasons to consider such mini-max type of equations is that nearly all fully nonlinear elliptic and parabolic partial differential equations can be written in the form (1.15). We refrain from going any further in detail about this topic here.

2 Maximum Principles

2.1 Linear Equations

Consider the linear elliptic operator

$$\mathcal{L}(u) = a_{ij}(x,t)u_{x_ix_j} - a_i(x,t)u_{x_i} - a_o(x,t)u,$$

and the associated linear parabolic equation

$$u_t - \mathcal{L}(u) = f(x,t) \quad \text{in } E_T.$$

The assumptions on $(a_{ij}(x,t))$, and $a_i(x,t)$, $i = 0,1,\dots,N$, and $f(x,t)$ are

H1) The functions $(x,t) \to a_{ij}(x,t)$, $a_i(x,t)$, $a_o(x,t)$, $f(x,t)$ are only bounded and measurable in E_T.

H2) There exist $0 < \lambda_o \le \Lambda_o < \infty$ such that

$$\lambda_o|\boldsymbol{\xi}|^2 \le a_{ij}(x,t)\xi_i\xi_j \le \Lambda_o|\boldsymbol{\xi}|^2 \quad \forall\, \boldsymbol{\xi} \in \mathbb{R}^N.$$

H3) We have

$$\|a_{ij}\|_{\infty;E_T} + \|a_i\|_{\infty;E_T} \le A$$

for a given constant A.

For $s \in (0,T]$ we let $E_s \equiv E \times (0,s)$, $S_s \equiv \displaystyle\bigcup_{\tau\in(0,s]} \partial E \times \{\tau\}$, and $\Gamma_s \equiv S_s \cup (E \times \{0\})$. Clearly, Γ_s is the parabolic boundary of E_s. If $s \in (0,T]$ is fixed, we let

$$A_o = \|a_o^-\|_{\infty;E_s}, \quad a_o^- \equiv \max\{0; -a_o\}. \tag{2.1}$$

2.1.1 The Dirichlet Problem

Let $(x,t) \to h(x,t) \in L^\infty(S_T)$ and $x \to u_o(x) \in L^\infty(E)$. Consider the Cauchy–Dirichlet problem

$$\begin{cases} u_t - \mathcal{L}(u) = f & \text{in } E_T \\ \quad u(x,t) = h(x,t) & \text{on } S_T \\ \quad u(x,0) = u_o(x) & \text{in } E. \end{cases} \tag{D}$$

We further assume that $h \in C(\bar{S}_T)$, $u_o \in C(\bar{E})$, and $u_o(x) = h(x,0)$ when $x \in \partial E$. We let $u \in C^{2,1}(E_T) \cap C(\bar{E}_T)$ be a classical solution of (D).

In order to unify the notation, we set

$$\psi(x,t) = \begin{cases} h(x,t) & \text{on } S_T, \\ u_o(x) & \text{in } E \times \{0\}. \end{cases}$$

Theorem 2.1. *Let $u \in C^{2,1}(E_T) \cap C(\bar{E}_T)$ be a classical solution of (D) and let H1), H2), H3) hold. Then, $\forall\, s \in (0,T]$*

$$\begin{aligned} &\sup_{\lambda > A_o} \min \left\{ 0; \min_{\Gamma_s} \psi\, e^{\lambda(s-t)}; \frac{1}{\lambda - A_o} \min_{E_s} f e^{\lambda(s-t)} \right\} \\ &\qquad \le u(x,s) \le \inf_{\lambda > A_o} \max \left\{ 0; \max_{\Gamma_s} \psi\, e^{\lambda(s-t)}; \frac{1}{\lambda - A_o} \max_{E_s} f e^{\lambda(s-t)} \right\}. \end{aligned} \tag{2.2}$$

Proof. Let $\lambda \in \mathbb{R}$ be chosen and let $(x,t) \to v(x,t)$ be defined by the exponential shift

$$u(x,t) = v(x,t) e^{\lambda t}.$$

Then, v satisfies

$$\begin{cases} v_t - a_{ij}(x,t) v_{x_i x_j} + a_i(x,t) v_{x_i} + (a_o(x,t) + \lambda) v = f e^{\lambda t} & \text{in } E_T, \\ v = \psi e^{\lambda t} \quad \text{on } \Gamma_T. \end{cases} \tag{D'}$$

If $v(x,t) \le 0$ in E_s or if $0 \le \max_{E_s} v(x,t) \le \max_{\Gamma_s} v(x,t)$, then there is nothing to prove.

If $0 < \max_{E_s} v(x,t) = v(x_o,t_o)$ for $(x_o,t_o) \in \mathring{E}_s \equiv \mathring{E} \times (0,s]$, then

$$v_t(x_o,t_o) \ge 0, \quad v_{x_i}(x_o,t_o) = 0, \quad \text{and } \; -a_{ij}(x_o,t_o) v_{x_i x_j}(x_o,t_o) \ge 0,$$

since $(-D^2 u(x_o,t_o))$ is a symmetric, positive definite $N \times N$ matrix.

Therefore, from (D') calculated at (x_o,t_o)

$$(a_o(x_o,t_o) + \lambda) v(x_o,t_o) \le \max_{E_s} f(x,t) e^{\lambda s},$$

and $\forall\, (x,t) \in E_s$

$$u(x,t) \le \inf_{\lambda > A_o} \frac{1}{\lambda - A_o} \max_{E_s} f(x,t) e^{\lambda(s-t)}.$$

The estimate below is proved analogously. ■

Remark 2.1 To prove the estimate above it could be enough to have a classical sub-solution of (D), i.e.,

$$\begin{cases} u \in C^{2,1}(E_T) \cap C(\bar{E}_T), \\ u_t - \mathcal{L}(u) \le f \quad \text{in } E_T, \\ u\Big|_{\Gamma_T} = \psi. \end{cases} \tag{D$_{sub}$}$$

Analogously, to prove the estimate below, it would suffice to have a classical super-solution of (D), i.e.

$$\begin{cases} u \in C^{2,1}(E_T) \cap C(\bar{E}_T), \\ u_t - \mathcal{L}(u) \ge f \quad \text{in } E_T, \\ u\Big|_{\Gamma_T} = \psi. \end{cases} \tag{D$_{super}$}$$

Let $u \in C^{2,1}(E_T) \cap C(\bar{E}_T)$ be a classical solution of (D) and $\forall\, s \in (0,T]$ set

$$M_1(s) = \|\psi e^{-A_o t}\|_{\infty;\Gamma_s}, \quad M_2(s) = \|f e^{-A_o t}\|_{\infty;E_s}.$$

Consider the functions

$$w^{\pm}(x,t) = M_1 + M_2 t \pm u(x,t)e^{-A_o t}, \quad (x,t) \in E_s.$$

By direct computation

$$\begin{aligned} w_t^{\pm} - \mathcal{L}(w^{\pm}) &= \pm f e^{-A_o t} + M_2 + a_o(x,t)[M_1 + M_2 t] \mp A_o u e^{-A_o t} \\ &\ge -A_o w^{\pm}, \end{aligned}$$

and therefore,

$$w_t^{\pm} - a_{ij}(x,t) w^{\pm}_{x_i x_j} + a_i(x,t) w^{\pm}_{x_i} + (a_o + A_o) w^{\pm} \ge 0.$$

Since $a_o + A_o \ge 0$ (see (2.1)) and since

$$w^{\pm}\Big|_{\Gamma_s} = M_1(s) + M_2(s)t \pm \psi(x,s)e^{-A_o s} \ge 0,$$

from Theorem 2.1 it follows that $w^{\pm} \ge 0$ in E_T and we have the following estimate.

Lemma 2.1 *Let $u \in C^{2,1}(E_T) \cap C(\bar{E}_T)$ be a classical solution of (D) in E_T. Then, $\forall (x,s) \in E_T$*

$$|u(x,s)| \le \|\psi e^{A_o(s-t)}\|_{\infty;\Gamma_s} + \|f e^{A_o(s-t)}\|_{\infty;E_s}.$$

As a simple consequence of Theorem 2.1 we have

Corollary 2.1 *Let $u \in C^{2,1}(E_T) \cap C(\bar{E}_T)$ be a classical solution of (D). Then, $\psi \le 0$ on Γ_T and $f \le 0$ in E_T imply $u(x,t) \le 0$ for all $(x,t) \in E_T$. Analogously, $\psi \ge 0$ on Γ_T and $f \ge 0$ in E_T imply $u(x,t) \ge 0$ for all $(x,t) \in E_T$. Finally, if $f \equiv a_o \equiv 0$, then*

$$\min_{\Gamma_t} \psi \le u(x,t) \le \max_{\Gamma_t} \psi.$$

All these estimates follow from (2.2); for the last one, first take $\lambda > 0$ and then let λ tend to zero.

2.1.2 The Neumann Problem

We assume that ∂E is of class C^2 and has outer unit normal

$$\boldsymbol{\nu} = (\nu_1(x), \nu_2(x), \dots, \nu_N(x)) \quad \text{along } \partial E.$$

Consider the problem

$$\begin{cases} u_t - \mathcal{L}(u) = f & \text{in } E_T \\ u(x,0) = u_o(x) & \text{in } E \\ \nu_i(x) u_{x_i}(x,t) + b(x,t)u = \psi(x,t) & \text{on } S_T. \end{cases} \tag{N}$$

As before, we assume $u_o \in C(\bar{E})$, $\psi \in C(S_T)$, and we suppose that $u \in C^{2,1}(E_T) \cap C^1(\bar{E} \times (0,T])$ is a classical solution of (N).

Strictly speaking, (N) is the Cauchy–Neumann problem only if $b(x,t) \equiv 0$. In order to derive a priori estimates, we first consider the case $b > 0$, and then the case $b(x,t) \ge -b_o$ with $b_o \ge 0$, to conclude the case of the Cauchy–Neumann problem.

H4) We assume that $b(x,t) \ge b_o > 0$ for all $(x,t) \in S_T$ and for some given $b_o > 0$.

Theorem 2.2. *Let $u \in C^{2,1}(E_T) \cap C^1(\bar{E} \times (0,T])$ be a classical solution of (N) and let H1), H2), H3), and H4) hold. Then, $\forall\, (x,s) \in E_T$*

$$\begin{aligned} &\sup_{\lambda > A_o} \min \left\{ 0; \min_{S_s} \frac{\psi\, e^{\lambda(s-t)}}{b(x,t)}; \min_E u_o(x) e^{\lambda s}; \frac{1}{\lambda - A_o} \min_{E_s} f e^{\lambda(s-t)} \right\} \\ &\le u(x,s) \\ &\le \inf_{\lambda > A_o} \max \left\{ 0; \max_{S_s} \frac{\psi\, e^{\lambda(s-t)}}{b(x,t)}; \max_E u_o(x) e^{\lambda s}; \frac{1}{\lambda - A_o} \max_{E_s} f e^{\lambda(s-t)} \right\}. \end{aligned}$$

Proof. As before, introduce

$$v(x,t) = u(x,t) e^{-\lambda t},$$

and analyze the location of the maximum and the minimum of v in E_T. The previous argument can be repeated here, except when the extremum is on S_s. In such a case, one uses the data on S_s, observing that at a maximum point on S_s $u_{x_i}\nu_i \ge 0$, and at a minimum point $u_{x_i}\nu_i \le 0$. ■

Now we deal with the general case.

H5) We assume that $b(x,t) \ge -b_o$, for all $(x,t) \in S_T$ and for some given $b_o > 0$.

Theorem 2.3. *Let $u \in C^{2,1}(E_T) \cap C^1(\bar{E} \times (0,T])$ be a classical solution of (N) and let H1), H2), H3), and H5) hold. Then there exist constants C_1, C_2 depending only on A, A_o, b_o such that*

$$\|u\|_{\infty;E_T} \le C_1 e^{C_2 T} \max\left\{ \|\psi\|_{\infty;S_T}; \|u_o\|_{\infty;E}; \|f\|_{\infty;E_T} \right\}.$$

Proof. Construct a function $x \to \varphi(x) \in C^2(\bar{E})$ such that

i) $\varphi(x) \ge \frac{1}{2}$ for all $x \in E$;
ii) $\varphi\big|_{\partial E} = 1$;
iii) $-\frac{\partial\varphi}{\partial\nu} = -(\varphi_{x_i}\nu_i) = m \ge (b_o + 1)$ on ∂E.

Such a function can obviously be constructed if ∂E is smooth (say, of class C^2). Consider the auxiliary function

$$w(x,t) = u(x,t)\varphi(x).$$

By direct calculation we have

$$\begin{aligned} w_t - \mathcal{L}(w) =& \varphi(x)[u_t - \mathcal{L}(u)] + a_j(x,t)u\varphi_{x_j} + 2a_{ij}(x,t)u_{x_i}\varphi_{x_j} \\ & - a_{ij}(x,t)\varphi_{x_ix_j}u \\ =& \varphi(x)f(x,t) + \Phi_i(x,t)w_{x_i} + \Phi_o(x,t)w, \end{aligned}$$

where

$$\Phi_i = 2a_{ij}\frac{\varphi_{x_j}}{\varphi}, \quad \Phi_o = -\left(\frac{a_{ij}\varphi_{x_ix_j}}{\varphi} + \frac{a_{ij}\varphi_{x_i}\varphi_{x_j}}{\varphi}\right) + \frac{a_i\varphi_{x_i}}{\varphi}.$$

It follows that w satisfies in E_T

$$w_t - a_{ij}(x,t)w_{x_ix_j} + A_i(x,t)w_{x_i} + A_o(x,t)w = \varphi(x)f(x,t),$$

with

$$A_i(x,t) = [a_i(x,t) - \Phi_i(x,t)], \qquad A_o = [a_o(x,t) - \Phi_o(x,t)].$$

Moreover, on S_T

$$\frac{\partial w}{\partial\nu} + bw = \varphi\left(\frac{\partial u}{\partial\nu} + bu\right) + u\frac{\partial\varphi}{\partial\nu} = \psi\varphi + \frac{w}{\varphi}\frac{\partial\varphi}{\partial\nu},$$

i.e.,

$$\frac{\partial w}{\partial\nu} + \left(b - \frac{\partial\varphi}{\partial\nu}\right)w = \psi\varphi \quad \text{on } S_T.$$

Since $b - \dfrac{\partial\varphi}{\partial\nu} \ge 1$, the result follows from Theorem 2.1. ■

2.2 Quasi-Linear Equations

Consider the elliptic operator

$$Q(u) = A_{ij}(x,t,u,Du)u_{x_ix_j} - B(x,t,u,Du)$$

and the associated parabolic equation

$$u_t - Q(u) = 0 \qquad \text{in } E_T. \tag{2.3}$$

The assumptions on $Q(u)$ are

Q1) $(x,t,z,\mathbf{p}) \to A_{ij}(x,t,z,\mathbf{p})$, $B(x,t,z,\mathbf{p})$ are bounded and measurable in $E_T \times \mathbb{R} \times \mathbb{R}^N$.

Q2) $\lambda_o|\boldsymbol{\xi}|^2 \le A_{ij}(x,t,z,\mathbf{p})\xi_i\xi_j \le \Lambda_o|\boldsymbol{\xi}|^2$, $\forall \boldsymbol{\xi} \in \mathbb{R}^N$, with $0 < \lambda_o \le \Lambda_o < \infty$, $\forall\,(x,t,z,\mathbf{p}) \in E_T \times \mathbb{R} \times \mathbb{R}^N$.

Q3) $uB(x,t,u,0) \ge -B_o - B_1u^2$ for two given non-negative constants B_o, B_1.

Remark 2.2 Regarding Q2), in what follows it will be enough to assume that the matrix $(A_{ij}(x,t,u,0))$ is non-negative definite.

Remark 2.3 Assumption Q3) would be satisfied if we had, for example,

Q3') $|B(x,t,u,Du)| \le A_o + A_1|u| + A_2|Du|$ for given non-negative constants A_l, $l = 0,1,2$.

Indeed, in such a case

$$uB(x,t,u,0) \ge -A_1u^2 - A_ou \ge -\left(A_1^2 + \frac{1}{2}\right)u^2 - \frac{1}{2}A_o^2.$$

2.2.1 The Dirichlet Problem

Consider the Cauchy–Dirichlet problem

$$\begin{cases} u_t - Q(u) = 0 & \text{in } E_T, \\ u = \psi & \text{on } \Gamma_T, \end{cases} \tag{$\mathcal{D}$}$$

where $(x,t) \to \psi(x,t) \in C(\overline{\Gamma}_T)$.

We assume that $(\mathcal{D})$ has a solution $u \in C^{2,1}(E_T) \cap C(\bar{E}_T)$. The same technique of proof of Theorem 2.1 gives the following result.

Theorem 2.4. *Let $u \in C^{2,1}(E_T) \cap C(\bar{E}_T)$ be a classical solution of $(\mathcal{D})$. Then, $\forall\, s \in (0,T]$*

$$\sup_{\lambda > B_1} \min\left\{0;\ \min_{(x,\tau)\in\Gamma_s} \psi(x,\tau)\,e^{\lambda(s-\tau)};\ -e^{\lambda s}\sqrt{\frac{B_o}{\lambda - B_o}}\right\}$$
$$\le u(x,s) \le \inf_{\lambda > B_1} \max\left\{0;\ \max_{(x,\tau)\in\Gamma_s} \psi(x,\tau)\,e^{\lambda(s-\tau)};\ e^{\lambda s}\sqrt{\frac{B_o}{\lambda - B_o}}\right\}.$$

Moreover,

$$\|u\|_{\infty;E_T} \le \inf_{\lambda > B_1} e^{\lambda T}\left\{\|\psi\|_{\infty;\Gamma_T};\ \sqrt{\frac{B_o}{\lambda - B_o}}\right\}.$$

We have already remarked that Q3) would be implied by Q3'). Suppose now that Q3') is replaced by

Q3") $|B(x,t,u,Du)| \le A_o + A_1|u|^{1+\alpha} + A_2|Du|$ for some $\alpha > 0$.

This means that the lower order terms in $Q(u)$ grow faster than linearly with respect to u. From Q3") we find

$$uB(x,t,u,0) \ge -\left(A_1 + \frac{1}{1+\alpha}\right)|u|^{1+\alpha}|u| - \frac{\alpha}{1+\alpha}A_o^{\frac{1+\alpha}{\alpha}}.$$

In general, we may consider lower order terms satisfying

$$uB(x,t,u,0) \ge -\varPhi(|u|)|u| - B_o,$$

where $B_o \ge 0$ and $s \to \varPhi(s)$ is a nondecreasing, positive function in $\mathbb{R}_+$ and

$$\varPhi(s) \le Cs^{1+\alpha}, \qquad \text{for some } \alpha > 0.$$

In such a case the method of proof of Theorems 2.1 and 2.4 would not apply, since at a positive interior maximum (for example) for $e^{\lambda t}u(x,t)$ we would be led to the inequality

$$e^{-\lambda t}(\lambda - \varPhi(u))u \le B_o,$$

which would not imply an estimate for u.

On the other hand, it is conceivable that under some integrability conditions on u, an upper bound could be derived. This will be accomplished by the Alexandrov maximum principle.

Let us now consider the case when $B(x,t,u,0)$ grows less than linearly with respect to u, that is, we assume

$$uB(x,t,u,0) \ge -\varPhi(|u|)|u| - B_o, \tag{2.4}$$

where $s \to \varPhi(s)$ is nondecreasing in $\mathbb{R}_+$ and

$$\int_0^\infty \frac{1}{\varPhi(s)}\,ds = \infty.$$

Without loss of generality we may assume

$$\varPhi(0) > 0. \tag{2.5}$$

Let φ be a twice differentiable, invertible function in $\mathbb{R}_+$ and set $u = \varphi(v)$. Writing (2.3) in terms of v, we find

$$\begin{aligned} v_t &- A_{ij}(x,t,u,Du)v_{x_ix_j} \\ &- A_{ij}(x,t,u,Du)v_{x_i}v_{x_j}\frac{\varphi''(v)}{\varphi'(v)} + B(x,t,u,Du)\frac{1}{\varphi'(v)} = 0. \end{aligned}$$

Next we make an exponential shift

$$w = e^{-\lambda t}v,$$

and we find that w satisfies

$$\begin{aligned} &w_t - A_{ij}(x,t,u,Du)w_{x_ix_j} \\ &\quad - A_{ij}(x,t,u,Du)e^{\lambda t}w_{x_i}w_{x_j}\frac{\varphi''(v)}{\varphi'(v)} + \lambda w + B(x,t,u,Du)\frac{e^{-\lambda t}}{\varphi'(v)} = 0. \end{aligned}$$

At an interior maximum point (x_o, t_o) for w

$$[B(x,t,u,0) + \lambda v\varphi'(v)]_{(x_o,t_o)} \le 0. \tag{2.6}$$

Assume that

$$u(x_o,t_o) \ge 0.$$

Then, multiplying (2.6) by $u(x_o,t_o)$ and making use of (2.4) yields

$$[-\Phi(u) + \lambda v\varphi'(v)]_{(x_o,t_o)}\, u(x_o,t_o) \le B_o. \tag{2.7}$$

Next we choose φ so that $s\varphi'(s) = \Phi(\varphi(s))$, i.e.,

$$\begin{cases} \displaystyle\int_0^{\varphi(s)} \frac{1}{\Phi(\tau)}\,d\tau = \ln s, \quad s \ge 1 \\ \varphi(1) = 0. \end{cases} \tag{2.8}$$

Remark 2.4 The function $s \to \varphi(s)$ is increasing and for $s \in (0,\infty)$ the range of φ is $(-\infty,\infty)$.

For the given choice of φ, we find from (2.7)

$$(\lambda - 1)u\,\Phi(u)\Big|_{(x_o,t_o)} \le B_o, \qquad \lambda > 1,$$

and since $\Phi(0) > 0$

$$u(x_o,t_o) \le \frac{B_o}{(\lambda-1)\Phi(0)}.$$

Now,

$$\max_{E_T} w(x,t) \le w(x_o,t_o) = \varphi^{-1}(u(x_o,t_o))e^{-\lambda t_o} \le \varphi^{-1}\left(\frac{B_o}{(\lambda-1)\Phi(0)}\right)e^{-\lambda t_o},$$

that is,

$$\max_{E_T} v(x,t) \le \max\left\{1; e^{\lambda T}\varphi^{-1}\left(\frac{B_o}{(\lambda-1)\Phi(0)}\right)\right\}.$$

Consequently, arguing also with $-u$ replacing u, we have the following.

Theorem 2.5. *Let $u \in C^{2,1}(E_T) \cap C(\bar{E}_T)$ be a classical solution of $(\mathcal{D})$ and let Q1), Q2),* (2.4)–(2.5) *hold. Then,*

$$\|u\|_{\infty;E_T} \le \inf_{\lambda>1} \varphi(\xi),$$

where

$$\xi \overset{def}{=} e^{\lambda T}\max\left\{1; \varphi^{-1}\left(\frac{B_o}{(\lambda-1)\Phi(0)}\right); \varphi^{-1}\left(\|\psi\|_{\infty;\Gamma_T}\right)\right\},$$

and φ^{-1} is the inverse function of $s \to \varphi(s)$ with $s > 0$, defined implicitly by (2.8)*.*

2.2.2 Variational Boundary Data

We assume that ∂E is of class C^2, denote by $\boldsymbol{\nu}$ the outer unit normal to ∂E, and consider the Cauchy–Neumann-type problem

$$\begin{cases} u_t - Q(u) = 0 \quad \text{in } E_T, & (2.9)\\ A_{ij}(x,t,u,Du)u_{x_j}\cos(\boldsymbol{\nu},x_i) + \psi(x,t,u)\Big|_{S_T} = 0, & (2.10)\\ u(x,0) = u_o(x), \quad x \in E. & (2.11)\end{cases}$$

Here we assume that Q1), Q2), Q3') hold and in addition on the datum $\psi(x,t,u)$ we impose

$$\begin{cases} \psi \text{ is continuous in } \bar{S}_T \times \mathbb{R} \text{ and} \\ u\,\psi(x,t,u) \ge -C_o - C_1 u^2 \text{ for } (x,t) \in S_T \end{cases} \tag{2.12}$$

for two given constants C_o, C_1. Finally, we require

$$u_o \in C(\bar{E}). \tag{2.13}$$

We have the following.

Theorem 2.6. *Let* $u \in C^{2,1}(E_T) \cap C^{1,0}(\bar{E} \times (0,T)) \cap C(\bar{E}_T)$ *be a classical solution of* (2.9)–(2.11), *and let* (2.12)–(2.13) *hold. Then,*

$$\|u\|_{\infty;E_T} \le \lambda_1 e^{\lambda T} \max\left\{\sqrt{A_o + A_1^2 + A_2^2};\, \|u_o\|_{\infty;E}\right\}, \tag{2.14}$$

where λ, λ_1 *are constants depending only upon* λ_o, Λ_o, A_3, C_1.

Proof. Construct a function $x \to \varphi(x) \in C^2(\bar{E})$ and determine a positive number λ such that

$$\begin{cases} \varphi(x) \ge 1, \quad \forall\, x \in \bar{E} \\ -A_{ij}(x,t,u,Du)\cos(\boldsymbol{\nu},x_i)\varphi_{x_j}(x)\Big|_{S_T} \ge 2C_1, \end{cases} \tag{2.15}$$

and

$$\min_{\substack{u\in\mathbb{R}\\ Du\in\mathbb{R}^N}} \left\{\lambda + A_{ij}(x,t,u,Du)\frac{\varphi_{x_ix_j}}{\varphi} - 2A_{ij}(x,t,u,Du)\frac{\varphi_{x_i}\varphi_{x_j}}{\varphi^2}\right\} \ge 1. \tag{2.16}$$

Remark 2.5 Since the matrix (A_{ij}) is positive definite, the vector of jth component $\sum_{i=1}^{N} A_{ij}\cos(\boldsymbol{\nu},x_i)$ forms an acute angle with $\boldsymbol{\nu}$.

The construction of such a φ is obvious if ∂E is of class C^2. Letting

$$w(x,t) = e^{-\lambda t}\varphi(x)u(x,t),$$

we find by direct calculation

$$\begin{aligned}&w_t - A_{ij}(x,t,u,Du)w_{x_ix_j}\\&+\left\{\lambda + A_{ij}(x,t,u,Du)\frac{\varphi_{x_ix_j}}{\varphi} - 2A_{ij}(x,t,u,Du)\frac{\varphi_{x_i}\varphi_{x_j}}{\varphi^2}\right\}w\\&+e^{-\lambda t}\varphi B(x,t,u,Du) + 2A_{ij}(x,t,u,Du)\frac{\varphi_{x_i}}{\varphi}w_{x_j} = 0,\end{aligned} \tag{2.17}$$

$$\begin{aligned}&A_{ij}(x,t,u,Du)w_{x_j}\cos(\boldsymbol{\nu},x_i) - wA_{ij}(x,t,u,Du)\frac{\varphi_{x_j}}{\varphi}\cos(\boldsymbol{\nu},x_i)\\&+e^{-\lambda t}\varphi(x)\psi(x,t,u)\Big|_{S_T} = 0,\end{aligned} \tag{2.18}$$

and

$$w(x,0) = u_o(x)\varphi(x).$$

Let $(x_o,t_o)\in\bar{E}_T$ be a point where $w^2(x,t)$ achieves its maximum. If $t_o=0$ estimate (2.14) follows. If $t_o>0$ and $x_o\in\partial E$, multiplying (2.18) by $w(x_o,t_o)$ and observing that $ww_{x_j} = \frac{1}{2}(w^2)_{x_j}$, we have $A_{ij}(w^2)_{x_j}\cos(\boldsymbol{\nu},x_i)\ge 0$ and using (2.15) and (2.12) we find

$$2C_1w^2(x_o,t_o) - C_1e^{2\lambda t_o}\varphi^2(x_o)u^2(x_o,t_o) - C_oe^{-2\lambda t_o}\varphi^2(x_o)\le 0,$$

that is,

$$|w(x_o,t_o)|\le\sqrt{C_o}\|\varphi\|_{\infty;E},$$

and therefore, (2.14) follows with $\lambda_1 = \|\varphi\|_{\infty;E}$.

If $(x_o,t_o)\in\mathring{E}_T$, then multiply (2.17) by $w(x_o,t_o)$ and observe that at (x_o,t_o)

$$ww_t = \frac{1}{2}(w^2)_t = 0,\quad ww_{x_i} = \frac{1}{2}(w^2)_{x_i} = 0,\quad -wA_{ij}w_{x_ix_j}\ge 0.$$

Moreover, using the choice of λ in (2.16)

$$w^2(x_o,t_o) - e^{-\lambda t_o}\varphi^2(x_o)\left[A_o + A_1|u(x_o,t_o)| + A_2|Du(x_o,t_o)|\right]\le 0. \tag{2.19}$$

At (x_o,t_o) we have

$$0 = Dw(x_o,t_o) = e^{-\lambda t_o}\left[u(x_o,t_o)D\varphi(x_o,t_o) + \varphi(x_o,t_o)Du(x_o,t_o)\right],$$

and therefore,

$$|Du|(x_o,t_o)\le|u|\frac{|D\varphi|}{\varphi}\Big|_{(x_o,t_o)}.$$

Consequently, (2.19) implies

$$w^2(x_o,t_o) \le e^{-\lambda t_o}\left[A_o\varphi^2 + A_1\varphi w + A_2|D\varphi|w\right]\Big|_{(x_o,t_o)},$$

that is,

$$|w(x_o,t_o)| \le 2\sqrt{A_o + A_1^2 + A_2^2}\,\left(\|\varphi\|_\infty + \|D\varphi\|_\infty\right),$$

and the theorem follows. ∎

3 The Aleksandrov Maximum Principle

The main ideas presented in this section are attributed to Aleksandrov, who introduced them in connection with elliptic equations in nondivergence form with bounded and measurable coefficients [8]. We give here the parabolic version of such ideas, worked out by Krylov [145]. We follow the approach of Reye [217] (see also Nazarov and Ural'tzeva [191]).

3.1 Basic Geometric Notions

3.1.1 The Upper Contact Set

Let $x \to u(x) \in C(E)$. We define the *upper contact set* of u by

$$\Gamma^+ \overset{\text{def}}{=} \{y \in E : u(x) \le u(y) + \mathbf{p}\cdot(x-y),\ \forall\, x \in E,\ \text{ for some } \mathbf{p} \in \mathbb{R}^N\}.$$

Let $u \in C^1(E)$. Then, $y \in \Gamma^+$ if the tangent hyperplane to the graph of u through $(y, u(y))$ is all above the graph of u. Hence, if $y \in \Gamma^+$ and $u \in C^1$ in a neighborhood of y, then $\mathbf{p} = \nabla u(y)$.

3.1.2 The Concave Hull

We define the *concave hull* of u to be the smallest concave function on E lying above u. We denote such a function by Ψ_u.

Using Figure 3.1 as a guideline,

one can see that

$$\Gamma^+ \equiv \{x \in E : u(x) \equiv \Psi_u(x)\}.$$

Remark 3.1 If $u \in C^2(E)$, then $(D^2u(x)) \le 0$ on Γ^+.

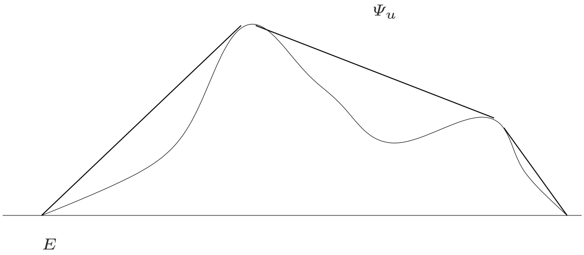

Fig. 3.1

3.1.3 The Normal Mapping

In the definition of the upper contact set, we have that $y \in \Gamma^+$ if we can find $\mathbf{p} \in \mathbb{R}^N$ such that

$$u(x) \leq u(y) + \mathbf{p} \cdot (x - y) \quad \forall\, x \in E.$$

If $y \notin \Gamma^+$, then we might set the corresponding $\mathbf{p}(y)$ to be the empty set. In such a way, we have the *normal mapping* $\chi : E \to \mathbb{R}^N$ defined by

$$\chi(y) \equiv \begin{cases} \{\mathbf{p} \in \mathbb{R}^N : u(x) \leq u(y) + \mathbf{p} \cdot (x - y), \forall x \in E\} \text{ if } y \in \Gamma^+, \\ \emptyset \text{ if } y \notin \Gamma^+. \end{cases}$$

In other words, $\chi(y)$ is the set of the "slopes" of the "tangent hyperplanes" to the graph of u at the point $(y, u(y))$. It is apparent that, in general, χ is not single valued, as indicated by Figure 3.2.

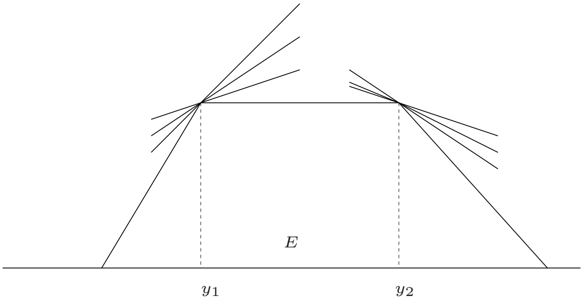

Fig. 3.2

We consider the following specific example.

3.1.4 The Normal Mapping of a Cone

Let $E = B_R(z) \equiv \{x \in \mathbb{R}^N : |x - z| < R\}$, and let $x \to u(x)$ be the function whose graph is the cone with base $B_R(z)$ and vertex (z, a) for some positive a, that is

$$u(x) = a\left(1 - \frac{|x - z|}{R}\right).$$

If $(y, u(y))$ is not the vertex (i.e., $y \neq z$), we have

$$\chi(y) = Du(y) = -\frac{a}{R}\frac{y - z}{|y - z|}, \quad y \neq z.$$

At the "vertex," there are infinitely many "tangent hyperplanes," whose "normals" fill a ball of center the origin and radius $\frac{a}{R}$. Therefore,

$$\chi(y) = \begin{cases} -\dfrac{a}{R}\dfrac{y - z}{|y - z|}, & \text{if } y \neq z, \\ B_{\frac{a}{R}}(0), & \text{if } y = z. \end{cases}$$

3.2 Increasing Concave Hull of u

Next we consider functions depending upon x and t. If $(x, t) \to u(x, t) \in C(\bar{E}_T)$, for each $t \in (0, T)$, we may define the upper contact set of $x \to u(x, t)$ and denote it by $\Gamma^+(t)$. Set

$$\Gamma_u^+ \overset{\text{def}}{=} \bigcup_{0 \le t \le T} \Gamma^+(t).$$

If $t \to u(\cdot, t) \in C^1(0, T)$, we define the increasing set of u as

$$I \overset{\text{def}}{=} \{(x, t) \in E_T : u_t(x, t) \ge 0\},$$

and set

$$F \overset{\text{def}}{=} \Gamma_u^+ \cap I.$$

If $u \ge 0$, to estimate the "largeness" of u in E_T, it will be enough to estimate the size of u on F.

Let $u \in C^{2,1}(E_T)$. The *increasing concave hull* of u is the smallest function in E_T, which is concave with respect to x for all $t \in (0, T)$, nondecreasing in t, and which lies above u.

We denote such a function by ξ_u. The function ξ_u has the following properties.

Proposition 3.1 *If $u \in C^{2,1}(E_T)$, then $\xi_u \in W_\infty^{2,1}(E_T)$, that is,*

$$\left\|\frac{\partial^2 \xi_u}{\partial x_i \partial x_j}\right\|_{\infty;E_T} + \left\|\frac{\partial \xi_u}{\partial t}\right\|_{\infty;E_T} \le C.$$

Here, the derivatives are distributional derivatives identified with $L^\infty(E_T)$ functions. The spaces $W_q^{k,m}(E_T)$ were defined in Section 1 of Chapter 11.

Proposition 3.2 *If* $u \in C^{2,1}(E_T)$, *then*

$$\partial_t \xi_u \det(D^2 \xi_u) = \begin{cases} \partial_t u \det(D^2 u) & \textit{a.e. on } [u = \xi_u] \equiv F, \\ 0 & \textit{otherwise.} \end{cases}$$

3.2.1 Proof of Proposition 3.1

Define

$$v(x,t) = \sup_{s \le t} u(x,s), \quad \forall\, x \in E,\ s \in [0,T].$$

Then,

$$\xi_u(\cdot,t) = \Psi_{v(\cdot,t)},$$

where $\Psi_{v(\cdot,t)}$ is the concave hull of $v(x,t)$. Since $u \in C^{2,1}(E_T)$, v is at least in $W_\infty^{1,1}(E_T)$. Indeed, for any $\epsilon > 0$ and any vector h such that $x + h \in E$, we have

$$\begin{aligned} &|v(x+h,t) - v(x,t)| + |v(x,t+\epsilon) - v(x,t)| \\ &\qquad \le \sup_{s \in (0,t)} |u(x+h,s) - u(x,s)| + \sup_{s \in (0,t)} |u(x,s+\epsilon) - u(x,s)| \\ &\qquad \le \|Du\|_\infty |h| + \|u_t\|_\infty \epsilon. \end{aligned}$$

Since $\Psi_{v(\cdot,t)} = \xi_u(\cdot,t)$ is concave, it can be looked upon as the pointwise infimum of all affine functions f, whose graph lies above the graph of v and $\|Df\|_{\infty;E} \le \|Du\|_{\infty;E}$.

Consequently, $D\xi_u(\cdot,t)$ exists for a.e. $x \in E$ and

$$\|D\xi_u\|_{\infty;E_T} \le \|Du\|_{\infty;E_T}.$$

In a similar way, we have

$$\|\partial_t \xi_u\|_{\infty;E_T} \le \|\partial_t u\|_{\infty;E_T}.$$

We now need to obtain the bound on the second-order derivatives $\dfrac{\partial^2 u}{\partial x_i \partial x_j}$. From the previous argument, it follows that for $t \in [0,T]$ fixed, for a.e. $x_o \in E$, the graph of ξ_u has a tangent hyperplane

$$\pi(x) = \xi_u(x_o) + D\xi_u(x_o) \cdot (x - x_o), \tag{3.1}$$

and by the concavity of ξ_u we also have

$$\xi_u(x) \le \pi(x) \quad \forall\, x \in E. \tag{3.2}$$

Since $t \in [0,T]$ is fixed, we drop it from the notation.

Suppose now that for a.e. $x_o \in E$ we can find a paraboloid of the form

$$\theta(x, x_o) = \xi_u(x_o) + D\xi_u(x_o) \cdot (x - x_o) - C|x - x_o|^2 \tag{3.3}$$

lying all below the graph of ξ_u, that is,

$$\xi_u(x) \geq \theta(x, x_o) \quad \forall\, x \in E. \tag{3.4}$$

This will then imply that $\xi_u(\cdot, t) \in W^{2,\infty}(E)$, where for any $p \geq 1$

$$W^{2,p}(E) = \{v \in L^p(E) :\ D^\alpha v \in L^p(E) \ \text{ for } \ |\alpha| \leq 2\}$$

and the derivatives are meant in the weak sense (see Section 1.1c of the Complements of Chapter 9). We rely on the following.

Theorem 3.1. [28] *Let $\mathcal{P}$ be the class of all polynomials of degree* 1*. Suppose that a function $w \in L^2(E)$ satisfies*

$$\inf_{P \in \mathcal{P}} \fint_{B_\rho(x_o)} |w(x) - P(x)|^2\, dx \leq C\rho^4$$

for a.e. x_o and $\rho > 0$ such that $B_\rho(x_o) \subset E$, and for a given constant $C > 0$. Then, $w \in W^{2,\infty}(E)$ and

$$\|w_{x_i x_j}\|_{\infty; E} \leq C\gamma,$$

where γ depends only upon the dimension N.

Let x_o be a point in E such that (3.1) and (3.3) both hold. Then, $\forall\, x \in E$ from (3.2) and (3.4)

$$|\xi_u(x) - P(x)|^2 \leq C|x - x_o|^2,$$

where $P(x) = \xi_u(x_o) + D\xi_u(x_o) \cdot (x - x_o)$. Integrating over $B_\rho(x_o)$ and taking the infimum over all $P \in \mathcal{P}$ the assumptions of the previous theorem are satisfied and its conclusion proves Proposition 3.1. Thus, it remains to prove that for a.e. $x_o \in E$ a paraboloid of the form (3.3) can be found. This proof requires the following fact.

Let $x_o \in E$ be such that $v(x_o) \neq \xi_u(x_o)$ (as before, $t \in [0, T]$ is assumed fixed and dropped from the notation). Since $\xi_u(\cdot, t) = \Psi_{v(\cdot,t)}$ the point $(x_o, \xi_u(x_o))$ in the graph of ξ_u is the linear combination of at most $N + 1$ points in the graph of v, that is, there exist $x_1, x_2, \dots, x_{N+1} \in E$ and $\alpha_1, \alpha_2, \dots, \alpha_{N+1} \in (0, 1)$ such that

$$\sum_{i=1}^{N+1} \alpha_i = 1, \tag{3.5}$$

$$x_o = \sum_{i=1}^{N+1} \alpha_i x_i, \tag{3.6}$$

$$\xi_u(x_o) = \Psi_v(x_o) = \sum_{i=1}^{N+1} \alpha_i v(x_i). \tag{3.7}$$

Moreover,

$$v(x_i) = \xi_u(x_i), \quad i = 1, 2, \dots, N+1.$$

Indeed, by definition of concave hull, $v(x) \le \xi_u(x)$ for any $x \in \bar{E}$. If for some i we had $v(x_i) < \xi_u(x_i)$, then

$$\xi_u(x_o) = \sum_{i=1}^{N+1} \alpha_i v(x_i) < \sum_{i=1}^{N+1} \alpha_i \xi_u(x_i) \le \xi_u(x_o).$$

Proof of Proposition 3.1 concluded By the regularity of u, for each $(x_o, t_o) \in E_T$, there exists a paraboloid

$$\theta_u(x, x_o, t_o) = u(x_o, t_o) + Du(x_o, t_o) \cdot (x - x_o) - c(x_o, t_o)|x - x_o|^2,$$

with $c \le \|D^2 u\|_{\infty; E_T}$, such that

$$u(x, t_o) \ge \theta_u(x, x_o, t_o) \quad \forall x \in E.$$

Since the supremum in the definition of v must be attained, it follows that v enjoys the same property with $Du(x_o, t_o)$ replaced by some vector bounded by $\|Du\|_{\infty; E_T}$. Let us now refer to the $N+1$ points x_i. If θ_i is the paraboloid lying under $v(x_i, t_o)$ with $\theta_i(x_i) = v(x_i, t_o)$, it follows that $\displaystyle\sum_{i=1}^{N+1} \alpha_i \theta_i(x, x_i, t_o)$ is a paraboloid with the same bound on its second-order derivatives, lying under $\Psi_{v(\cdot, t_o)}$ and equal to it at x. This allows us to conclude. ■

3.2.2 Proof of Proposition 3.2

We need some introductory tools.

Lemma 3.1 *Each of the following line segments* $\displaystyle\left(x_i, \frac{1}{1-\alpha_i} \sum_{j \ne i} \alpha_j x_j \right)$ *contains* x_o *and* ξ_u *is affine along them.*

Proof. Let $i = 1$ and set

$$\bar{x} = \frac{1}{1 - \alpha_1} \sum_{j=2}^{N+1} \alpha_j x_j.$$

Then from (3.6)

$$x_o = \alpha_1 x_1 + (1 - \alpha_1)\bar{x}.$$

From (3.7) we have

$$\xi_u(x_o) = \alpha_1 \xi_u(x_1) + (1-\alpha_1) \sum_{j=2}^{N+1} \frac{1}{1-\alpha_1} \alpha_j v(x_j). \tag{3.8}$$

Since $(1-\alpha_1)^{-1} \sum_{j=2}^{N+1} \alpha_j = 1$, we have that $\displaystyle\sum_{j=2}^{N+1} (1-\alpha_1)^{-1} \alpha_j v(x_j)$ belongs to the convex hull of the graph of v and by definition of ξ_u

$$\sum_{j=2}^{N+1} \frac{1}{1-\alpha_1} \alpha_j v(x_j) \le \xi_u(\bar{x}).$$

Next, by the concavity of ξ_u along the line segment $(x_1, \bar{x})$ we must have

$$\xi_u(x_o) \ge \alpha_1 \xi_u(x_1) + (1-\alpha_1) \xi_u(\bar{x}). \tag{3.9}$$

Combining (3.8)–(3.9) yields

$$\xi_u(x_o) = \alpha_1 \xi_u(x_1) + (1-\alpha_1) \xi_u(\bar{x}). \tag{3.10}$$

To prove that ξ_u is affine along the segment $(x_1, \bar{x})$, we show that $\forall\, \tau \in (0,1)$

$$\xi_u(\tau x_1 + (1-\tau)\bar{x}) = \tau \xi_u(x_1) + (1-\tau) \xi_u(\bar{x}).$$

Since ξ_u is concave

$$\xi_u(\tau x_1 + (1-\tau)\bar{x}) \ge \tau \xi_u(x_1) + (1-\tau) \xi_u(\bar{x}), \tag{3.11}$$

and we must show that equality must occur in (3.11). Suppose that for some $\bar{\tau} \in (0,1)$ (3.11) holds with strict inequality. Without loss of generality we may assume that $\bar{\tau} > \alpha_1$, so that

$$x_o = \beta(\bar{\tau} x_1 + (1-\bar{\tau})\bar{x}) + (1-\beta)\bar{x}, \qquad \beta = \frac{\alpha_1}{\bar{\tau}} \in (0,1).$$

Then, by concavity

$$\begin{aligned} \xi_u(x_o) &\ge \beta \xi_u(\bar{\tau} x_1 + (1-\bar{\tau})\bar{x}) + (1-\beta)\xi_u(\bar{x}) \\ &> \beta\bar{\tau} \xi_u(x_1) + \beta(1-\bar{\tau})\xi_u(\bar{x}) + (1-\beta)\xi_u(\bar{x}) \\ &= \alpha_1 \xi_u(x_1) + (1-\alpha_1)\xi_u(\bar{x}) = \xi_u(x_o) \end{aligned}$$

by (3.10). The contradiction proves the lemma. ■

Corollary 3.1 *Suppose that there are at least three distinct points x_j for which* (3.5)–(3.7) *hold. Then, ξ_u is affine in a neighborhood of x_o.*

Proof. Indeed, the graph of ξ_u must contain in the neighborhood of x_o, two distinct line segments intersecting at $(x_o, \xi_u(x_o))$. ■

Lemma 3.2 *At a.e.* $x_o \in E$ *such that* $\xi_u(x_o) \neq v(x_o)$, $\det(D^2\xi_u(x_o)) = 0$.

Proof. If for $x_o \in E$, $\xi_u(x_o) \neq v(x_o)$, there exist $x_1, \dots, x_{N+1} \in E$, $\alpha_1, \dots, \alpha_{N+1} \in (0,1)$ such that (3.5)–(3.7) hold. Then, by Lemma 3.1, ξ_u is affine along the segment $(x_1, \bar{x} = (1-\alpha_1)^{-1}\sum_{j=2}^{N+1}\alpha_j x_j)$ containing x_o.

After a rotation and a translation, we may assume that $x_o = 0$, and that $(x_1, \bar{x})$ is a portion of the x_1-axis, so that $\dfrac{\partial^2\xi_u}{\partial x_1^2}(0) = 0$, since ξ_u is affine on such a portion. Moreover, by the concavity of the matrix

$$(D^2\xi_u)(0) = \begin{pmatrix} 0 & \frac{\partial^2\xi_u}{\partial x_1\partial x_2} & \dots & \frac{\partial^2\xi_u}{\partial x_1\partial x_N} \\ \frac{\partial^2\xi_u}{\partial x_2\partial x_1} & \dots & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial^2\xi_u}{\partial x_N\partial x_1} & \dots & \dots & \frac{\partial^2\xi_u}{\partial x_N\partial x_N} \end{pmatrix}(0)$$

is negative semi-definite. Let $\mathbb{X}$ be a $N \times N$ unitary matrix that diagonalizes $(D^2\xi_u(0))$, that is, such that

$$\mathbb{X}^t \cdot (D^2\xi_u)(0) \cdot \mathbb{X} = \begin{pmatrix} \lambda_1 & \dots & 0 \\ 0 & \ddots & 0 \\ 0 & \dots & \lambda_N \end{pmatrix},$$

where λ_i, $i = 1, \dots, N$ are the real eigenvalues of the symmetric matrix $(D^2\xi_u)(0)$. By direct calculation, setting $\mathbf{e}_1^t = \begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix}$, we have

$$\begin{aligned} 0 &= \mathbf{e}_1^t (D^2\xi_u)(0)\mathbf{e}_1 \\ &= \mathbf{e}_1^t \mathbb{X}\mathbb{X}^t (D^2\xi_u)(0)\mathbb{X}\mathbb{X}^t\mathbf{e}_1 \\ &= (\mathbb{X}^t\mathbf{e}_1)^t \begin{pmatrix} \lambda_1 & \dots & 0 \\ 0 & \ddots & 0 \\ 0 & \dots & \lambda_N \end{pmatrix} (\mathbb{X}^t\mathbf{e}_1). \end{aligned} \tag{3.12}$$

Therefore, denoting with (η_i) the entries of the vector $(\mathbb{X}^t\mathbf{e}_1)$, we have from (3.12)

$$0 = \sum_{i=1}^N \lambda_i \eta_i^2.$$

Since at least for some index i we have $\eta_i^2 \neq 0$, we must have $\lambda_i = 0$ for some index i, since $\lambda_i \leq 0$ (the matrix $(D^2\xi_u)(0)$ is negative semi-definite).

We conclude that the largest eigenvalue of $(D^2\xi_u)(0)$ is zero; hence,

$$\det(D^2\xi_u)(0) = \lambda_1 \cdot \lambda_2 \cdot \dots \cdot \lambda_N = 0.$$

■

Lemma 3.3 *If* $f \in W^{1,\infty}(E)$*, then* $|Df| = 0$ *a.e. on the set* $\{f = 0\}$*.*

Proof. First divide f into its positive and negative parts, to reduce the lemma to the case $f \geq 0$. Then the proof follows by a standard approximation process, and the definition of a weak derivative. ■

Proof of Proposition 3.2 concluded If for some $(x,t) \in E_T$, $\xi_u(x,t) \neq v(x,t)$, then by Lemma 3.2 we have that $\det(D^2\xi_u)(x,t) = 0$.

Let $(x,t) \in E_T$ be such that $\xi_u(x,t) = v(x,t) \neq u(x,t)$. Then for some $s < t$

$$u(x,s) = v(x,s) = v(x,t).$$

Hence, since $t \to \xi_u(\cdot,t)$ is nondecreasing, $\xi_u(x,s) = \xi_u(x,t)$, and $\frac{\partial \xi_u}{\partial t}(x,t) = 0$. Finally, consider those points $(x,t) \in E_T$ where

$$\xi_u(x,t) = v(x,t) = u(x,t).$$

On such a set $(\xi_u - u) = 0$, $\frac{\partial(\xi_u - u)}{\partial t} = 0$, and $D_{x_i}(\xi_u - u) = 0$. Moreover, since $D(\xi_u - u) \in W^{1,0}_\infty(E_T)$, we have $D_{x_i}D_{x_j}(\xi_u - u) = 0$ on $[\xi_u = u]$, and the proposition follows. ■

3.3 Auxiliary Lemmas

Lemma 3.4 *Let* $w \in C^3(E)$*. Then,*

$$\sum_{j=1}^{N} \frac{\partial}{\partial x_j}\left\{\frac{\partial}{\partial w_{x_i x_j}} \det(D^2 w)\right\} = 0, \quad i = 1,\dots,N.$$

Proof. Fix $i = 1$ and for notational simplicity, set $w_{x_i x_j} = w_{ij}$. Then, $(D^2 w) = (w_{ij})$, and if A_{ij} is the algebraic complement of w_{ij}

$$\det(D^2 w) = \sum_{j=1}^{N}(-1)^{1+j} w_{1j} \det(A_{1j}),$$

$$\sum_{j=1}^{N} \frac{\partial}{\partial x_j}\left[\frac{\partial}{\partial w_{ij}} \det(D^2 w)\right] = \sum_{j=1}^{N}(-1)^{1+j} \frac{\partial}{\partial x_j} \det(A_{1j}).$$

For $j = 1,\dots,N$ fixed, we have

$$\frac{\partial}{\partial x_j} \det(A_{1j}) = \sum_{h \in \mathcal{H}} \sum_{l=2}^{N} (-1)^{1+j+(h)} w_{lh_l j} \prod_{i \neq l}^{N} w_{ih_i},$$

where $\mathcal{H}$ is the set of all permutations $(h_2, h_3, \dots, h_N)$, $h_2 \neq h_3 \neq \dots \neq h_N \neq j$ and (h) is the index of the permutation h. Then,

$$\sum_{j=1}^{N}\frac{\partial}{\partial x_j}\left[\frac{\partial}{\partial w_{ij}}\det(D^2w)\right]=\sum_{j=1}^{N}\sum_{h\in\mathcal{H}}\sum_{l=2}^{N}(-1)^{1+j+(h)}w_{lh_lj}\prod_{i\neq l}w_{ih_i}.$$

Since $w\in C^3(E)$, $w_{lh_lj}=w_{ljh_l}$, and rearranging the sums, the lemma follows. ■

Let E be convex and bounded, and consider the cylindrical domain $Q=E\times(t_1,t_2)$, $0\le t_1<t_2$.

Lemma 3.5 *Let $w\in C^\infty(Q)$ be such that $w=0$ on $\partial E\times[t_1,t_2]$. Then,*

$$\begin{aligned}\int_{t_1}^{t_2}\int_E \partial_t w\det(D^2w)\,dxdt =&\frac{1}{N+1}\left[\int_{E\times\{t_2\}} w\det(D^2w)\,dx\right.\\ &\left.-\int_{E\times\{t_1\}} w\det(D^2w)\,dx\right].\end{aligned}\tag{3.13}$$

Proof. Integrating the left-hand side of (3.13) with respect to t, we get

$$\begin{aligned}\iint_Q \partial_t w\det(D^2w)\,dxdt &= \int_{E\times\{t_2\}} w\det(D^2w)\,dx\\ &\quad-\int_{E\times\{t_1\}} w\det(D^2w)\,dx\\ &\quad-\sum_{i,j=1}^{N}\iint_Q w\frac{\partial^3 w}{\partial t\partial x_i\partial x_j}\left(\frac{\partial}{\partial w_{x_ix_j}}\det(D^2w)\right)dxdt.\end{aligned}\tag{3.14}$$

Using Lemma 3.4, the last term can be integrated twice by parts in x_i, x_j, to give

$$\begin{aligned}-\sum_{i,j=1}^{N}\iint_Q \partial_t w\, w_{x_ix_j}\left(\frac{\partial}{\partial w_{x_ix_j}}\det(D^2w)\right)dxdt&\\ =-N\iint_Q \partial_t w\,\det(D^2w)\,dxdt.&\end{aligned}$$

Putting this in (3.14) proves the lemma. ■

In Lemma 3.5 consider the following special situation. Let $t_1=0$, $t_2=1$, and

$$w(x,t)=tw_1(x)+(1-t)w_2(x),\quad t\in(0,1),$$

where $x\to w_i(x)$ $i=1$, 2 are two concave functions in E, such that

$$\begin{aligned}w_i=0\quad&\text{on }\partial E,\ i=1,2,\\ w_1(x)\le w_2(x)\quad&\forall\,x\in E.\end{aligned}$$

Since the convex combination of concave functions is concave, if w_1 and w_2 are smooth, we have

$$\det(D^2 w) \le 0, \quad \partial_t w = w_1 - w_2 \le 0,$$

and

$$\partial_t w \det(D^2 w) \ge 0.$$

Putting this in (3.13) yields

$$\int_E w_2 \det(D^2 w_2)\, dx \le \int_E w_1 \det(D^2 w_1)\, dx. \tag{3.15}$$

We have therefore shown the following result.

Lemma 3.6 *Let E be convex with smooth boundary ∂E. Let $w_1, w_2 \in C^2(E)$ be concave, satisfying $w_1 = w_2 = 0$ on ∂E and $w_1(x) \le w_2(x)\ \forall\, x \in E$. Then,* (3.15) *holds.*

3.4 Embedding by Normal Mapping

Let E be a bounded, convex domain in $\mathbb{R}^N$ with a smooth boundary and let $w \in C^2(E)$ be concave, such that $w = 0$ on ∂E.

Let $\chi_w : E \to \mathbb{R}^N$ be the normal mapping relative to w. If $w \in C^2(E)$, then χ_w is single valued, but need not be one-to-one. For $y \in E$ we have

$$\chi_w(y) = Dw(y).$$

The approximations $\chi_\epsilon(y) = \chi_w(y) - \epsilon y$, i.e.,

$$\chi_\epsilon(y) = Dw(y) - \epsilon y,$$

are one-to-one.

Indeed, if $\chi_\epsilon(E)$ is the image of E under χ_ϵ, the determinant of the Jacobian of the inverse mapping is given by

$$\det\ J(\chi_\epsilon^{-1}) = \det\left(D^2 w(y) - \epsilon \mathbb{I}\right), \tag{3.16}$$

and since w is concave, the eigenvalues of $(D^2 w(y))$ are nonpositive for any $y \in E$ and hence, the right-hand side of (3.16) is not zero $\forall\, y \in E$. Since E is convex, this implies that χ_ϵ is one-to-one.

Since $w \in C^2(E)$, both χ_ϵ and χ_ϵ^{-1} are continuous, so that $\chi_\epsilon(E)$ is open in $\mathbb{R}^N$.

Consider the integral $\displaystyle\int_{\chi_\epsilon(E)} \tilde{w}(\xi)\, d\xi$, where $\tilde{w}(\xi) = \tilde{w}(\chi_\epsilon(y)) = w(y)$. Changing variables, and writing it as an integral over E we have

$$\int_{\chi_\epsilon(E)} \tilde{w}(\xi)\, d\xi = \int_E w(y) \det(-D^2 w + \epsilon \mathbb{I})\, dy. \tag{3.17}$$

Letting $\epsilon \to 0$, the right-hand side tends to $\displaystyle\int_E w(y) \det(-D^2 w)\, dy$. As for the left-hand side, let $\xi \in \mathring{\chi}_w(E)$, where $\mathring{\chi}_w(E)$ denotes the interior (possibly empty) of $\chi_w(E)$. Then, for ϵ small enough,

$$\chi_\epsilon(y) = Dw(y) - \epsilon y = \chi_w(y) - \epsilon y = \xi - \epsilon y, \qquad (\xi = \chi_w(y))$$

belongs to a neighborhood of ξ in $\overset{\circ}{\chi}_w(E)$. Therefore, if I_Σ denotes the characteristic function of Σ, i.e.,

$$I_\Sigma(\xi) = \begin{cases} 1 & \xi \in \Sigma \\ 0 & \xi \text{ otherwise} \end{cases}$$

we see that

$$\liminf_{\epsilon\to 0} I_{\chi_\epsilon(E)}(\xi) \geq I_{\overset{\circ}{\chi}_w(E)}(\xi).$$

Next, observe that for every net $\{\epsilon'\}$, there exists a subnet $\{\epsilon''\}$ such that $\chi_\epsilon^{-1}(\xi) \to y \in \chi_w^{-1}(\xi)$. Indeed, since $\{\chi_{\epsilon'}^{-1}(\xi)\}$ is compact, there is a subnet $\{\epsilon''\}$ such that $\chi_{\epsilon''}^{-1}(\xi) \to \eta \in E$. Since $\chi_{\epsilon''}$ is continuous

$$\chi_{\epsilon''}(\chi_{\epsilon''}^{-1}(\xi)) = \xi \quad \Rightarrow \quad \chi_w(\eta) = \xi \;\text{ i.e. }\; \eta \in \chi_w^{-1}(\xi).$$

Since w is continuous, we must have

$$\liminf_{\epsilon\to 0} w(\chi_\epsilon^{-1}(\xi)) \geq \inf_{\{y\in E:\, \chi_w(y)=\xi\}} w(y) =: \bar{w}(y).$$

Indeed, if $\liminf_{\epsilon\to 0} w(\chi_\epsilon^{-1}(\xi)) < \bar{w}(\xi)$, there must be a net $\{\epsilon'\}$ such that $w(\chi_{\epsilon'}^{-1}(\xi)) < \bar{w}(\xi) - \sigma$ for some $\sigma \in (0,1)$, and then for a subnet $\{\epsilon''\}$ we would have $\chi_{\epsilon''}^{-1}(\xi) \to y \in \chi_w^{-1}(\xi)$ and $w(\chi_{\epsilon'}^{-1}(\xi)) \to \bar{\bar{w}}(\xi) \geq \bar{w}(\xi)$, which is a contradiction.

Using these facts we pass to the limit on the left-hand side of (3.17), using Fatou's Lemma to obtain

$$\begin{aligned}\liminf_{\epsilon\to 0} \int_{\chi_\epsilon(E)} \tilde{w}(\xi)\, d\xi &\geq \int_{\mathbb{R}^N} \liminf_{\epsilon\to 0} w(\chi_\epsilon^{-1}(\xi)) I_{\chi_\epsilon(E)}(\xi)\, d\xi \\ &\geq \int_{\mathbb{R}^N} \bar{w}(\xi) I_{\overset{\circ}{\chi}_w(E)}(\xi)\, d\xi = \int_{\overset{\circ}{\chi}_w(E)} \bar{w}(\xi)\, d\xi.\end{aligned}$$

Combining this and (3.17), we have

$$\int_{\overset{\circ}{\chi}_w(E)} \bar{w}(\xi)\, d\xi \leq \int_E w(y) \det(-D^2 w(y))\, dy. \tag{3.18}$$

To proceed, let $C_{x_o} : E \to \mathbb{R}$ be the function whose graph is the cone of base E and vertex at $(x_o, w(x_o))$, so that

$$\max C_{x_o}(x) = C_{x_o}(x_o) = w(x_o).$$

Lemma 3.7

$$\chi_{C_{x_o}}(E) \subset \chi_w(E).$$

Proof. Since $\chi_{C_{x_o}}(x) \subset \chi_{C_{x_o}}(x_o)\ \forall\, x \in E$, it will be enough to show that for every $\mathbf{p} \in \chi_{C_{x_o}}(x_o)$ there exists $y \in E$ such that $Dw(y) = \mathbf{p} = \chi_w(y)$. Let

$$\pi(x) = w(x_o) + \mathbf{p} \cdot (x - x_o)$$

be the hyperplane through $(x_o, w(x_o))$ and "slope" $\mathbf{p}$. Since $\mathbf{p} \in \chi_{C_{x_o}}$

$$\pi(x) \ge C_{x_o}(x) \qquad \forall\, x \in E. \tag{3.19}$$

The set $\Sigma \equiv \{x \in \bar{E} : w(x) \ge \pi(x)\}$ is closed and convex, and by virtue of (3.19) $w(x) - \pi(x) = 0$ on $\partial\Sigma$. By Rolle's Theorem there exists $y \in \Sigma$ such that $Dw(y) - D\pi(x) = 0$, i.e., $Dw(y) = \mathbf{p}$. ■

Set

$$R = k \operatorname{diam}(E), \qquad k \ge 1,$$

and consider the function D_{x_o} defined on $B_R(x_o)$, whose graph is the circular cone of base $B_R(x_o)$ and vertex $(x_o, w(x_o))$ (see Figure 3.3).

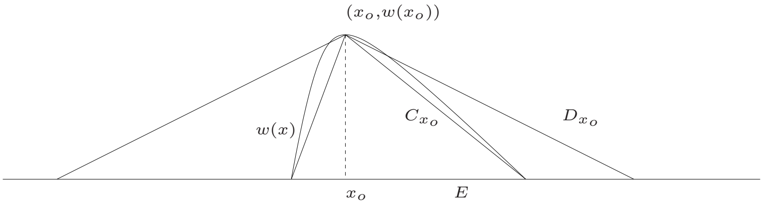

Fig. 3.3

Lemma 3.8

$$\chi_{D_{x_o}}(B_R(x_o)) \subset \chi_{C_{x_o}}(E).$$

Proof. This is obvious, once we observe that $D_{x_o}(x) \ge C_{x_o}(x)\ \forall\, x \in E$, $D_{x_o}(x_o) = C_{x_o}(x_o) = w(x_o)$. Therefore, every "tangent" hyperplane to the graph of D_{x_o} through $(x_o, w(x_o))$ is also a "tangent" hyperplane to the graph of C_{x_o} through $(x_o, w(x_o))$. ■

Combining the two lemmas in (3.18) we have

$$\int_{\mathring{\chi}_{D_{x_o}}(B_R(x_o))} \bar{w}(\xi)\, d\xi \le \int_E w(y) \det(-D^2 w(y))\, dy.$$

From the calculation of the normal mapping of a cone (see Section 3.1.4) we finally find

$$\int_{B_{\frac{w(x_o)}{k \operatorname{diam}(E)}}(0)} \bar{w}(\xi)\, d\xi \le \int_E w(y) \det(-D^2 w(y))\, dy, \tag{3.20}$$

where

$$\bar{w}(\xi) = \inf_{\{y \in E:\, \chi_w(y) = \xi\}} w(y). \tag{3.21}$$

Inequality (3.20) and the definition (3.21) will be the basis of the following embedding theorem.

Theorem 3.2. *Let E be convex with a smooth boundary ∂E, and let $w \in C^2(E)$ be concave and such that $w = 0$ on ∂E. Then,*

$$\sup_{x \in E} w(x) \le 2\omega_N^{-\frac{1}{N+1}} (\operatorname{diam}(E))^{\frac{N}{N+1}} \left\{ \int_E w(y) \det(-D^2 w(y))\, dy \right\}^{\frac{1}{N+1}}, \tag{3.22}$$

where ω_N is the volume of the unit ball in $\mathbb{R}^N$.

Proof. From (3.21) it follows that $\bar{w}(0) = w(x_o)$. We write

$$\begin{aligned} \bar{w}(\xi) &= \bar{w}(0) + \bar{w}(\xi) - \bar{w}(0) \\ &= w(x_o) + \inf_{\{y \in E:\, \chi_w(y) = \xi\}} [w(y) - w(x_o)] \\ &\ge w(x_o) - \sup_{\{y \in E:\, \chi_w(y) = \xi\}} |w(y) - w(x_o)|. \end{aligned} \tag{3.23}$$

In estimating the last term on the right-most side of (3.23), we first observe that since $\{y \in E : \chi_w(y) = \xi\}$ is closed, the supremum is achieved at some $\bar{y}$ such that $\chi_w(\bar{y}) = Dw(\bar{y}) = \xi$.

Along the line segment $t\bar{y} + (1-t)x_o$, $t \in (0,1)$, the function $t \to w(t\bar{y} + (1-t)x_o)$ is concave, and therefore, $|w_t|$ achieves its maximum at $t = 1$.

Therefore,

$$|w(\bar{y}) - w(x_o)| \le |w_t(t=1)| = |Dw(\bar{y})||\bar{y} - x_o| = |\xi| \operatorname{diam}(E).$$

Consequently, $\forall\, \xi \in B_{\frac{w(x_o)}{k \operatorname{diam}(E)}}(0)$

$$\bar{w}(\xi) \ge w(x_o) - \frac{w(x_o)}{k \operatorname{diam}(E)} \operatorname{diam}(E) \ge \left(1 - \frac{1}{k}\right) w(x_o). \tag{3.24}$$

Take $k = 2$. Then from (3.20) and (3.24)

$$\frac{1}{2} w(x_o) \frac{(w(x_o))^N \omega_N}{2^N (\operatorname{diam}(E))^N} \le \int_E w(y) \det(-D^2 w(y))\, dy,$$

which is (3.22). ■

By approximation, we also have the following.

Corollary 3.2 *The conclusion of Theorem 3.2 holds if $w \in W_o^{2,N+1}(E)$.*

Now let $(x,t) \to w(x,t)$ belong to $C^{2,1}(E_T)$ and such that

$$\begin{cases} x \to w(x,t) \;\text{ is concave } \forall\, t \in [0,T], \\ w(x,t) = 0 \quad x \in \partial E, \forall\, t \in [0,T], \\ w(x,0) = 0 \quad \forall\, x \in E, \\ w_t(x,t) \geq 0 \quad (x,t) \in E_T. \end{cases} \tag{$\mathcal{H}$}$$

Theorem 3.2 and Lemma 3.5 give

Theorem 3.3. *Let* $w \in C^{2,1}(E_T)$ *satisfy* ($\mathcal{H}$). *Then,*

$$\sup_{E_T} w \leq C_N (\operatorname{diam}(E))^{\frac{N}{N+1}} \left\{ \iint_{E_T} w_t \det(-D^2 w)\, dxdt \right\}^{\frac{1}{N+1}}, \tag{3.25}$$

where

$$C_N = 2 \left(\frac{N+1}{\omega_N} \right)^{\frac{1}{N+1}}. \tag{3.26}$$

In particular, C_N is independent of T.

Corollary 3.3 *By approximation, Theorem 3.3 remains valid for functions* w *satisfying*

$$w \in W^{1,N+1}(0,T; L^{N+1}(E)) \cap L^{N+1}(0,T; W_o^{2,N+1}(E)),$$

and the conditions in ($\mathcal{H}$) *are meant in a weak sense.*

Finally, we wish to find an embedding theorem of the type of Theorem 3.3 for functions that are not necessarily concave in the space variables. Namely, we assume

$$\begin{cases} u \in C^{2,1}(E_T), \\ u(x,t) = 0 \;\; \forall\, x \in \partial E \;\text{ and }\; \forall\, t \in (0,T), \\ u(x,0) = 0 \;\; \forall\, x \in E, \end{cases}$$

and recall that we have denoted with ξ_u the increasing concave hull of u.

By virtue of Propositions 3.1 and 3.2, and Corollary 3.3, estimate (3.25) is valid for ξ_u. Upon observing that

$$\sup_{E_T} u = \sup_{E_T} \xi_u,$$

we have the following.

Theorem 3.4. *Let* $u \in C^{2,1}(E_T)$, *such that* u *vanishes on the parabolic boundary of* E_T, *i.e.,* $\partial E_T \backslash \{t = T\}$. *Then,*

$$\sup_{E_T} u \leq C_N (\operatorname{diam}(E))^{\frac{N}{N+1}} \left\{ \iint_{[u=\xi_u]\cap E_T} u_t \det(-D^2 u)\, dxdt \right\}^{\frac{1}{N+1}}, \tag{3.27}$$

where C_N *is given by* (3.26).

Corollary 3.4 *By approximation, Theorem 3.4 remains valid for functions*

$$\begin{aligned}&u \in W^{1,N+1}(0,T;L^{N+1}(E)) \cap L^{N+1}(0,T;W_o^{2,N+1}(E)),\\ &u(x,0)=0 \qquad \textit{(in the sense of traces)}\ ,\end{aligned}$$

provided that the domain of integration in (3.27) *is replaced by* E_T*.*

3.5 Estimates of the Supremum of a Function

It is straightforward to see that

$$W^{2,1}_{N+1}(E_T) \equiv W^{1,N+1}(0,T;L^{N+1}(E)) \cap L^{N+1}(0,T;W^{2,N+1}(E));$$

we let

$$\begin{aligned}&\mathring{W}^{2,1}_{N+1}(E_T) \equiv \{u \in W^{2,1}_{N+1}(E_T) :\ u(\cdot,t)=0 \ \text{ on } \ \partial E \ \forall\, t \in [0,T]\},\\ &W^{2,1}_{o,N+1}(E_T) \equiv \{u \in W^{2,1}_{N+1}(E_T) :\ u=0 \ \text{ on } \ \Gamma_T = \partial E_T \backslash \{t=T\}\}.\end{aligned}$$

In this section, we give estimates of the supremum of a function $u \in W^{2,1}_{N+1}(E_T)$ in terms of a parabolic operator applied to it.

For $\lambda_o > 0$ we denote with

$$\begin{aligned}&A(\lambda_o)\ \text{all the positive definite, } N\times N \text{ symmetric matrices with}\\ &\text{measurable entries } (a_{ij}(x,t)),\ (x,t)\in E_T,\ i,j=1,2,\dots,N,\\ &\text{whose eigenvalues are larger than, or equal to } \lambda_o.\end{aligned} \tag{3.28}$$

For $u \in C^{2,1}(E_T)$, let

$$\mathcal{L}_o = a_{ij}(x,t)u_{x_ix_j}, \tag{3.29}$$

$$\mathcal{L} = a_{ij}(x,t)u_{x_ix_j} + b_i(x,t)u_{x_i} + c(x,t)u. \tag{3.30}$$

Throughout this section, $(x,t) \to u(x,t)$ is a given function satisfying

$$u \in C^{2,1}(\bar{E}_T), \quad u\Big|_{\Gamma_T} \le 0, \tag{3.31}$$

where we recall that Γ_T is the parabolic boundary of E_T.

Denote with ξ_u^+ the smallest, non-negative function in E_T, concave in x for all $t \in [0,T]$, nondecreasing in t, and which lies above u. This definition is analogous to that of the increasing concave hull of u (see Section 3.2), except that now we allow u to be negative on Γ_T, and force the increasing concave hull to vanish on the parabolic boundary of E_T.

It is apparent that all the results of the previous sections apply to the present situation upon observing that

$$\sup_{E_T} u_+ = \sup_{E_T} \xi_u^+.$$

Lemma 3.9 *Let u satisfy* (3.31) *and let E be convex. Then, for every matrix* $(a_{ij}) \in A(\lambda_o)$

$$\sup_{E_T} u \le \gamma_N \left[\frac{\operatorname{diam}(E)}{\lambda_o}\right]^{\frac{N}{N+1}} \left[\iint_{[u=\xi_u^+]\cap E_T} (u_t - \mathcal{L}_o(u))_+^{N+1}\,dxdt\right]^{\frac{1}{N+1}},$$

where $\gamma_N = \dfrac{2}{N+1}\left(\dfrac{N+1}{\omega_N}\right)^{\frac{1}{N+1}}$.

Proof. Since $(-D^2u)$ is symmetric, there exists a unitary matrix $\mathbb{X}$ that diagonalizes it, that is,

$$\mathbb{X}\cdot(-D^2u)\cdot\mathbb{X}^t = \begin{pmatrix} -C_1(u) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & -C_N(u) \end{pmatrix}$$

where $C_i(u)$, $i = 1,\dots,N$ are the eigenvalues of (D^2u). At points (x,t) of concavity $[u = \xi^+ u]$, $C_i(u) \le 0$ for $i = 1,\dots,N$ and $u_t \ge 0$.

If $(a_{ij}(x,t)) = \mathbb{A} \in A(\lambda_o)$, we have

$$\begin{aligned} u_t - a_{ij}u_{x_ix_j} &= u_t + \operatorname{tr}\mathbb{A}(-D^2u) \\ &= u_t + \operatorname{tr}\left\{\mathbb{X}\mathbb{A}\mathbb{X}^t \begin{pmatrix} -C_1(u) & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & -C_N(u) \end{pmatrix}\right\} \\ &= u_t - \sum_{i=1}^N \lambda_i C_i(u), \end{aligned}$$

where λ_i are the diagonal elements of $\mathbb{X}\mathbb{A}\mathbb{X}^t$. We also have (see Section 1.2)

$$\lambda_i \ge \lambda_o, \quad i = 1,\dots,N.$$

Therefore, since the geometric mean of $(N+1)$ non-negative numbers is less than their arithmetic mean, we have

$$\begin{aligned} [(u_t - \mathcal{L}_o(u))_+(x,t)]^{N+1} &= \left[u_t - \sum_{i=1}^N \lambda_i C_i(u)\right]^{N+1} \\ &\ge \left[\frac{u_t + \lambda_o(-C_1(u)) + \dots + \lambda_o(-C_N(u))}{N+1}\right]^{N+1} (N+1)^{N+1} \\ &\ge \left[\lambda_o^N u_t |C_1(u)\cdot\dots\cdot C_N(u)|\right](N+1)^{N+1} \\ &= \lambda_o^N (N+1)^{N+1} u_t \det(-D^2u). \end{aligned}$$

From Theorem 3.4 and (3.26) we find

$$\sup_{E_T} u \le \gamma_N \left(\frac{\operatorname{diam}(E)}{\lambda_o}\right)^{\frac{N}{N+1}} \left\{\iint_{[u=\xi_u^+]\cap E_T} (u_t - \mathcal{L}_o(u))_+^{N+1} dxdt\right\}^{\frac{1}{N+1}} \tag{3.32}$$

and $\gamma_N = \frac{2}{N+1}\left(\frac{N+1}{\omega_N}\right)^{\frac{1}{N+1}}$. ■

Next, we derive an estimate like (3.32) with $\mathcal{L}_o(u)$ replaced by the full operator $\mathcal{L}(u)$ in (3.30).

We assume that the origin $0 \in E$ and consider the ball $B_R(0)$ of radius

$$R = k(\operatorname{diam}(E)), \qquad k \ge 1,$$

and the cylinder

$$Q_R^T = B_{k\operatorname{diam}(E)}(0) \times [0,T].$$

First, we extend u with $\tilde{u} \in C^{2,1}(\bar{Q}_R^T)$ nonpositive outside E_T, and then we work with $\tilde{u}$ defined in the whole Q_R^T and $\tilde{u} \le 0$ on the parabolic boundary of Q_R^T.

If $\xi_{\tilde{u}}^+$ is the nondecreasing, non-negative concave hull of $\tilde{u}$, the contact set $[\tilde{u} = \xi_{\tilde{u}}^+]$ occurs within E_T. Therefore, for $(x_o,t_o) \in [\tilde{u} = \xi_{\tilde{u}}^+]$

$$|\tilde{u}_{x_i}| = |u_{x_i}| \le \frac{u(x_o,t_o)}{\operatorname{dist}(x_o,\partial B_R)} \le \frac{u(x_o,t_o)}{(k-1)\operatorname{diam}(E)}. \tag{3.33}$$

Now consider the function

$$v(x,t) = \tilde{u}(x,t)e^{-Ct}, \qquad (x,t) \in Q_R^T,$$

where C is a positive constant to be chosen.

From Lemma 3.9 we have that for every matrix $(a_{ij}(x,t)) \in A_{\lambda_o}$

$$\begin{aligned}\sup_{Q_R^T} v \le \gamma_N &\left(\frac{2k\operatorname{diam}(E)}{\lambda_o}\right)^{\frac{N}{N+1}} \\ &\times \left\{\iint_{[v=\xi_v^+]\cap Q_R^T} \left[e^{-Ct}\left[u_t - \mathcal{L}_o(u) - Cu\right]_+\right]^{N+1} dxdt\right\}^{\frac{1}{N+1}}.\end{aligned} \tag{3.34}$$

Next, write

$$e^{-Ct}(u_t - \mathcal{L}_o(u) - Cu)_+ = e^{-Ct}(u_t - \mathcal{L}(u) + b_i u_{x_i} - c(x,t)u - Cu)_+ \tag{3.35}$$

and observe that for $(x,t) \in [v = \xi_v^+]$ by (3.33) we have

$$\begin{aligned}e^{-Ct}(u_t - \mathcal{L}_o(u))_+ &\le e^{-Ct}(u_t - \mathcal{L}(u))_+ \\ &+ \left[\frac{B_o}{(k-1)\operatorname{diam}(E)} + c^-(x,t) - C\right] u(x,t)e^{-Ct},\end{aligned} \tag{3.36}$$

where with B_o we have denoted an upper bound for b_i and $c^-(x,t) = \max\{0, -c(x,t)\}$.

We deduce that if the negative part of $c(x,t)$ is bounded by a constant B_1, then we can select C so that the term in brackets on the right-hand side of (3.36) is nonpositive. Taking $k = 2$ we conclude

Lemma 3.10 *Let $u \in C^{2,1}(E_T)$, $u\Big|_{\Gamma_T} \le 0$, and E convex. Assume*

$$\sum_{i=1}^N \|b_i\|_{\infty;E_T} \le B_o, \qquad \|c^-\|_{\infty;E_T} \le B_1$$

for two given constants B_o and B_1. Then,

$$\begin{aligned}\sup_{E_T} u_+ \le \bar{\gamma}_N \exp\left(T\left[B_o + \frac{B_1}{R}\right]\right)\left(\frac{R}{\lambda_o}\right)^{\frac{N}{N+1}} \\ \times \left(\iint_{E_T} (u_t - \mathcal{L}(u))_+^{N+1} dxdt\right)^{\frac{1}{N+1}}\end{aligned} \tag{3.37}$$

for every $R \ge \operatorname{diam}(E)$, where $\bar{\gamma}_N = 4^{\frac{N}{N+1}}\gamma_N$.

Remark 3.2 If $c(x,t) > 0$ and

$$\sum_{i=1}^N \|b_i\|_{\infty;E_T} \le \frac{1}{2} R\, c(x,t), \qquad (x,t) \in E_T,$$

then in (3.36) we might take $C = 0$ and conclude that (3.37) holds without the exponential term. In particular, the coefficient of the integral does not depend on T.

Assume now that $\mathbf{b}(x,t) = (b_1(x,t), \dots, b_N(x,t))$ is not bounded, but

$$\left(\sum_{i=1}^N \iint_{E_T} |b_i|^{N+1}\, dxdt\right)^{\frac{1}{N+1}} \le B,$$

for a given constant B. Then an estimate similar to (3.37) can be obtained relying on the freedom given by the parameter $k \ge 1$ in (3.35)–(3.36).

Indeed, in (3.35), taking $C = \|c^-\|_{\infty;E_T}$ and using this in (3.34), by (3.33) we obtain for all $R \ge \operatorname{diam}(E)$

$$\begin{aligned}\sup_{E_T} v \le & \bar{\gamma}_N \left(\frac{kR}{\lambda_o}\right)^{\frac{N}{N+1}} \left\{\iint_{E_T} (u_t - \mathcal{L}(u))_+^{N+1} dxdt\right\}^{\frac{1}{N+1}} \\ & + \bar{\bar{\gamma}}_N \left(\frac{kR}{\lambda_o}\right)^{\frac{N}{N+1}} \frac{1}{(k-1)R} B \sup_{E_T} v.\end{aligned}$$

Therefore, if k is so large that

$$\bar{\bar{\gamma}}_N \left(\frac{kR}{\lambda_o}\right)^{\frac{N}{N+1}} \frac{1}{(k-1)R} B \le \frac{1}{2}$$

we obtain the following.

Lemma 3.11 *Let $u \in C^{2,1}(E_T)$, $u\Big|_{\Gamma_T} \le 0$, assume that E is convex, and*

$$\|c^-\|_{\infty;E_T} \le B_1, \qquad \left(\sum_{i=1}^N \iint_{E_T} |b_i|^{N+1}\,dxdt\right)^{\frac{1}{N+1}} \le B,$$

for two given constants B_1 and B.

Then there exists a constant $\gamma = \gamma(N, B, R, \lambda_o)$ such that

$$\sup_{E_T} u \le \gamma e^{B_1 T} \left(\iint_{E_T} (u_t - \mathcal{L}(u))_+^{N+1}\,dxdt\right)^{\frac{1}{N+1}}.$$

We generalize these results to the case when $u\Big|_{\Gamma_T}$ is not required to be non-positive, E is not required to be convex, and u is only in $W^{2,1}_{N+1}(E_T)$.

In a precise way we have the following.

Theorem 3.5. *Let E be a bounded, open set in $\mathbb{R}^N$ with boundary ∂E of class C^2 and let $u \in W^{2,1}_{N+1}(E_T)$. For any matrix $(a_{ij}(x,t)) \in A(\lambda_o)$, any vector $\mathbf{b}(x,t)$ and scalar $c(x,t)$ such that*

$$\|\mathbf{b}\|_{\infty;E_T} \le B_o, \qquad \|c^-\|_{\infty;E_T} \le B_1,$$

for two given constants B_o, B_1, consider the operator

$$\mathcal{L}(u) = a_{ij}(x,t)u_{x_i x_j} + b_i(x,t)u_{x_i} - c(x,t)u.$$

Then, for every $R \ge \operatorname{diam}(E)$, $\forall\,(x_o,t_o) \in E_T$

$$\begin{aligned} u(x_o,t_o) \le \|e^{B_1(t_o-s)}u_+\|_{\infty;\Gamma_{t_o}} + \bar{\gamma}_N \left(\frac{R}{\lambda_o}\right)^{\frac{N}{N+1}} \cdot \\ \cdot \left\{\iint_{E_{t_o}\cap\mathcal{E}^+_{t_o}} \left[e^{[B_1+\frac{B_o}{\operatorname{diam}(E)}](t_o-t)}(u_t - \mathcal{L}(u))_+\right]^{N+1} dxdt\right\}^{\frac{1}{N+1}}, \end{aligned} \tag{3.38}$$

where

$$\begin{aligned} &E_{t_o} \equiv E \times (0,t_o), \qquad \Gamma_{t_o} = \partial E_{t_o} \backslash \{t = t_o\}, \\ &\mathcal{E}^+_{t_o} = \{(x,t) \in E_{t_o} : u(x,t) > \|e^{B_1(t-s)}u_+\|_{\infty;\Gamma_{t_o}}\}, \end{aligned}$$

and

$$\bar{\gamma}_N = 4^{\frac{N}{N+1}} \frac{1}{N+1} \left(\frac{N+1}{\omega_N}\right)^{\frac{1}{N+1}}.$$

Proof. First, we observe that u is not required to be the solution of a partial differential equation, and therefore, $(a_{ij}(x,t)) \in A(\lambda_o)$, $\mathbf{b}(x,t)$, $c(x,t) \in L^\infty(E_T)$ being given, and $u \in W^{2,1}_{N+1}(E_T)$ being fixed, the quantity $(u_t - \mathcal{L}(u))_+$ can be approximated a.e. by similar quantities, where

$$a_{ij}(x,t),\ \ \mathbf{b}(x,t),\ \ c(x,t) \in C^\infty(\mathbb{R}^{N+1}),\ \ i,j = 1,\dots,N,$$

$$u \in C^\infty(\mathbb{R}^{N+1}).$$

Let $t_o \in (0,T]$ be fixed, and let $k > 0$ be a constant to be determined. Consider the function

$$v(x,t) = e^{-kt}u(x,t) - e^{(B_1-k)t}\|e^{-B_1 s}u_+\|_{\infty;\Gamma_{t_o}}, \tag{3.39}$$

where

$$\|e^{-B_1 s}u_+\|_{\infty;\Gamma_{t_o}} = \sup_{\substack{y\in\partial E;\,0<s\le t_o\\ y\in E\times\{0\}}} e^{-B_1 s}u_+(y,s).$$

By direct calculation

$$\begin{aligned} v_t - \mathcal{L}(v) =& e^{-kt}(u_t - \mathcal{L}(u)) - kv \\ & + [-B_1 - c(x,t)]e^{(B_1-k)t}\|e^{-B_1 s}u_+\|_{\infty;\Gamma_{t_o}}. \end{aligned}$$

By our choice of the constant B_1

$$v_t - \mathcal{L}(v) \le e^{-kt}(u_t - \mathcal{L}(u)) - kv \quad \text{in } E_{t_o}, \tag{3.40}$$

$$v\Big|_{\Gamma_{t_o}} \le 0. \tag{3.41}$$

Since $v \in C^\infty(\mathbb{R}^{N+1})$, we may extend v in $Q^{t_o}_{2R}$, $R \ge \operatorname{diam}(E)$ in such a way that

$$v\Big|_{\partial_p Q^{t_o}_{2R}} \le 0,$$

where $\partial_p Q^{t_o}_{2R} \equiv \partial Q^{t_o}_{2R}\backslash\{t = t_o\}$ is the parabolic boundary of $Q^{t_o}_{2R}$.

By Lemma 3.9 with E_T replaced by $Q^{t_o}_{2R}$

$$\sup_{Q^{t_o}_{2R}} v \le \gamma_N\left(\frac{2R}{\lambda_o}\right)^{\frac{N}{N+1}}\left\{\iint_{[v=\xi_v^+]\cap Q^{t_o}_{2R}} (v_t - \mathcal{L}_o(v))_+^{N+1}\,dxdt\right\}^{\frac{1}{N+1}}, \tag{3.42}$$

with $\gamma_N = \frac{2}{N+1}\left(\frac{N+1}{\omega_N}\right)^{\frac{1}{N+1}}$.

The set $[v = \xi_v^+]$ occurs within E_{t_o} and on such a set $v > 0$. Moreover, $\forall (x,t) \in [v = \xi_v^+]$, arguing as in the proof of Lemma 3.10,

$$\begin{aligned} (v_t - \mathcal{L}_o(v))_+ &\le (v_t - \mathcal{L}(u))_+ + (b_i v_{x_i} - c(x,t)v)_+ \\ &\le (v_t - \mathcal{L}(v))_+ + \left[B_1 + \frac{B_o}{\operatorname{diam}(E)}\right]v. \end{aligned}$$

Therefore, choosing $k = B_1 + \frac{B_o}{\operatorname{diam}(E)}$, from (3.40) we find

$$(v_t - \mathcal{L}_o(v))_+ \chi_{[v=\xi_v^+]} \le e^{-kt}(u_t - \mathcal{L}(u))_+ \chi_{[v>0]}.$$

We replace this last estimate in (3.42) and use the definition (3.39) of v to deduce

$$\begin{aligned} e^{-kt_o}&\Big(u(x_o,t_o) - \|e^{B_1(t_o-s)}u_+\|_{\infty;\Gamma_{t_o}}\Big) \\ &\le \bar{\gamma}_N \left(\frac{R}{\lambda_o}\right)^{\frac{N}{N+1}} \left\{ \iint_{E_{t_o}\cap\mathcal{E}_{t_o}^+} \left[e^{-kt}(u_t - \mathcal{L}(u))_+\right]^{N+1} dxdt \right\}^{\frac{1}{N+1}}, \end{aligned}$$

and this obviously implies (3.38). ■

The theorem can be generalized in a straightforward way to include the case when $b_i \in L^{N+1}(E_T)$, $i = 1, \dots, N$.

First we work with the cylinder $Q_{nR}^{t_o}$, $n \ge 2$, and v extended to be nonpositive outside E_{t_o}. Then we write (3.42) with $2R$ replaced by nR and estimate on $[v = \xi_v^+]$

$$\begin{aligned} (v_t - \mathcal{L}_o(v))_+ \le& (v_t - \mathcal{L}(v))_+ + b_i v_{x_i} - cv \\ \le& e^{-kt}(u_t - \mathcal{L}(u))_+ + \frac{\sum_{i=1}^N |b_i|}{n\operatorname{diam}(E)} v \\ &+ [-k - c(x,t)]v. \end{aligned}$$

Choosing $k \ge \|c^-\|_{\infty;E_T}$ we find

$$\begin{aligned} \sup_{E_{t_o}} v \le& \bar{\gamma}_N \left(\frac{nR}{\lambda_o}\right)^{\frac{N}{N+1}} \left\{ \iint_{[v=\xi_v^+]\cap Q_{nR}^{t_o}} e^{-B_1 t}(u_t - \mathcal{L}(u))_+^{N+1} dxdt \right\}^{\frac{1}{N+1}} \\ &+ \bar{\bar{\gamma}}_N \left(\frac{nR}{\lambda_o}\right)^{\frac{N}{N+1}} \frac{\sum_{i=1}^N \|b_i\|_{N+1;E_T}}{nR} \sup_{E_{t_o}} v. \end{aligned}$$

Now choose n so large that

$$\bar{\bar{\gamma}}_N \frac{\lambda_o^{-\frac{N}{N+1}}}{(nR)^{\frac{1}{N}}} \sum_{i=1}^N \|b_i\|_{N+1;E_T} \le \frac{1}{2},$$

to conclude.

Theorem 3.6. *Let E be a bounded, open set in $\mathbb{R}^N$ with C^2 boundary ∂E and let $u \in W_{N+1}^{2,1}(E_T)$. Let $(a_{ij}(x,t)) \in A(\lambda_o)$, and $\mathbf{b}(x,t)$ and $c(x,t)$ a vector field and a scalar satisfying*

$$b_i \in L^{N+1}(E_T),\ i = 1,\dots,N, \qquad \|c^-\|_{\infty;E_T} \le B_1,\ B_1 \ \textit{given}.$$

Then for every operator $\mathcal{L}(\cdot)$ *of the type*

$$\mathcal{L}(u) = a_{ij}u_{x_ix_j} + b_iu_{x_i} - cu,$$

for every $(x_o, t_o) \in E_T$ *and every* $R \geq \operatorname{diam}(E)$ *there exists a constant* $\gamma = \gamma(N, \operatorname{diam}(E), \|b_i\|_{N+1;E_T}, B_1)$ *such that*

$$\begin{aligned} u(x_o, t_o) &\leq \|e^{B_1(t_o-s)}u_+\|_{\infty;\Gamma_{t_o}} \\ &+ \gamma\left(\frac{R}{\lambda_o}\right)^{\frac{N}{N+1}}\left[\iint_{E_{T_o}\cap\mathcal{E}^+_{t_o}}\left[e^{B_1(t_o-t)}(u_t - \mathcal{L}(u))_+\right]^{N+1}dxdt\right]^{\frac{1}{N+1}}, \end{aligned} \tag{3.43}$$

where $\mathcal{E}^+_{t_o} \equiv \left\{(x,t) \in E_{t_o} : u(x,t) > \|e^{B_1(t-s)}u_+\|_{\infty;\Gamma_{t_o}}\right\}$.

3.6 Maximum Principle for Nonlinear Operators

Following the discussion about nonlinear equations of Section 1, let

$$F[u] = F(x, t, u, Du, D^2u),$$

where $F : E_T \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^{N^2} \to \mathbb{R}$ is measurable in (x,t) for each (u, Du, D^2u), and a.e. differentiable in u, u_{x_i}, $u_{x_ix_j}$. Let

$$(a_{ij}) = (F_{u_{x_ix_j}}), \quad b_i = F_{u_{x_i}}, \quad c = -F_u.$$

Set

$$a_{ij}[u] = a_{ij}(x, t, u, Du, D^2u),$$

and define analogously $b_i[u]$ and $c[u]$.

Theorem 3.7 (Comparison Principle). *Let* u, $v \in W^{2,1}_{N+1}(E_T)$ *satisfy*

$$u \leq v \quad \text{on } \partial_pE_T \equiv \Gamma_T \equiv \partial E_T\backslash\{t = T\}, \tag{3.44}$$

$$u_t - F[u] \leq v_t - F[v] \quad \text{in } E_T. \tag{3.45}$$

Moreover, assume that

$$(\tilde{a}_{ij}(x,t)) \equiv \left(\int_0^1 a_{ij}[su + (1-s)v]\,ds\right) \in A(\lambda_o), \tag{3.46}$$

$$\tilde{b}_i(x,t) \equiv \int_0^1 b_i[su + (1-s)v]\,ds \in L^\infty(E_T) \quad \text{and} \quad \|\tilde{b}_i\|_{\infty;E_T} \leq B_o, \tag{3.47}$$

$$\tilde{c}(x,t) = \int_0^1 c[su + (1-s)v]\,ds \quad \text{satisfies} \quad \|\tilde{c}^-\|_{\infty;E_T} \leq B_1, \tag{3.48}$$

where B_o *and* B_1 *are given non-negative constants. Then,* $u(x,t) \leq v(x,t)$ *for any* $(x,t) \in \bar{E}_T$.

Requirements (3.46)–(3.48) are formulated in terms of u and v given. If, in particular, they hold true for all u, $v \in W^{2,1}_{N+1}(E_T)$, then we have the following uniqueness result.

Corollary 3.5 *Suppose* (3.46)–(3.48) *in Theorem 3.7 hold uniformly in* u, $v \in W^{2,1}_{N+1}(E_T)$. *Then, the Cauchy–Dirichlet problem*

$$\begin{cases} u_t = F[u] & \textit{in } E_T, \\ u\Big|_{\Gamma_T} = \psi \in C(\Gamma_T), \end{cases}$$

has at most one solution in $W^{2,1}_{N+1}(E_T)$.

Proof of Theorem 3.7. Setting $w = u - v$, we have $w\Big|_{\Gamma_T} \le 0$ and $w_t - \tilde{\mathcal{L}}(w) \le 0$ in E_T, where

$$\tilde{\mathcal{L}}(w) = \tilde{a}_{ij}(x,t)w_{x_i x_j} + \tilde{b}_i(x,t)w_{x_i} - \tilde{c}(x,t)w.$$

Therefore, the theorem follows from (3.38). ■

Since $\tilde{b}_i$ depends upon $w_s = su + (1-s)v$, it is desirable to allow $\tilde{b}_i$ not to be bounded. However, since u, $v \in W^{2,1}_{N+1}(E_T)$, a reasonable requirement is

$$\tilde{b}_i \in L^{N+1}(E_T), \quad i = 1, \dots, N. \tag{3.49}$$

Using (3.43) and Theorem 3.5, we have that Theorem 3.7 remains true, if (3.47) is replaced by (3.49).

4 Local Estimates and the Harnack Inequality

In this section we deal with quasi-linear equations

$$\begin{aligned} &u_t - Q[u] = 0, \quad \text{in } E_T \subset \mathbb{R}^{N+1}, \\ &Q[u] = A_{ij}(x,t,u,Du)u_{x_i x_j} + B(x,t,u,Du). \end{aligned}$$

The structure conditions are specified in accordance with the various results.

We derive two basic estimates. The first one is a local estimate of a subsolution only in terms of its integral average over a portion of E_T and is the object of Section 4.1.

The second one, which is more involved, is the object of the remaining sections. In an attempt to gain in clarity, we prove first the Harnack inequality for linear equations, and then we generalize it to quasi-linear equations.

4.1 A Local Maximum Principle

Take $R > 0$; we denote with Q_R the cylinder

$$Q_R \equiv B_R(0) \times (0, R^2).$$

If $(x_o, t_o) \in E \times (0, T - R^2)$, then $(x_o, t_o) + Q_R \subset E_T$ denotes the cylinder $B_R(x_o) \times (t_o, t_o + R^2)$.

We assume that $u \in C^{2,1}(E_T)$ satisfies

$$u_t - A_{ij}(x,t,u,Du)u_{x_i x_j} - B(x,t,u,Du) \le 0 \quad \text{in } \ E_T, \tag{4.1}$$

i.e., it is a sub-solution. On the matrix (A_{ij}) and the scalar term B we impose the following structure conditions:

- If $\Lambda = \Lambda(x,t,u,Du)$ is the largest eigenvalue of $A_{ij}(x,t,u,Du)$, then

  $$\Lambda(x,t,u,Du) \le \varphi_o(x,t)[\det(A_{ij}(x,t,u,Du))]^{\frac{1}{N+1}}, \tag{4.2}$$

  where $(x,t) \to \varphi_o(x,t)$ is a given non-negative function satisfying

  $$\varphi_o \in L^q(E_T) \tag{4.3}$$

  for some $q > N + 1$.
- As for B,

  $$|B(x,t,u,Du)| \le [\psi_o|Du| + \psi_1|u| + \psi_2][\det(A_{ij}(x,t,u,Du))]^{\frac{1}{N+1}}, \tag{4.4}$$

  where $(x,t) \to \psi_i(x,t)$, $i = 0, 1, 2$, are given non-negative functions satisfying (4.3).
- Moreover,

  $$[\det(A_{ij}(x,t,u,Du))]^{-\frac{1}{N+1}} \in L^q(E_T), \quad q > N + 1. \tag{4.5}$$

- Finally, as usual,

  $$(A_{ij}(x,t,u,Du)) \ \text{ is assumed positive semi-definite and symmetric.} \tag{4.6}$$

Remark 4.1 Condition (4.5) is obviously satisfied if $(A_{ij}(x,t,u,Du)) = (a_{ij}(x,t)) \in A(\lambda_o)$, $\lambda_o > 0$, where $A(\lambda_o)$ is defined in (3.28). In such a case, the operator in (4.1) is uniformly parabolic.

Theorem 4.1. *Let* (4.1)–(4.6) *hold. Then for every* $(x_o, t_o) + Q_R \subset E_T$*, for every* $\sigma \in (0,1)$*, for every* $p > 0$

$$\begin{aligned}\sup_{B_{\sigma R}(x_o)\times(t_o+(1-\sigma)R^2, t_o+R^2)} u_+ \le{} & C_1 \left(⨍\!\!\!\!\!\!⨍_{(x_o,t_o)+Q_R} u_+^p \, dxdt \right)^{\frac{1}{p}} \\ & + C_2 R^{\frac{N}{N+1}},\end{aligned}$$

where C_1, C_2 are two constants depending upon p, σ, N, and the quantities

$$⨍\!\!⨍_{(x_o,t_o)+Q_R} |\varphi_o|^q \, dxdt, \qquad ⨍\!\!⨍_{(x_o,t_o)+Q_R} |\psi_i|^q \, dxdt, \; i = 0,\, 1,\, 2.$$

Here, we have denoted with $⨍\!\!⨍_{(x_o,t_o)+Q_R} |f| \, dxdt$ the integral average of $f \in L^1((x_o,t_o)+Q_R)$ over $(x_o,t_o)+Q_R$.

Remark 4.2 Since the theorem is local in nature, conditions (4.1)–(4.6) do not have to hold throughout E_T. The theorem remains valid in a subdomain Q of E_T if (4.1)–(4.6) hold true in Q.

Let q^* be defined by

$$\frac{1}{q} + \frac{1}{q^*} = \frac{1}{N+1},$$

and let $p \in (0,1)$ be fixed. Then, C_1 depends on the data as follows.

First, set

$$\Phi_o = \left\{ \varphi_o + \psi_o + \psi_1 + [\det(A_{ij}(x,t,u,Du))]^{-\frac{1}{N+1}} \right\};$$

then

$$C_1 = \gamma(N,p,q,\sigma) \left(⨍\!\!⨍_{(x_o,t_o)+Q_R} |\Phi_o|^q \, dxdt \right)^{\frac{1}{q}\left(\frac{q^*}{p}\right)},$$

where $\gamma(N,p,q,\sigma)$ is a constant independent of φ_o, ψ_i, u.

The dependence of C_2 on the data is

$$C_2 = \gamma(N,p,q,\sigma) \left(⨍\!\!⨍_{E_T} |\psi_2|^{N+1} \, dxdt \right)^{\frac{1}{N+1}},$$

for a constant γ independent of u.

Proof of Theorem 4.1. After a suitable translation, we may assume that $(x_o,t_o) = (0,0)$. In Q_R consider the cutoff function

$$\eta(x,t) = \left(\frac{R^2 - |x|^2}{R^2} \right)^{\beta} t^{\frac{\beta}{2}}, \tag{4.7}$$

where $\beta > 0$ has to be chosen, and set

$$v = \eta u.$$

If ξ_v is the increasing concave hull of v (see Section 3.2), on the upper contact set $[v = \xi_v]$, $v > 0$ and $|Dv| = |D\xi_v| \le v/(R - |x|)$. Therefore, from $\eta Du = Dv - uD\eta$, we have

$$\begin{aligned}|Du| &\le u\left(\frac{1}{R-|x|}+\frac{|D\eta|}{\eta}\right)\\ &\le u\left(\frac{R+|x|}{R^2-|x|^2}+2\beta\frac{(R^2-|x|^2)^{\beta-1}}{R(R^2-|x|^2)^{\beta}}\right)\le 2(1+\beta)\frac{R}{R^2-|x|^2}\,u.\end{aligned}$$

Therefore, on $[v=\xi_v]$

$$|Du|\le 2(1+\beta)t^{\frac12}\eta^{-\frac{1}{\beta}}u. \tag{4.8}$$

Moreover, on $[v=\xi_v]$, using (4.1) we have

$$\begin{aligned}v_t-A_{ij}(x,t,u,Du)v_{x_ix_j} =&\eta[u_t-A_{ij}(x,t,u,Du)u_{x_ix_j}]\\ &-2A_{ij}u_{x_i}\eta_{x_j}+u[\eta_t-A_{ij}\eta_{x_i}\eta_{x_j}]\\ \le&\eta B(x,t,u,Du)+2\Lambda(x,t,u,Du)|Du||D\eta|\\ &+u\left[\frac{\beta}{2}\eta t^{-1}+2\beta A_{ij}\delta_{ij}\left(\frac{R^2-|x|^2}{R^2}\right)^{\beta-1}\frac{t^{\frac{\beta}{2}}}{R^2}\right.\\ &\left.-4\beta(\beta-1)\left(\frac{R^2-|x|^2}{R^2}\right)^{\beta-2}A_{ij}\frac{x_ix_j}{R^4}t^{\frac{\beta}{2}}\right].\end{aligned}$$

The following estimates are obvious

$$\begin{aligned}-4\beta(\beta-1)\left(\frac{R^2-|x|^2}{R^2}\right)^{\beta-2}A_{ij}\frac{x_ix_j}{R^4}t^{\frac{\beta}{2}}&\le 0\\ 2\beta A_{ij}\delta_{ij}\left(\frac{R^2-|x|^2}{R^2}\right)^{\beta-1}\frac{t^{\frac{\beta}{2}}}{R^2}&\le 2\beta\Lambda t^{\frac12}\eta^{1-\frac{1}{\beta}}\\ \frac{\beta}{2}\eta\, t^{-1}&\le\frac{\beta}{2}\eta^{1-\frac{2}{\beta}}.\end{aligned}$$

Moreover, using (4.8)

$$2\Lambda|Du||D\eta|\le 4\Lambda(1+\beta)t^{\frac12}\eta^{-\frac1\beta}\beta t^{\frac12}\eta^{1-\frac1\beta}u\le C_\beta\Lambda\eta^{1-\frac2\beta}u.$$

Combining these estimates, we find

$$\begin{aligned}v_t-A_{ij}(x,t,u,Du)v_{x_ix_j}\le&\eta B(x,t,u,Du)\\ &+\gamma_\beta(1+\Lambda(x,t,u,Du))\eta^{1-\frac2\beta}u,\end{aligned}\tag{4.9}$$

where γ_β is a constant depending only upon β.

Next, we use the structure conditions (4.2) and (4.4) to obtain

$$\begin{aligned}&\eta|B(x,t,u,Du)|[\det(A_{ij}(x,t,u,Du)]^{-\frac{1}{N+1}}\\ &\quad\le\eta[2\psi_o(1+\beta)\eta^{-\frac1\beta}t^{\frac12}u+\psi_1u+\psi_2]\le\bar\psi_o\eta^{-\frac2\beta}v+\eta\psi_2,\end{aligned}$$

where $\bar\psi_o=2\psi_o(1+\beta)+\psi_1\in L^q(E_T)$, and

$$\begin{aligned}&\gamma_\beta(1+\Lambda(x,t,u,Du))[\det(A_{ij}(x,t,u,Du))]^{-\frac{1}{N+1}}\\&\quad\le\gamma_\beta\left(\varphi_o(x,t)+[\det(A_{ij}(x,t,u,Du))]^{-\frac{1}{N+1}}\right)\eta^{-\frac{2}{\beta}}v=\bar\varphi_o\eta^{-\frac{2}{\beta}}v,\end{aligned}$$

where $\bar\varphi_o=\gamma_\beta(\varphi_o(x,t)+[\det(A_{ij}(x,t,u,Du))]^{-\frac{1}{N+1}})\in L^q(E_T)$.

Combining these in (4.9), we find on $[v=\xi_v]$

$$\begin{aligned}&[\det(A_{ij}(x,t,u,Du))]^{-\frac{1}{N+1}}(v_t-A_{ij}(x,t,u,Du)v_{x_ix_j})_+\\&\qquad\qquad\le\Phi_o(x,t)\eta^{-\frac{2}{\beta}}v+\eta\,\psi_2,\end{aligned}$$

where $\Phi_o(x,t)=\bar\psi_o+\bar\varphi_o\in L^q(E_T)$.

Now we apply Lemma 3.1c of the Complements, to find

$$\begin{aligned}&\sup_{Q_R}v_+\\&\le\gamma_NR^{\frac{N}{N+1}}\left\{\iint_{[v=\xi_v]\cap Q_R}[\det(A_{ij})]^{-1}(v_t-A_{ij}v_{x_ix_j})_+^{N+1}\,dxdt\right\}^{\frac{1}{N+1}}\\&\le\bar\gamma_NR^{\frac{N}{N+1}}\left(\iint_{Q_R}\Phi_o^{N+1}(\eta^{-\frac{2}{\beta}}v)^{N+1}\,dxdt\right)^{\frac{1}{N+1}}\\&\quad+\bar{\bar\gamma}_NR^{\frac{N}{N+1}}\left(\iint_{Q_R}\eta^{N+1}\psi_2^{N+1}\,dxdt\right)^{\frac{1}{N+1}}.\end{aligned}\tag{4.10}$$

We estimate the first integral on the right-hand side of (4.10) as follows. By the previous definition of q^*, we have

$$\left(\iint_{Q_R}\Phi_o^{N+1}\left(\eta^{-\frac{2}{\beta}}v\right)^{N+1}\,dxdt\right)^{\frac{1}{N+1}}\le\|\Phi_o\|_{q;Q_R}\|\eta^{-\frac{2}{\beta}}v\|_{q^*;Q_R}$$

and

$$\begin{aligned}\|\eta^{-\frac{2}{\beta}}v\|_{q^*;Q_R}^{q^*}&=\iint_{Q_R}\eta^{-\frac{2q^*}{\beta}}v^{q^*(\alpha+1-\alpha)}\,dxdt\\&=\iint_{Q_R}\eta^{-\frac{2q^*}{\beta}}\eta^{q^*}u^{q^*(1-\alpha)}u^{\alpha q^*}\,dxdt\\&=\iint_{Q_R}(\eta u)^{q^*(1-\alpha)}u^{\alpha q^*}\,dxdt,\end{aligned}$$

if we choose β so that $1-\frac{2}{\beta}=\alpha$, i.e.,

$$\beta=\frac{2}{\alpha},\qquad\forall\,\alpha\in(0,1).$$

With this choice

$$\|\eta^{-\frac{2}{\beta}}\|_{q^*;Q_R} \le \left(\sup_{Q_R} v_+\right)^{1-\alpha} \|u_+\|_{\alpha q^*;Q_R}.$$

The second integral on the right-hand side of (4.10) is easily estimated by recalling the definition of η in (4.7). We have

$$\bar{\bar{\gamma}}_N R^{\frac{N}{N+1}} \left(\iint_{Q_R} \eta^{N+1}\psi_2^{N+1}\, dxdt\right)^{\frac{1}{N+1}} \le \tilde{\gamma} R^{\frac{N}{N+1}+\beta},$$

where

$$\tilde{\gamma} = \bar{\bar{\gamma}}_N \|\psi_2\|_{N+1;E_T}.$$

Substituting these estimates in (4.10)

$$\sup_{Q_R} v_+ \le \gamma_N R^{\frac{N}{N+1}} \|\Phi_o\|_{q;Q_R} \|u_+\|^{\alpha}_{\alpha q^*;Q_R} \left(\sup_{Q_R} v_+\right)^{1-\alpha} + \tilde{\gamma} R^{\frac{N}{N+1}+\beta}. \tag{4.11}$$

By Young's inequality

$$\begin{aligned} &\gamma_N R^{\frac{N}{N+1}} \|\Phi_o\|_{q;Q_R} \|u_+\|^{\alpha}_{\alpha q^*;Q_R} \left(\sup_{Q_R} v_+\right)^{1-\alpha} \\ &\le \frac{1}{2}\sup_{Q_R} v_+ + \gamma(N,\alpha) R^{\frac{N}{N+1}\frac{1}{\alpha}+\frac{N+2}{\alpha q}+\frac{N+2}{\alpha q^*}} \\ &\quad \cdot \left(\fint\!\!\!\fint_{Q_R} |\Phi_o|^q\, dxdt\right)^{\frac{1}{\alpha q}} \left(\fint\!\!\!\fint_{Q_R} |u_+|^{\alpha q^*}\, dxdt\right)^{\frac{1}{\alpha q^*}}. \end{aligned} \tag{4.12}$$

By the definition of q^*, we have

$$R^{\frac{N}{N+1}\frac{1}{\alpha}+\frac{N+2}{\alpha q}+\frac{N+2}{\alpha q^*}} = R^{\frac{N}{N+1}\frac{1}{\alpha}+\frac{N+2}{\alpha}\left(\frac{1}{q}+\frac{1}{q^*}\right)} = R^{\frac{2}{\alpha}} = R^{\beta}.$$

Setting

$$\gamma(N,\alpha)\left(\fint\!\!\!\fint_{Q_R} |\Phi_o|^q\, dxdt\right)^{\frac{1}{\alpha q}} = \bar{C}_1,$$

we obtain from (4.11) and (4.12)

$$\sup_{Q_R} v_+ \le 2\bar{C}_1 R^{\beta} \left(\fint\!\!\!\fint_{Q_R} |u_+|^{\alpha q^*}\, dxdt\right)^{\frac{1}{\alpha q^*}} + 2\tilde{\gamma} R^{\frac{N}{N+1}+\beta}.$$

For $(x,t) \in Q_R^{\sigma} = B_{\sigma R} \times ((1-\sigma)R^2, R^2)$

$$\sup_{Q_R} v_+ \ge \eta(x,t)u_+(x,t) \ge (1-\sigma)^{\frac{3\beta}{2}} R^{\beta} u_+(x,t),$$

and this implies the theorem for the choice $\alpha q^* = p$. ■

4.2 A Covering Lemma

In this section we will be working with cubes in $\mathbb{R}^{N+1}$. For $x = (x_1, \dots, x_N)$ let $\|x\| = \max_{i=1,\dots,N} |x_i|$, and consider the unit cube $K_1 = \{x \in \mathbb{R}^N : \|x\| < 1\}$. Denote with Q_1 the cylinder $Q_1 \equiv K_1 \times (0,1)$, and within Q_1 consider cylinders of the type $(x_o, t_o) + Q_R \equiv \{\|x - x_o\| < R\} \times (t_o, t_o + R^2)$, i.e., we will assume that $(x_o, t_o) \in Q_1$ and that R is so small that $(x_o, t_o) + Q_R \subset Q_1$.

If $(x_o, t_o) = (0,0)$, we write $(0,0) + Q_R = Q_R$ and we obviously have

$$|(x_o, t_o) + Q_R| = |Q_R|.$$

We are given

(a) a measurable set $\mathcal{E} \subset Q_1$,
(b) numbers δ, $\eta \in (0,1)$,

and consider the collection $\mathcal{B}(\delta)$ of those subcubes $(x_o, t_o) + Q_R$ of Q_1 satisfying

$$|((x_o, t_o) + Q_R) \cap \mathcal{E}| \geq \delta |Q_R|.$$

For each $(x_o, t_o) + Q_R \in \mathcal{B}(\delta)$, we construct two boxes as follows

$$\begin{aligned} Q^{(1)} &\equiv \{\|x - x_o\| < 3R\} \times (t_o - 3R^2, t_o + 4R^2) \cap Q_1, \\ Q^{(2)} &\equiv \{\|x - x_o\| < 3R\} \times \left(t_o + R^2, t_o + \left(1 + \frac{4}{\eta}\right) R^2\right), \end{aligned}$$

as depicted in Figure 4.4.

The construction of $Q^{(1)}$ is self explanatory. The box $Q^{(2)}$ lies on top of $(x_o, t_o) + Q_R$ and has length $\frac{4}{\eta} R^2$.

By definition, $Q^{(1)}$ is all contained in Q_1, but $Q^{(2)}$ might extend beyond the ceiling of Q_1.

Remark 4.3 The length of $Q^{(2)}$ in the t-direction is $\frac{4}{\eta} R^2$. The length of $Q^{(1)} \cup Q^{(2)}$ in the t-direction is

$$4R^2 + \frac{4}{\eta} R^2 = (1 + \eta) \frac{4}{\eta} R^2,$$

i.e., $\text{length}(Q^{(1)} \cup Q^{(2)}) \leq (1 + \eta)\, \text{length}(Q^{(2)})$.

Using $Q^{(1)}$ and $Q^{(2)}$ as building blocks, define two open sets

$$D^{(i)} = \bigcup_{(x_o, t_o) + Q_R \in \mathcal{B}(\delta)} Q^{(i)}, \quad i = 1, 2.$$

The main object of this section is to prove the following facts.

Lemma 4.1 (Krylov and Safonov [147])

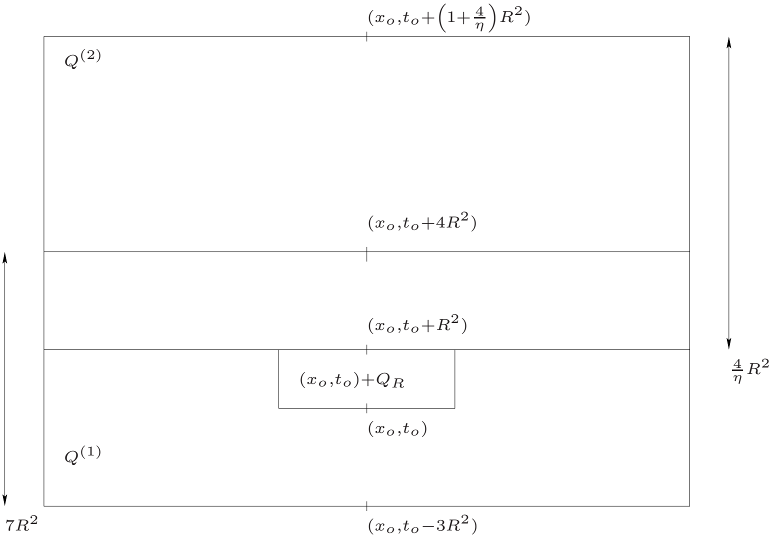

Fig. 4.4

(i) $|\mathcal{E}\backslash D^{(1)}| = 0$*;*
(ii) If $|\mathcal{E}| < \delta|Q_1|$*, then* $|\mathcal{E}| < \delta|D^{(1)}|$*;*
(iii) $|D^{(1)}| \le (1+\eta)|D^{(2)}|$*.*

Corollary 4.1 *Let* $\mathcal{E}$ *be a measurable subset of* Q_1 *and let* $\delta, \eta \in (0,1)$ *be given. Then,*

(I) either $D^{(1)} \equiv Q_1$*,*
(II) or $|D^{(2)}| \ge \frac{|\mathcal{E}|}{\delta(1+\eta)}$*.*

Proof of Lemma 4.1. We first consider (i). Let $\chi(\mathcal{E})$ be the characteristic function of $\mathcal{E}$. Since $\mathcal{E}$ is measurable, $\chi(\mathcal{E})$ is integrable, and by the Lebesgue theorem

$$\lim_{R\to 0} \frac{1}{|Q_R|} \iint_{(x_o,t_o)+Q_R} \chi(\mathcal{E})\, dxdt = 1$$

for a.e. $(x_o,t_o) \in \mathcal{E}$. Hence, except at most for a set of measure zero, we have

$$\lim_{R\to 0} \frac{|((x_o,t_o)+Q_R)\cap\mathcal{E}|}{|Q_R|} = 1,$$

and for R small enough $|((x_o,t_o)+Q_R)\cap\mathcal{E}| \ge \delta|Q_R|$. It follows that almost every point of $\mathcal{E}$ belongs to some $(x_o,t_o)+Q_R \in \mathcal{B}(\delta)$.

Let us now deal with (ii). We represent the open set $D^{(1)}$, up to a set of measure zero, as the union of binary boxes of the form $(x_o,t_o)+Q_R$ as follows.

Partition Q_1 with hyperplanes

$$t = \frac{1}{4},\ t = \frac{1}{2},\ t = \frac{3}{4},\ x_i = 0,\ x_i = \pm\frac{1}{2},\ i = 1, \ldots, N.$$

We obtain in this way 2^{2N+2} boxes each congruent to

$$\left\{ (x,t) :\ 0 < t < \frac{1}{4},\ 0 < x_i < \frac{1}{2},\ i = 1, \ldots, N \right\},$$

and Q_1 can be represented, up to a set of measure zero, as the union of such boxes.

We call the collection of such boxes by $\mathcal{C}_1$ and define

$$\Sigma_1 \equiv \left\{ \text{Collection of the binary boxes in } \mathcal{C}_1 \text{ that are all contained in } D^{(1)} \right\}.$$

If no boxes in $\mathcal{C}_1$ are all contained in $D^{(1)}$, we set

$$\Sigma_1 = \emptyset.$$

Consider next all those boxes in $\mathcal{C}_1$ that are not in Σ_1, and partition each of them into 2^{2N+2} boxes, each congruent to

$$\left\{ (x,t) :\ 0 < t < \frac{1}{4^2},\ 0 < x_i < \frac{1}{2^2},\ i = 1, \ldots, N \right\}.$$

Let us call $\mathcal{C}_2$ such a collection, and define

$$\Sigma_2 \equiv \left\{ \text{Collection of the binary boxes in } \mathcal{C}_2 \text{ that are all contained in } D^{(1)} \right\}.$$

If no boxes in $\mathcal{C}_2$ are all contained in $D^{(1)}$, we set Σ_2 to be the empty set.

Proceeding in this fashion, if $\mathcal{C}_n$ and Σ_n have been defined, we partition the boxes in $\mathcal{C}_n$ into 2^{2N+2} boxes, each congruent to

$$\left\{ (x,t) :\ 0 < t < \frac{1}{4^n},\ 0 < x_i < \frac{1}{2^n},\ i = 1, \ldots, N \right\},$$

and define $\mathcal{C}_{n+1}$ as the collection of such boxes. Set

$$\Sigma_{n+1} \equiv \left\{ \text{Collection of the binary boxes in } \mathcal{C}_{n+1} \text{ that are all contained in } D^{(1)} \right\},$$

and if no boxes in $\mathcal{C}_{n+1}$ are all contained in $D^{(1)}$, we set $\Sigma_{n+1} = \emptyset$.

We can immediately see that distinct boxes of Σ_n are disjoint, and that elements of Σ_n are disjoint from elements of Σ_m if $m \neq n$. Moreover,

$$|D^{(1)} \setminus \bigcup_{n \geq 1} \bigcup_{Q \in \Sigma_n} Q| = 0$$

$$|D^{(1)}| = \sum_{n=1}^{\infty} \sum_{Q\in\Sigma_n} |Q|$$

$$|\mathcal{E}| = |\mathcal{E} \cap D^{(1)}| = \sum_{n=1}^{\infty} \sum_{Q\in\Sigma_n} |\mathcal{E} \cap Q|. \tag{4.13}$$

To prove (ii) let us start by considering Σ_1. If there exists a box in Σ_1, say $\tilde{Q}$, such that

$$|\mathcal{E} \cap \tilde{Q}| \geq \delta|\tilde{Q}|, \quad \text{i.e., } \tilde{Q} \in \mathcal{B}(\delta),$$

then the corresponding cube $\tilde{Q}^{(1)}$, constructed as we did before with $Q^{(1)}$, would cover all Q_1, and hence,

$$D^{(1)} = Q_1.$$

We conclude that for all cubes $Q \in \Sigma_1$

$$|\mathcal{E} \cap Q| < \delta|Q|.$$

Next, consider Σ_{n+1}, $n \geq 1$. Suppose that there exists a box Q^* in Σ_{n+1} such that

$$|\mathcal{E} \cap Q^*| \geq \delta|Q^*|. \tag{4.14}$$

The box Q^* results from a partition of a cube in $\mathcal{C}_n$, which is not all contained in $D^{(1)}$. Let us call Q° such a cube.

By (4.14), $Q^* \in \mathcal{B}(\delta)$ and the corresponding $Q^{*(1)}$ constructed starting from Q^* as before, belongs to $D^{(1)}$ and covers Q° (by construction). Hence, $Q^\circ \subset D^{(1)}$, which is a contradiction. We conclude that

$$|\mathcal{E} \cap Q| < \delta|Q|, \qquad \forall\, Q \in \Sigma_n,\ \forall\, n \in \mathbb{N}.$$

From (4.13) now it follows that

$$|\mathcal{E}| < \delta \sum_{n=1}^{\infty} \sum_{Q\in\Sigma_n} |Q| = \delta|D^{(1)}|$$

and (ii) of the lemma follows.

We conclude with the proof of (iii). For this, we need a preliminary fact.

Lemma 4.2 *Let A be the set of all subintervals of $(-\infty, +\infty)$, $B \subset A$ and let $g : A \to A$ be a set-valued function satisfying*

$$|g(I)| \leq (1+\eta)|I|, \quad \eta > 0,$$

for all $I \in A$, where $|g(I)|$ and $|I|$ denote the one-dimensional Lebesgue measure of $g(I)$ and I. Assume that

$$g(I_1) \subset g(I_2), \quad \textit{whenever } I_1 \subset I_2.$$

Then,

$$|\bigcup_{I\in B} g(I)| \leq (1+\eta)|\bigcup_{I\in B} I|.$$

Proof. The set $\bigcup\{I : I \in B\}$ is open and it can be represented as the union of non-intersecting intervals I_n. Therefore,

$$\begin{aligned}|\bigcup_{I\in B} g(I)| &\le |\bigcup_n \bigcup_{I\subset I_n} g(I)| \\ &\le |\bigcup_n g(I_n)| \le \sum_n (1+\eta)|I_n| = (1+\eta)|\bigcup_{I\in B} I|.\end{aligned}$$

■

Proof of (iii). If $\chi(D^{(1)}\cup D^{(2)})$ is the characteristic function of $D^{(1)}\cup D^{(2)}$, by Fubini's Theorem

$$\begin{aligned}|D^{(1)}\cup D^{(2)}| &= \iint_{\mathbb{R}^{N+1}} \chi(D^{(1)}\cup D^{(2)})\,dxdt \\ &= \int_{\mathbb{R}^N}\left(\int_{-\infty}^{+\infty}\chi(D^{(1)}\cup D^{(2)})(x,t)\,dt\right)dx.\end{aligned}$$

Let $B(x)$ with $\|x\|<1$ be the set of one-dimensional intervals defined by

$$\Big\{t:\ (t,x)\in Q^{(2)},\ Q\in\mathcal{B}(\delta)\Big\}$$

and define for $I\in B(x)$ the function $g(I)$ as the result of expanding I of a factor $(1+\eta)$ by keeping the right-hand point fixed. Then, obviously,

$$\begin{aligned}\left|D^{(1)}\cup D^{(2)}\right| &= \int_{\mathbb{R}^N}\left(\int_{\bigcup_{I\in B(x)} g(I)}\chi\left(D^{(1)}\cup D^{(2)}\right)(x,t)\,dt\right)dx \\ &\le \int_{\mathbb{R}^N}|\bigcup_{I\in B(x)} g(I)|\,dx \\ &\le \int_{\mathbb{R}^N}(1+\eta)|\bigcup_{I\in B(x)} I|\,dx \\ &= (1+\eta)|D^{(2)}|.\end{aligned}$$

Consequently,

$$|D^{(1)}|\le(1+\eta)|D^{(2)}|$$

and the lemma is proved. ■

4.3 Two Technical Lemmas

To simplify the presentation, we restrict ourselves to linear operators and prove two lemmas for non-negative super-solutions of

$$\begin{aligned}u_t-\mathcal{L}(u)&\ge f \quad \text{in } E_T \\ \mathcal{L}(u)&=a_{ij}(x,t)u_{x_ix_j}-b_i(x,t)u_{x_i}+c(x,t)u.\end{aligned}\tag{4.15}$$

We assume $u\in C^{2,1}(E_T)$ and

$(a_{ij}(x,t))$ is positive definite $\forall\,(x,t)\in E_T$ and if $\lambda_N(x,t)\ge\lambda_{N-1}(x,t)\ge\cdots\ge\lambda_1(x,t)$ are the ordered eigenvalues, there exists $\lambda_o>0$ such that

$$\lambda_i(x,t)\ge\lambda_o\quad\forall\,(x,t)\in E_T,\quad\forall\,i=1,\dots,N. \tag{4.16}$$

There exist constants B_o, B_1 such that

$$\sum_{i=1}^N\|b_i\|_{\infty;E_T}\le B_o;\ \ \|c\|_{\infty;E_T}\le B_1. \tag{4.17}$$

$$f\in L^{N+1}(E_T). \tag{4.18}$$

Remark 4.4 Because of the local nature of our estimates, assumptions (4.16)–(4.18) need not be true in the whole E_T. If they are satisfied over compact subsets $\mathcal{K}\subset E_T$, then our estimates will be valid within $\mathcal{K}$.

The case of quasi-linear operators $Q[u]$ (see the beginning of the Section) will be treated later.

If $(x_o,t_o)\in E_T$, we will be working with the box $(x_o,t_o)+Q_\rho$, where, for $\rho>0$

$$Q_\rho\equiv K_\rho\times(0,\rho^2)\equiv\{x\in\mathbb{R}^N:\ \max_{1\le i\le N}|x_i|<\rho\}\times(0,\rho^2), \tag{4.19}$$

and will assume ρ to be so small that $(x_o,t_o)+Q_\rho\subset E_T$.

Lemma 4.3 *Let $k>0$ be fixed. For every $\sigma\in(0,1)$, there exist $\delta\in(0,1)$ and a constant γ depending only upon N, B_o, B_1, $\|f\|_{N+1;E_T}$ and independent of k, such that if*

$$\begin{aligned}|[u>k]\cap((x_o,t_o)+Q_\rho)|&\equiv|\{(x,t)\in(x_o,t_o)+Q_\rho:\ u(x,t)>k\}|\\&\ge\delta|Q_\rho|\end{aligned} \tag{4.20}$$

then $\forall\,(x,t)\in((x_o,t_o+(1-\sigma)\rho^2)+Q_{\sigma\rho}$

$$u(x,t)\ge\theta k-\gamma\rho^{\frac{N}{N+1}},$$

where $\theta=\dfrac{1}{2}e^{-B_1}$ and (see Figure 4.5)

$$(x_o,t_o+(1-\sigma)\rho^2)+Q_{\sigma\rho}\equiv\{\|x-x_o\|<\sigma\rho\}\times(t_o+(1-\sigma)\rho^2,t_o+\rho^2).$$

Proof. Without loss of generality, we may assume $(x_o,t_o)=(0,0)$. Since $u\ge 0$, the function $\tilde v=ue^{B_1t}$ satisfies

$$\tilde v_t-a_{ij}(x,t)\tilde v_{x_ix_j}+b_i(x,t)\tilde v_{x_i}\ge fe^{B_1t}\quad(c=0),$$

and $v=k-\tilde v$ satisfies

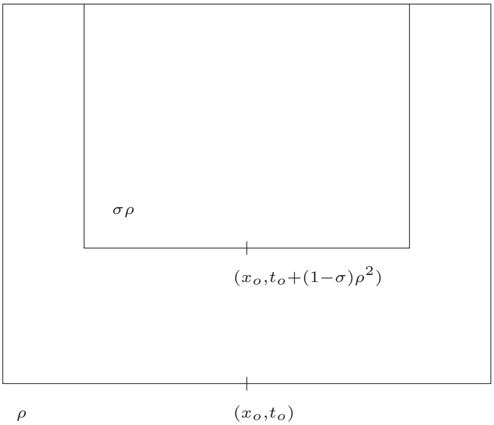

Fig. 4.5

$$v_t - a_{ij}(x,t)v_{x_ix_j} + b_i(x,t)v_{x_i} \le -fe^{B_1t}.$$

By Theorem 4.1 with $p = 1$

$$\sup_{Q_{\sigma\rho}} v_+ \le C_1 \iint_{Q_\rho} v_+\,dxdt + C_2\rho^{\frac{N}{N+1}},$$

where C_1, C_2 are two constants depending upon N, σ, B_o, B_1, $\|f\|_{N+1;E_T}$. Now the set $[v_+ > 0]\cap Q_\rho$ coincides with $\big\{(x,t) \in Q_\rho : e^{B_1t}u(x,t) < k\big\}$, which is included in the set $\{(x,t) \in Q_\rho : u(x,t) < k\}$. Therefore, for all $(x,t) \in Q_{\sigma\rho}$, using the assumption (4.20)

$$k - e^{B_1t}u(x,t) \le C_1k(1-\delta) + C_2\rho^{\frac{N}{N+1}},$$

that is,

$$u(x,t) \ge k\,[1 - C_1(1-\delta)]\,e^{-B_1} - C_2\rho^{\frac{N}{N+1}}\|f\|_{N+1;E_T}.$$

Choosing $(1-\delta) = \frac{1}{2C_1}$, $\theta = \frac{1}{2}e^{-B_1}$, $\gamma = C_2\|f\|_{N+1;E_T}$, the lemma follows. ■

In the next lemma we assume that (4.15)–(4.19) continue to hold and in addition

$$0 < \lambda_o \le \lambda_1(x,t) \le \cdots \le \lambda_N(x,t) \le B, \quad \forall\,(x,t) \in E_T, \tag{4.21}$$

for a given constant B. Here, as in (4.16), $\lambda_i(x,t)$ are the eigenvalues of $(a_{ij}(x,t))$.

Lemma 4.4 (Krylov and Safonov [147]) *Assume $u \ge 0$ and that*

$$u(x,t_o) \ge k, \quad \forall\,x \in B_{\varepsilon\rho}(x_o)$$

for $k > 0$ fixed and some $\varepsilon \in (0,1)$. Then for every $\sigma \in (0,1)$ there exist constants $\xi \in (0,1)$, $m \gg 1$ depending only upon λ_o, B_o, B_1, B, N, σ, and independent of k and ε, such that

$$u(x,t) \geq k\xi\varepsilon^m - C\rho^{\frac{N}{N+1}}\|f\|_{N+1;E_T}, \quad \forall\,(x,t) \in Q_\rho(\sigma),$$

where

$$Q_\rho(\sigma) \equiv B_{(1-\sigma)\rho}(x_o) \times (t_o + \sigma\rho^2, t_o + \sigma^{-1}\rho^2).$$

Remark 4.5 Lemma 4.4 can be seen as a statement about the expansion of positivity. Therefore, in the framework of parabolic equations in nondivergence form, it is analogous to Proposition 10.1 of Chapter 12 given for parabolic DeGiorgi classes.

Proof. Without loss of generality, we may assume that $(x_o, t_o) = (0,0)$, $\rho = 1$, and $k = 1$.

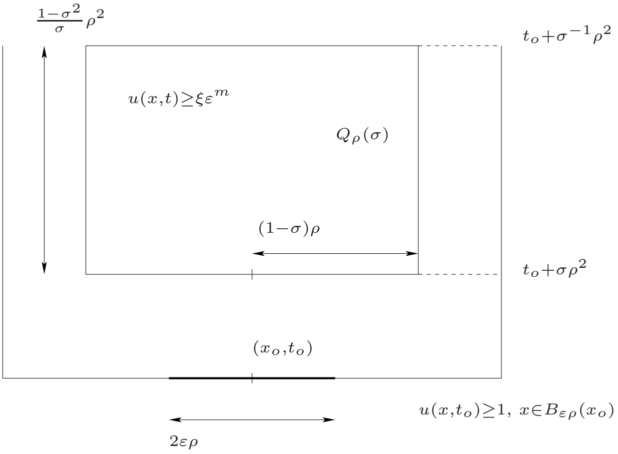
$\frac{1-\sigma^2}{\sigma}\rho^2$
$t_o+\sigma^{-1}\rho^2$
$u(x,t)\geq\xi\varepsilon^m$
$Q_\rho(\sigma)$
$(1-\sigma)\rho$
$t_o+\sigma\rho^2$
(x_o,t_o)
$u(x,t_o)\geq 1,\ x\in B_{\varepsilon\rho}(x_o)$
$2\varepsilon\rho$

Fig. 4.6

Also, without loss of generality, we may assume $\varepsilon^2 = \dfrac{1}{2}\sigma^2$.

We first prove the lemma for points $(x^*, t^*) \in Q_1(\sigma)$ satisfying

$$|x^*| < (1-\sigma), \qquad t^* = \sigma^{-1} \quad (\rho = 1).$$

Let such a point be fixed and consider the two sets

$$C \equiv \left\{(x,t) :\ 0 < t < \sigma^{-1},\ \|x - \sigma t x^*\|^2 < \frac{1}{2}\sigma^2 + \varepsilon^2\right\}$$

$$D \equiv \left\{(x,t) :\ 0 < t < \sigma^{-1},\ \|x - \sigma t x^*\|^2 < \frac{1}{2}\sigma^3 t + \varepsilon^2\right\}.$$

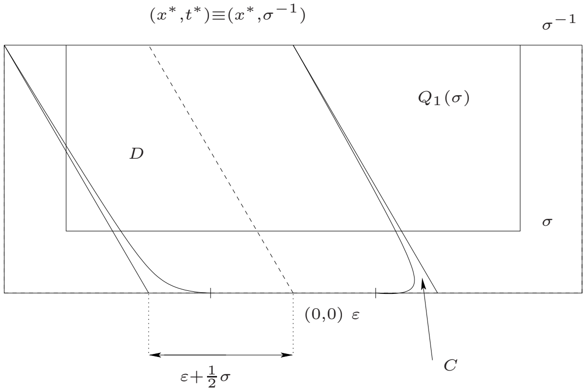
$(x^*,t^*)\equiv(x^*,\sigma^{-1})$
σ^{-1}
$Q_1(\sigma)$
D
σ
$(0,0)$ ε
C
$\varepsilon+\frac{1}{2}\sigma$

Fig. 4.7

Obviously, we have $D \subset C \subset Q_1$ (see Figure 4.7).

The parabolic boundary of D consists of two parts

$$\partial_1 D \equiv \left\{(x,t) : 0 < t < \sigma^{-1},\ \|x - \sigma t x^*\|^2 = \frac{1}{2}\sigma^3 t + \varepsilon^2\right\}$$
$$\partial_2 D \equiv \{(x,t) : t = 0,\ \|x\| < \varepsilon\}.$$

Consider the following functions

$$z(x,t) = \frac{x + \sigma t x^*}{\sqrt{\frac{1}{2}\sigma^3 t + \varepsilon^2}}, \quad z : \mathbb{R}^{N+1} \to \mathbb{R}^N,$$
$$\psi(x,t) = \left(1 - \|z(x,t)\|^2\right)_+^2 \Big/ \left(\frac{1}{2}\sigma^3 t + \varepsilon^2\right)^n,$$

where n is a positive constant to be chosen. Notice that by definition of $\partial_1 D$ we have $\|z(x,t)\| = 1$ on $\partial_1 D$ and

$$\begin{aligned} &(i) \quad \psi(x,t) \geq 0, \quad (x,t) \in D,\\ &(ii) \quad \psi(x,t) = 0, \quad (x,t) \in \partial_1 D,\\ &(iii) \quad 0 \leq \psi(x,0) \leq \varepsilon^{-2n}, \quad (x,t) \in \partial_2 D. \end{aligned} \tag{4.22}$$

We choose the constant n so that

$$\psi_t - \mathcal{L}(\psi) \leq 0 \ \text{ in } \ D.$$

By direct computation we have

$$
\begin{aligned}
\left(\frac{1}{2}\sigma^3 t+\varepsilon^2\right)^{n+1} & [\psi_t-\mathcal{L}(\psi)] \\
& = \left(1-\|z\|^2\right)^2\left[\left(\frac{1}{2}\sigma^3 t+\varepsilon^2\right)c(x,t)-\frac{n\sigma^3}{2}\right] \\
& \quad - 8a_{ij}\frac{\left(x_i-\sigma t x_i^*\right)\left(x_j-\sigma t x_j^*\right)}{\left(\frac{1}{2}\sigma^3 t+\varepsilon^2\right)} \\
& \quad - 2\left(\|z\|^2-1\right)\Big\{a_{ij}\delta_{ij}+2b_i\left(x_i-\sigma t x_i^*\right) \\
& \quad + \left(x_i-\sigma t x_i^*\right)x_i^*+\frac{\sigma^3}{2}\|z\|^2\Big\}.
\end{aligned}
$$

Set

$$
\begin{aligned}
A_1 &= \left(1-\|z\|^2\right)^2\left[\left(\frac{1}{2}\sigma^3 t+\varepsilon^2\right)c(x,t)-\frac{n\sigma^3}{2}\right] \\
A_2 &= -8a_{ij}(x,t)\frac{\left(x_i-\sigma t x_i^*\right)\left(x_j-\sigma t x_j^*\right)}{\left(\frac{1}{2}\sigma^3 t+\varepsilon^2\right)} \\
A_3 &= -2\left(\|z\|^2-1\right)\left\{a_{ij}\delta_{ij}+2b_i\left(x_i-\sigma t x_i^*\right)+\left(x_i-\sigma t x_i^*\right)x_i^*+\frac{\sigma^3}{2}\|z\|^2\right\}.
\end{aligned}
$$

We show that we can select n large enough so that

$$A_1+A_2+A_3<0.$$

We estimate the A_i's by using (4.15)–(4.19) and (4.21). We have

$$
\begin{aligned}
A_2 &\le -8\lambda_o\|z\|^2, \\
A_3 &\le 2\left(1-\|z\|^2\right)\left[B+2B_o\left(\varepsilon^2+\frac{1}{2}\sigma\right)^{\frac{1}{2}}+(1-\sigma)\left(\varepsilon^2+\frac{1}{2}\sigma\right)^{\frac{1}{2}}+\frac{\sigma^3}{2}\right] \\
&\le \gamma\left(1-\|z\|^2\right), \\
A_1 &\le \left(1-\|z\|^2\right)^2\left[B_1\sigma^2-\frac{n\sigma^3}{2}\right].
\end{aligned}
$$

Therefore,

$$A_1+A_2+A_3=-\left(8\lambda_o+\gamma\right)\|z\|^2+\gamma+\left(1-\|z\|^2\right)^2\left(\gamma-\frac{n\sigma^3}{2}\right),$$

where γ is a constant depending only upon the data and independent of ε.

Consider the following two cases.

Case 1. $1>\|z\|^2>\gamma/\left(8\lambda_o+\gamma\right)$.
In such a case the sum of the first two terms is nonpositive and

$$A_1+A_2+A_3\le\left(1-\|z\|^2\right)\left(\gamma-\frac{n\sigma^3}{2}\right).$$

Therefore, selecting $n \geq 2\gamma/\sigma^3$ we have

$$A_1 + A_2 + A_3 \leq 0.$$

Case 2. $\|z\|^2 < \gamma/\left(8\lambda_o + \gamma\right)$. Then,

$$\begin{aligned} A_1 + A_2 + A_3 &\leq \gamma + \left(1 - \frac{\gamma}{8\lambda_o + \gamma}\right)\left(\gamma - \frac{n\sigma^3}{2}\right) \\ &\leq 2\gamma - \frac{n\sigma^3}{2}\left(\frac{8\lambda_o}{8\lambda_o + \gamma}\right) \leq 0 \end{aligned}$$

for the choice $n > \dfrac{4\gamma}{\sigma^3}\left(\dfrac{8\lambda_o + \gamma}{8\lambda_o}\right)$.

Remark 4.6 The value of n depends upon σ and it deteriorates as the modulus of ellipticity λ_o tends to zero.

The function ψ is in $C^{2,1}(\bar{D})$ and it can be extended in a $C^{2,1}$ fashion with a function $\widetilde{\psi}$ defined in all Q_1 and such that $\tilde{\psi} \leq 0$ in $Q_1 \backslash D$.

Setting

$$w(x,t) = \varepsilon^{2n}\tilde{\psi}(x,t) - u(x,t) \in C^{2,1}\left(Q_1\right)$$

and recalling (4.22)-(iii), we have that $w \leq 0$ on the parabolic boundary of Q_1. Moreover, if ξ_w^+ is the positive increasing concave hull of w (see Section 3.5), the contact set $[\xi_w^+ = w]$ occurs within D, so that

$$w_t - \mathcal{L}(w) \leq -f \quad \text{in} \quad D.$$

By Theorem 3.5 with $\rho = 1$, $(x_o, t_o) \equiv \left(x^*, \sigma^{-1}\right)$

$$w\left(x^*, \sigma^{-1}\right) \leq \gamma_N \lambda_o^{-\frac{N}{N+1}} e^{B_o + B_1} \|f\|_{N+1;Q_1},$$

i.e.,

$$u\left(x^*, \sigma^{-1}\right) \geq \psi\left(x^*, \sigma^{-1}\right)\varepsilon^{2n} - C\|f\|_{N+1;Q_1}.$$

From the definition of ψ it follows that

$$\psi\left(x^*, \sigma^{-1}\right) \geq \sigma^{-2n}\left(1 - \left\|z\left(x^*, \sigma^{-1}\right)\right\|^2\right) \geq \sigma^{-2n}\sqrt{2}(\sqrt{2} - 1),$$

since $\sigma \in (0,1)$, $|x^*| < (1-\sigma)$, and $\varepsilon^2 < \frac{1}{2}\sigma^2$.

Finally, to prove the lemma for all points in $Q_1(\sigma)$, let $(x^*, t^*) \in Q_1(\sigma)$, and consider the comparison function

$$\begin{aligned} \bar{\psi}(x,t) &= \left(1 - \|\bar{z}(x,t)\|^2\right)_+^2 \Big/ \left(\frac{1}{2}\sigma^2\frac{t}{t^*} + \varepsilon^2\right)^n, \\ \bar{z}(x,t) &= \left(x + \frac{t}{t^*}x^*\right) \Big/ \sqrt{\frac{1}{2}\sigma^2\frac{t}{t^*} + \varepsilon^2}. \end{aligned}$$

Considerations in all analogous to those above, since $t^* \in \left(\sigma, \sigma^{-1}\right)$, yield the lemma for all $(x^*, t^*) \in Q_1(\sigma)$. ∎

4.4 The Harnack Inequality for Linear Equations

We will be dealing with linear operators

$$\mathcal{L}(u) = a_{ij}(x,t)u_{x_i x_j} - b_i(x,t)u_{x_i} + c(x,t)u,$$

where $u \in C^{2,1}(E_T)$ and

$$(x,t) \to a_{ij}(x,t) \in L^\infty(E_T), \quad i,j = 1,\dots,N, \tag{4.23}$$

and if $\lambda_1(x,t) \le \lambda_2(x,t) \le \cdots \le \lambda_N(x,t)$ are the ordered eigenvalues of $(a_{ij}(x,t))$, there exist $0 < \lambda_o \le \Lambda < \infty$ such that

$$0 < \lambda_o \le \lambda_1(x,t) \le \lambda_2(x,t) \le \cdots \le \lambda_N(x,t) < \Lambda \quad \text{for a.e. } (x,t) \in E_T.$$

There exist positive constants B_o, B_1 such that

$$\sum_{i=1}^N \|b_i\|_{\infty;E_T} \le B_o, \quad \|c\|_{\infty;E_T} \le B_1. \tag{4.24}$$

We are also given a function f satisfying

$$f \in L^{N+1}(E_T). \tag{4.25}$$

We will prove a weak Harnack inequality for non-negative super-solutions of

$$v_t - \mathcal{L}(v) = f, \quad v \in C^{2,1}(E_T), \tag{4.26}$$

and a strong Harnack inequality for non-negative solutions of (4.26).

Let $(x_o,t_o) \in E_T$ and consider the following boxes

$$\begin{cases} Q_R \equiv \{\|x-x_o\| < R\} \times \big(t_o - R^2, t_o + R^2\big) \\ Q_R^o \equiv \{\|x-x_o\| < R\} \times \big(t_o - R^2, t_o\big) \\ Q_R^+(\sigma) \equiv \{\|x-x_o\| < \sigma R\} \times \big(t_o + (1-\sigma)R^2, t_o + R^2\big) \\ Q_R^-(\sigma) \equiv \{\|x-x_o\| < \sigma R\} \times \big(t_o - (1-\sigma)R^2, t_o\big), \end{cases} \tag{4.27}$$

with $\sigma \in (0,1)$ (see Figure 4.8).

We assume R so small that $Q_{4R} \subset E_T$. Assume also that u is a super-solution (or a solution) of (4.26) in a neighborhood of Q_R, say for definiteness Q_{4R} (by a super-solution, we mean u that satisfies $u_t - \mathcal{L}(u) \ge f$).

Theorem 4.2 (The Weak Harnack Inequality). *Let $u \in C^{2,1}(Q_{4R})$ satisfy*

$$\begin{aligned} &u \ge 0 \quad \textit{in } Q_{4R}, \\ &u_t - \mathcal{L}(u) \ge f, \quad \textit{a.e. in } Q_{4R}, \end{aligned}$$

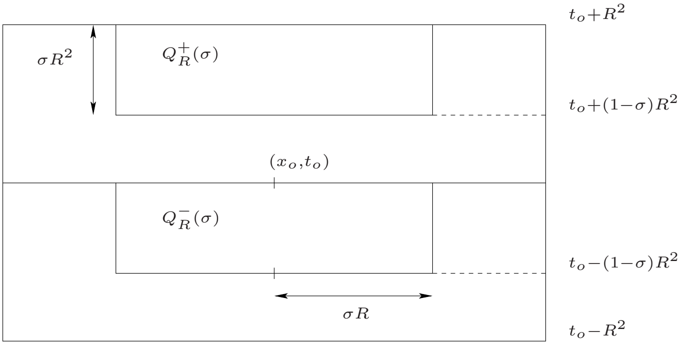

Fig. 4.8

and let (4.23)–(4.25) *hold. Fix* $\sigma \in (0,1)$ *and construct the boxes* $Q_R^\pm(\sigma)$ *as in* (4.27)*. There exist positive constants* $p \in (0,1)$*,* $\gamma_o \in (0,1)$*,* $\gamma_1 \gg 1$*, depending upon* N*,* B_i*,* $i = 0,1$*,* λ_o*,* Λ*,* σ*, and independent of* (x_o,t_o)*,* R*,* u*,* f *such that*

$$\inf_{Q_R^+(\sigma)} u + \gamma_1 R^{\frac{N}{N+1}} \|f\|_{N+1;E_T} \ge \gamma_o \left(\fint\!\!\!\fint_{Q_R^o} u^p \, dxdt \right)^{\frac{1}{p}} .$$

By combining Theorem 4.2 and Theorem 4.1 we have

Theorem 4.3 (The Strong Harnack Inequality). *Let* $u \in C^{2,1}(Q_{4R})$ *satisfy*

$$\begin{aligned} &u \ge 0 \quad \textit{in } Q_{4R}, \\ &u_t - \mathcal{L}(u) = f, \quad \textit{a.e. in } Q_{4R}, \end{aligned}$$

and let (4.23)–(4.25) *hold. Fix* $\sigma \in (0,1)$ *and construct the boxes* $Q_R^\pm(\sigma)$ *as in* (4.27)*.*

There exist positive constants $\gamma_o \in (0,1)$*,* $\gamma_1 \gg 1$ *depending upon* N*,* B_i*,* $i = 0,1$*,* λ_o*,* Λ*,* σ*, and independent of* (x_o,t_o)*,* R*,* u*,* f*, such that*

$$\inf_{Q_R^+(\sigma)} u + \gamma_1 \|f\|_{N+1;E_T} R^{\frac{N}{N+1}} \ge \gamma_o \sup_{Q_R^-(\sigma)} u \tag{4.28}$$

Remark 4.7 Inequality (4.28) could be formulated in a slightly more general geometrical setting. In particular, $Q_R^+(\sigma)$ and $Q_R^-(\sigma)$ could be replaced by

$$\begin{aligned} &Q_R^+(\sigma_1,\theta_1) \equiv \{\|x-x_o\| < \sigma_1 R\} \times \left(t_o + (1-\theta_1)R^2, t_o + R^2\right), \\ &Q_R^-(\sigma_2,\theta_2) \equiv \{\|x-x_o\| < \sigma_2 R\} \times \left(t_o - (1-\theta_2)R^2, t_o\right), \\ &\sigma_i,\, \theta_i \in (0,1), \quad i = 1,\, 2, \end{aligned}$$

provided the constants γ_o, γ_1 are suitably modified.

Remark 4.8 If $f = 0$, we obtain the classical version of the Harnack inequality.

Proof of Theorem 4.2 Without loss of generality we may assume $(x_o, t_o) = (0, 0)$ and define accordingly the boxes Q_R, Q^o_R, $Q^{\pm}_R(\sigma)$.

Case i): Let $k > 0$ be fixed and consider the set

$$\mathcal{E}_o \equiv \{(x,t) \in Q^o_R : \, u(x,t) > k\} .$$

Let $\delta \in (0,1)$ be the number claimed by Lemma 4.3 for the choice $\sigma = \frac{1}{2}$, and define

> $\mathcal{B}(\delta)$ the collection of sub-cubes of Q^o_R of the form $(\bar{x}, \bar{t}) + Q_\rho \subset Q^o_R$, $(\bar{x}, \bar{t}) \in Q^o_R$ and $Q_\rho \equiv \{\|x\| < \rho\} \times \big(0, \rho^2\big)$,

such that

$$|\mathcal{E}_o \cap ((\bar{x}, \bar{t}) + Q_\rho)| \geq \delta \, |Q_\rho| .$$

The number δ being fixed, motivated by Corollary 4.1 (II), we fix η so that

$$(1+\eta)\delta = \sqrt{\delta} = \delta_o. \tag{4.29}$$

By Lemma 4.3 (with $\sigma = \frac{1}{2}$), if $(\bar{x}, \bar{t}) + Q_\rho \in \mathcal{B}(\delta)$, then

$$u(x,t) > \theta k - \gamma \rho^{\frac{N}{N+1}} \|f\|,$$

$\forall \, (x,t) \in (\bar{x}, t^*) + Q_{\frac{1}{2}\rho}$, where $t^* = \bar{t} + \frac{1}{2}\rho^2$ and $\|f\| = \|f\|_{N+1;E_T}$; in particular, setting $v(x,t) = u(x,t) + \gamma \rho^{\frac{N}{N+1}} \|f\|$, we have

$$v\,(x, t^*) > \theta k, \qquad \text{in } \; \|x - \bar{x}\| < \frac{1}{2}\rho.$$

The function v satisfies a.e. in Q_{4R}

$$v_t - \mathcal{L}(v) \geq f + c(x,t)\, \gamma \rho^{\frac{N}{N+1}} \|f\|.$$

Therefore, applying Lemma 4.4 with ρ replaced 6ρ, $\varepsilon = \frac{1}{12}$, $\sigma = \eta$ (η fixed in (4.29)), we obtain

$$u(x,t) \geq \theta \xi 12^{-m} k - \gamma \rho^{\frac{N}{N+1}} \left\| f + c(x,t) \gamma \rho^{\frac{N}{N+1}} \|f\| \right\|, \tag{4.30}$$

in the box

$$Q^{(2)} \equiv \{\|x - \bar{x}\| < 3\rho\} \times \left(\bar{t} + \rho^2, \bar{t} + \frac{4}{\eta}\rho^2 \right),$$

where ξ, m are the constants claimed by Lemma 4.4 for the indicated choice of σ.

Without loss of generality, we may assume that R is so small that

$$\gamma R^{\frac{N}{N+1}} \left(\iint_{(\bar{x}, \bar{t}) + Q_{3\rho}} c^{N+1}(x,t)\, dx dt \right)^{\frac{1}{N+1}} \leq \|f\|,$$

so that from (4.30) it follows that

$$u(x,t) \ge \lambda k - 2\gamma \|f\| R^{\frac{N}{N+1}} \ \text{ in } \ Q^{(2)},$$

where $\lambda = \theta\xi 12^{-m}$.

Let

$$E_1 \equiv \Big\{(x,t) \in Q_R : \, u(x,t) > \lambda k - 2\gamma R^{\frac{N}{N+1}} \|f\|\Big\}$$

and

$$\mathcal{E}_1 \equiv \Big\{(x,t) \in Q_R^o : \, u(x,t) > \lambda k - 2\gamma R^{\frac{N}{N+1}} \|f\|\Big\}.$$

For each box $(\bar{x},\bar{t}) + Q_\rho \in \mathcal{B}(\delta)$ we construct $Q^{(1)}$ and $Q^{(2)}$ as in Section 4.2, and define accordingly the sets $D^{(1)}$, $D^{(2)}$.

The previous arguments show that $E_1 \subset D^{(2)}$, and by Corollary 4.1 (see (4.29))

$$|E_1| \ge |\mathcal{E}_o| / \delta_o. \tag{4.31}$$

Now let

$$\alpha = 1 - \sqrt{\delta_o}, \qquad \delta_* = \sqrt{\delta_o},$$

and consider the two sets $\mathcal{E}_1$ and $E_1 \backslash \mathcal{E}_1$. Then, in view of (4.31), we must have either

$$|\mathcal{E}_1| \ge \frac{1-\alpha}{\delta_o} |\mathcal{E}_o| = \frac{1}{\delta_*} |\mathcal{E}_o|, \tag{4.32}$$

or

$$|E_1 \backslash \mathcal{E}_1| > \frac{\alpha}{\delta_o} |\mathcal{E}_o|. \tag{4.33}$$

Case ii): The case when (4.33) occurs.

In such a case there must be a box $Q^{(2)} \subset D^{(2)}$ whose intersection with $t > \frac{\alpha}{\delta_o}(|\mathcal{E}_o| \, / \, |B_R|)R^2$, is not empty.

By construction, $Q^{(2)}$ is originated by a box $(\bar{x},\bar{t}) + Q_\rho \in \mathcal{B}(\delta)$ and the corresponding $Q^{(2)}$ is defined by

$$\{\|x - \bar{x}\| < 3\rho\} \times \left(\bar{t} + \rho^2, t_o + \frac{4}{\eta}\rho^2\right),$$

when $-R^2 \le t_o \le 0$. Therefore,

$$t_o + \rho^2 + \frac{4}{\eta}\rho^2 \ge \frac{\alpha}{\delta_o} \frac{|\mathcal{E}_o|}{|Q_R^o|} R^2,$$

i.e.,

$$\rho^2 \ge \left(\frac{\alpha\eta}{(4+\eta)\delta_o} \frac{|\mathcal{E}_o|}{|Q_R^o|}\right) R^2.$$

It follows that at the level $t = 0$, there exists a ball $B_\rho(\bar{x})$ of radius

$$\rho = \beta \left(\frac{|\mathcal{E}_o|}{|Q_R^o|} \right)^{1/2} R, \quad \beta = \left(\frac{\alpha\eta}{(4+\eta)\delta_o} \right)^{\frac{1}{2}},$$

all contained in B_R, such that

$$u(x,0) \ge \lambda k - 2\gamma \|f\| R^{\frac{N}{N+1}}.$$

Setting $v(x,t) = u(x,t) + 2\gamma R^{\frac{N}{N+1}} \|f\|$, we have

$$v_t - \mathcal{L}(v) \ge f + 2c(x,t)\gamma R^{\frac{N}{N+1}} \|f\| \quad \text{in } Q_{4R},$$

and

$$v(x,0) \ge \lambda k \quad \text{in } \{\|x - \bar{x}\| < \rho\} \subset B_R.$$

By Lemma 4.4 applied to v, with ρ replaced by $3R$ and

$$\varepsilon = \frac{1}{3} \left(\frac{\alpha\eta}{(4+\eta)\delta_o} \frac{|\mathcal{E}_o|}{|Q_R^o|} \right)^{\frac{1}{2}}, \quad \sigma = \frac{1}{3},$$

we obtain

$$v(x,t) \ge \xi \varepsilon^m \lambda k - C R^{\frac{N}{N+1}} \|f\|$$

for all $(x,t) \in Q_R^+(\sigma)$, i.e.,

$$\begin{aligned} u(x,t) &\ge C_o k \left(\frac{|\mathcal{E}_o|}{|Q_R^o|} \right)^{\frac{m}{2}} - C_1 R^{\frac{N}{N+1}} \|f\|, \\ C_o &= 3^{-m} \left(\frac{\alpha\eta}{(4+\eta)\delta_o} \right)^{\frac{m}{2}} \xi\lambda, \quad C_1 = (2\gamma + C). \end{aligned}$$

In particular,

$$\inf_{Q_R^+(\sigma)} u \ge C_o k \left(\frac{|\mathcal{E}_o|}{|Q_R^o|} \right)^{\frac{m}{2}} - C_1 R^{\frac{N}{N+1}} \|f\|. \tag{4.34}$$

Case iii): The iteration process.
We now repeat the process by starting with the set $\mathcal{E}_1$ previously defined. Proceeding as before, we find that either

$$\inf_{Q_R^+(\sigma)} u \ge C_o k \left(\frac{|\mathcal{E}_1|}{|Q_R^o|} \right)^{\frac{m}{2}} - (1 + C_o)\, C_1 R^{\frac{N}{N+1}} \|f\|$$

for the same constants C_o, C_1 as in (4.34), or

$$\begin{aligned} |\mathcal{E}_2| &= \left| \left\{ (x,t) \in Q_R^o : \, u(x,t) > \lambda^2 k - 2\gamma(1+\lambda) R^{\frac{N}{N+1}} \|f\| \right\} \right| \\ &\ge \frac{1-\alpha}{\delta_o} |\mathcal{E}_1| = \frac{1}{\delta_*} |\mathcal{E}_1| . \end{aligned}$$

Proceeding in this fashion we define sets

$$\mathcal{E}_s \equiv \Big\{(x,t) \in Q_R^o : \, u(x,t) > \lambda^s k - 2\gamma \left(1 + \lambda + \cdots + \lambda^s\right) R^{\frac{N}{N+1}} \|f\| \Big\}$$

and have the alternative of either

$$\inf_{Q_R^+(\sigma)} u \geq C_o \lambda^s k \left(\frac{|\mathcal{E}_s|}{|Q_R^o|} \right)^{\frac{m}{2}} - C_1 \left(1 + C_o + C_o^2 + \cdots + C_o^s\right) R^{\frac{N}{N+1}} \|f\|, \tag{4.35}$$

or

$$\frac{|\mathcal{E}_{s+1}|}{|Q_R^o|} > \delta_*^{-(s+1)} \frac{|\mathcal{E}_o|}{|Q_R^o|}. \tag{4.36}$$

Choose s^* from the equation

$$\delta_*^{-(s^*+1)} \frac{|\mathcal{E}_o|}{|Q_R^o|} = \delta.$$

Without loss of generality, we may assume that s^* is an integer. Then,

$$\lambda^{s^*} = C \left(\frac{|\mathcal{E}_o|}{|Q_R^o|} \right)^{\bar{c}}, \quad \bar{c} = \frac{\log_\lambda e^4}{\ln \delta} \tag{4.37}$$

for a constant C independent of k, u, R.

Therefore, if for some $s = 1, 2, \ldots, s^*$, (4.35) holds, since $|\mathcal{E}_s| \geq |\mathcal{E}_o|$, we have

$$\inf_{Q_R^+(\sigma)} u \geq \bar{C} k \left(\frac{|\mathcal{E}_o|}{|Q_R^o|} \right)^q - \bar{\bar{C}} R^{\frac{N}{N+1}} \|f\|,$$

where

$$\bar{C} = C_o C, \quad q = \bar{c} + \frac{m}{2}, \quad \bar{\bar{C}} = C_1 \left(\sum_{i=1}^{\infty} C_o^i \right).$$

Suppose now that (4.35) is not verified for any $s = 1, 2, \ldots, s^*$. Then, (4.36) must hold for all $s = 1, 2, \ldots, s^*$ and hence,

$$|\mathcal{E}_{s^*}| = \left| \Big\{(x,t) \in Q_R^o : \, u(x,t) > \lambda^{s^*} k - \bar{\gamma} R^{\frac{N}{N+1}} \|f\| \Big\} \right| \geq \delta \, |Q_R^o| \,,$$

where $\bar{\gamma} = 2\gamma \left(\sum_{i=1}^{\infty} \lambda^i \right)$.

Then, by Lemma 4.3 (with $\sigma = \frac{1}{2}$)

$$u(x,t) \geq \theta \lambda^{s^*} k - 2\bar{\gamma}(1+\theta) R^{\frac{N}{N+1}} \|f\|$$

in the box $\left\{\|x\| < \frac{R}{2}\right\} \times \left(-\frac{1}{2}R^2, 0\right)$. In particular, for $\|x\| < \frac{R}{2}$,

$$u(x,0) > \theta \lambda^{s^*} k - 2\bar{\gamma}(1+\theta) R^{\frac{N}{N+1}} \|f\|.$$

Define $v(x,t) = u(x,t) + 2\bar{\gamma}(1+\theta) R^{\frac{N}{N+1}} \|f\|$, which satisfies

$$v_t - \mathcal{L}(v) \ge f + 2\bar{\gamma} c(x,t)(1+\theta) R^{\frac{N}{N+1}} \|f\|$$

a.e. in Q_{4R}, and

$$v(x,0) \ge \theta \lambda^{s^*} k, \qquad \text{in } \|x\| < \frac{R}{2}.$$

By Lemma 4.4 we deduce that

$$u(x,t) \ge K_o \lambda^{s^*} k - K_1 R^{\frac{N}{N+1}} \|f\| \tag{4.38}$$

for all $(x,t) \in Q_R^+(\sigma)$, for two constants K_o, K_1 depending only upon the data.

From (4.38) and (4.37) it follows that

$$\inf_{Q_R^+(\sigma)} u \ge \bar{K} \left(\frac{|\mathcal{E}_o|}{|Q_R^o|} \right)^{\bar{c}} k - K_1 R^{\frac{N}{N+1}} \|f\|, \quad \bar{K} = K_o C.$$

Case iv): Proof of Theorem 4.2 concluded.
The previous arguments show that in either case there exist constants

$$A_o = \min\{\bar{K}, \bar{C}\}, \quad A_1 = \max\left\{ \bar{\bar{C}}; K_1 \right\}, \quad q \gg 1,$$

all depending upon the data and independent of u, R, f, such that

$$\inf_{Q_R^+(\sigma)} u \ge A_o k \left(\frac{|\mathcal{E}_o|}{|Q_R^o|} \right)^q - A_1 R^{\frac{N}{N+1}} \|f\|.$$

Set

$$D = A_o^{-1} \left(\inf_{Q_R^+(\sigma)} u + A_1 R^{\frac{N}{N+1}} \|f\| \right).$$

Then, recalling that $\mathcal{E}_o \equiv [u > k] \cap Q_R^o$, we have

$$\frac{|[u>k]|}{|Q_R^o|} \le k^{-\frac{1}{q}} D^{\frac{1}{q}}, \qquad \forall\, k > 0. \tag{4.39}$$

We multiply both sides by $k^{\frac{1}{2q}-1}$ and integrate with respect to k over $\mathbb{R}^+$ to get

$$\begin{aligned} \frac{1}{|Q_R^o|} \iint_{Q_R^o} u^{\frac{1}{2q}}\, dxdt &= \int_0^\infty k^{\frac{1}{2q}-1} \frac{|[u>k]|}{|Q_R^o|}\, dk \\ &= \int_0^D k^{\frac{1}{2q}-1} \frac{|[u>k]|}{|Q_R^o|}\, dk + \int_D^\infty k^{\frac{1}{2q}-1} \frac{|[u>k]|}{|Q_R^o|}\, dk \\ &\le 2q D^{\frac{1}{2q}} + D^{\frac{1}{q}} \int_D^\infty k^{-\frac{1}{q}-1}\, dk = q D^{\frac{1}{2q}}, \end{aligned}$$

where we have used (4.39).

If $\frac{1}{2q} = p \in (0,1)$ we have

$$\left(\iint_{Q_R^o} u^p \, dxdt \right)^{\frac{1}{p}} \le q^{2q} A_o^{-1} \left(\inf_{Q_R^+(\sigma)} u + A_1 R^{\frac{N}{N+1}} \|f\| \right).$$

■

4.5 The Harnack Inequality for Quasi-Linear Equations

In this section we adapt and modify the arguments of Section 4.4 to obtain the weak Harnack inequality for $C^{2,1}(E_T)$ non-negative super-solutions of

$$\begin{aligned} &u_t - Q[u] \ge 0 \quad \text{in } E_T \\ &Q[u] = A_{ij}(x,t,u,Du) u_{x_i x_j} + B(x,t,u,Du). \end{aligned} \tag{4.40}$$

The structural assumptions on $A_{ij} : \mathbb{R}^{2N+2} \to \mathbb{R}^{N \times N}$ and $B : \mathbb{R}^{2N+2} \to \mathbb{R}$ are more restrictive than the ones in Section 4.1. Namely, we assume the following.

> The matrix $(A_{ij}(x,t,u,Du))$ is positive definite in $E_T \times \mathbb{R}^{N+1}$ and if $\lambda_1(x,t,u,Du) \le \lambda_2(x,t,u,Du) \le \cdots \le \lambda_N(x,t,u,Du)$ are the (ordered) eigenvalues, there exist numbers $0 < \lambda_o \le \Lambda_o < \infty$, such that
>
> $$0 < \lambda_o \le \lambda_i(x,t,u,Du) \le \Lambda_o, \quad i = 1, \dots, N. \tag{4.41}$$
>
> $$B(x,t,u,Du) \ge -\gamma_* |Du| - \gamma_* u - f(x,t), \tag{4.42}$$
>
> where γ_* is a non-negative given constant and $f \in L^{N+1}(E_T)$.

For notational simplicity we set

$$f_o = \|f\|_{N+1;E_T}.$$

With this notation, the weak Harnack inequality is stated exactly as in Theorem 4.2. By combining Theorem 4.2 and Theorem 4.1, valid for sub-solutions of quasi-linear equations, we obtain the strong Harnack inequality stated exactly as in Theorem 4.3.

We have seen in the previous sections that Theorem 4.2 (the weak Harnack inequality) is derived only by making use of the measure-theoretical lemma of Krylov–Safonov (see Section 4.2) and the two technical lemmas of Section 4.3.

Therefore, to extend Theorem 4.2 to super-solutions of quasi-linear equations, it will be enough to prove results analogous to Lemmas 4.3 and 4.4.

As before, we will be working with boxes $(x_o, t_o) + Q_\rho \subset E_T$, congruent to

$$Q_\rho \equiv B_\rho \times \left(0, \rho^2\right) \equiv \{|x| < \rho\} \times \left(0, \rho^2\right).$$

Lemma 4.5 *Let $k > 0$ be fixed. For every $\sigma \in (0,1)$ there exist $\delta \in (0,1)$ and $\bar{\gamma} > 1$ depending only upon N, γ_*, λ_o, Λ_o, and independent of k, f_o, ρ, such that if*

$$\left| \{(x,t) \in ((x_o,t_o) + Q_\rho) : \, u(x,t) + f_o > k\} \right| > \delta \, |Q_\rho| ,$$

then

$$u(x,t) \ge \frac{1}{2} k - \bar{\gamma} \rho^{\frac{N}{N+1}} f_o, \quad \forall \, (x,t) \in Q_{\sigma\rho} \left(x_o, t_o \right),$$

where $Q_{\sigma\rho} \left(x_o, t_o \right) \equiv \big(x_o, t_o + (1-\sigma)\rho^2 \big) + Q_{\sigma\rho}$.

Proof. Setting

$$v = k - \left(u + f_o \rho^{\frac{N}{N+1}} \right),$$

we have

$$v_t - A_{ij} \left(x, t, u, Du \right) v_{x_i x_j} \le \tilde{B}[v] \quad \text{ in } \; E_T,$$

where, by (4.42)

$$\begin{aligned} \tilde{B}[v] &= -B \left(x, t, k + f_o \rho^{\frac{N}{N+1}} - v, -Dv \right) \\ &\le \gamma_* |Dv| + \gamma_* v + k + f_o \rho^{\frac{N}{N+1}} + f. \end{aligned}$$

By Theorem 4.1 applied with $p = 1$, if $\sigma \in (0,1)$

$$\begin{aligned} \sup_{Q_{\sigma\rho}(x_o,t_o)} v_+ \le & C_1 \iint_{(x_o,t_o)+Q_\rho} v_+(x,t) \, dxdt \\ & + \gamma \rho^{\frac{N}{N+1}} \left\| f + f_o + k \right\|_{N+1; E_T} . \end{aligned} \tag{4.43}$$

From Remark 4.2, in view of the hypotheses (4.41)–(4.42), it follows that $C_1 = C_1(N)$.

Now the set $[v > 0] \cap ((x_o,t_o) + Q_\rho)$ coincides with the set $[u + f_o > k] \cap ((x_o,t_o) + Q_\rho)$. Therefore, using the assumptions of the lemma in (4.43) we find

$$\sup_{Q_{\sigma\rho}(x_o,t_o)} v_+ \le \gamma\delta \left(k - f_o \rho^{\frac{N}{N+1}} \right) + 2\gamma f_o \rho^{\frac{N}{N+1}} + \gamma k \rho^{\frac{N}{N+1}},$$

i.e.,

$$u(x,t) \ge k \left(1 - \gamma(1-\delta) - \gamma \rho^{\frac{N}{N+1}} \right) - f_o (1 + 2\gamma) \rho^{\frac{N}{N+1}}.$$

We assume ρ to be so small that $\rho^{\frac{N}{N+1}} < 1 - \delta$ and then choose $(1-\delta)$ so small that

$$1 - \gamma(1-\delta) - \gamma \rho^{\frac{N}{N+1}} \ge 1 - 2\gamma(1-\delta) = \frac{1}{2}.$$

Setting

$$\bar{\gamma} = 1 + 2\gamma$$

the lemma follows. ■

Lemma 4.6 *Let* $u \geq 0$ *be a smooth, non-negative super-solution of* (4.40) *in* E_T. *Let* $(x_o, t_o) + Q_\rho \subset E_T$ *and assume that at the level* $t = t_o$

$$u(x, t_o) \geq k, \qquad \textit{in } \|x - x_o\| < \varepsilon\rho$$

for $k > 0$ *fixed,* $\varepsilon \in (0, 1)$ *given.*

Then for every $\sigma \in (0, 1)$, *there exist constants* $\xi \in (0, 1)$, $m \gg 1$, $\gamma > 0$ *depending only upon* N, λ_o, Λ_o, γ_*, *and independent of* f, ρ, k, u, *such that*

$$u(x, t) \geq k\xi\varepsilon^m - \gamma\rho^{\frac{N}{N+1}} f_o, \qquad \forall\, (x, t) \in Q_\rho(\sigma),$$

where $Q_\rho(\sigma) \equiv \{\|x - x_o\| < (1 - \sigma)\rho\} \times \big(t_o + \sigma\rho^2, t_o + \sigma^{-1}\rho^2\big)$, *provided*

$$\rho^{\frac{N}{N+1}} < \gamma^{-1}\varepsilon^m. \tag{4.44}$$

Remark 4.9 By following step by step the proof in Section 4.4, it is easily seen that the requirement (4.44) does not affect the argument, except for minor changes.

Proof of Lemma 4.6 As before, we may prove the lemma for $(x_o, t_o) = (0, 0)$ and momentarily make a change of variable, so that $\rho = 1$.

We refer to Figure 4.6 introduced in the proof of Lemma 4.4, construct the same domains C, D, and consider the same comparison function

$$\tilde{\psi}(x, t) = \big(1 - \|z(x, t)\|^2\big)_+^2 \Big/ \left(\frac{1}{2}\sigma^3 t + \varepsilon^2\right)^n$$

with $z(x, t) = (x - \sigma t x^*) \Big/ \left(\frac{1}{2}\sigma^3 t + \varepsilon^2\right)^n$, and n to be chosen.

As before, the choice of such a $\tilde{\psi}$ is made to prove the lemma at the point $(x^*, t^*) = (x^*, \sigma^{-1})$, with $|x^*| < 1 - \sigma$.

Define

$$\psi(x, t) = \varepsilon^{2n} k \tilde{\psi}(x, t)$$

and observe that

$$\begin{cases} \psi \geq 0 & \text{in } D, \\ \psi(x, t) = 0 & (x, t) \in \partial_1 D, \\ \psi(x, t) \leq k & (x, t) \in \partial_2 D. \end{cases} \tag{4.45}$$

We choose the constant n so that

$$\psi_t - \mathcal{L}_o(\psi) \leq 0 \quad \text{ in } D, \tag{4.46}$$

where

$$\mathcal{L}_o(\psi) = A_{ij}\,(x, t, u, Du)\, \psi_{x_i x_j}.$$

Since $\mathcal{L}_o(\cdot)$ is linear and $(A_{ij}(x, t, u, Du))$ satisfies (4.41), by repeating the calculations of Section 4.3 we see that there exists $n = n\,(N, \sigma, \lambda_o, \Lambda_o)$ such that (4.46) holds in D.

Set

$$w = \psi - u,$$

and observe that by (4.45) $w \le 0$ on $\partial_1 D$ and $\partial_2 D$. We consider w extended in Q_2 in a $C^{2,1}$ fashion, so that $w \le 0$ outside D.

By calculation and (4.46)

$$\begin{aligned} w_t - \mathcal{L}_o(w) &\le -\left[u_t - A_{ij}(x,t,u,Du)u_{x_i x_j}\right] \\ &\le B(x,t,u,Du) \le \gamma(u + |Du|) + f \end{aligned}$$

on the set $\left[\xi^+_{w_+} = w_+\right]$, $u \le \psi$ and $|Dw| \le \psi$, i.e.,

$$|Du| \le \psi + |D\psi|.$$

Therefore, on $\left[\xi^+_{w_+} = w_+\right]$

$$w_t - \mathcal{L}_o(w) \le \gamma(\psi + |D\psi|) + |f|.$$

By the maximum principle, Theorem 3.5

$$w\left(x^*, t^*\right) \le \gamma f_o + \gamma \left(\iint_D |\psi + D\psi|^{N+1}\, dxdt \right)^{\frac{1}{N+1}}.$$

Estimating the last integral we have

$$|D\psi| \le \varepsilon^{2n} k \frac{\left(1 - \|z\|^2\right)_+}{g^{n+1/2}} \le \left(\varepsilon^n k\right) \frac{\left(1 - \|z\|^2\right)_+}{g^{1/2}},$$

where

$$g(t) = \left(\frac{1}{2}\sigma^3 t + \varepsilon^2 \right).$$

Moreover,

$$\psi \le \varepsilon^n k \frac{\left(1 - \|z\|^2\right)_+}{g^{1/2}}.$$

Therefore,

$$\begin{aligned} I &= \iint_D (\psi + |D\psi|)^{N+1}\, dxdt \\ &\le \left(\varepsilon^n k\right)^{N+1} \int_0^{\sigma^{-1}} \int_{D(\tau)} \left[\frac{\left(1 - \|z\|^2\right)_+}{g^{1/2}(\tau)} \right]^{N+1} dxd\tau. \end{aligned}$$

Notice that $g^{1/2}(\tau)$ is the radius of the ball $D(\tau)$ so that

$$I \le \left(\varepsilon^n k\right)^{N+1} \int_0^{\sigma^{-1}} [g(\tau)]^{\frac{N}{2} - \frac{N+1}{2}}\, d\tau$$

$$= (\varepsilon^n k)^{N+1} \int_0^{\sigma^{-1}} \frac{1}{(\frac{1}{2}\sigma^3 t + \varepsilon^2)^{1/2}}\, d\tau = \gamma(\sigma)(\varepsilon^n k)^{N+1}.$$

Therefore, the last integral in the estimate of $w\,(x^*, t^*)$ is bounded by $\gamma f_o + \gamma\varepsilon^n k$. Scaling back to the cube of dimension ρ, we find

$$u\,(x^*, t^*) \ge \left(\frac{\varepsilon}{\sigma}\right)^{2n} k - \gamma\rho^{\frac{N}{N+1}}\varepsilon^n k - \gamma\rho^{\frac{N}{N+1}} f_o.$$

The lemma follows if ρ is so small to satisfy (4.44), for the point (x^*, t^*).

For the remaining points in $Q_\rho(\sigma)$ the argument of Lemma 4.4 applies to the present situation. ■

4.6 Local Hölder Continuity of Solutions

We let $u \in W^{2,1}_{N+1}\,(E_T)$ (see Section 3.5 for the definition of such a space) be a solution of

$$u_t - a_{ij}(x,t)u_{x_i x_j} + b_i(x,t)u_{x_i} = f, \tag{4.47}$$

where $(a_{ij}(x,t))$ and $\mathbf{b}(x,t)$ satisfy (4.23)–(4.24).

We show that the Harnack inequality proved in the previous sections implies an a priori estimate of the Hölder modulus of continuity of u over compact subsets of E_T, depending only upon λ_o, Λ, B_o, N. The proof we provide closely resembles the one given in Section 11 of Chapter 12. We repeat it here to keep the two chapters self-contained.

We will treat later the case when the operator in (4.47) contains the term $c(x,t)u$ and when u solves a quasi-linear equation.

Let $(x_o, t_o) \in E_T$ and R_o so small that

$$Q_{R_o} \equiv B_R(x_o) \times \left(t_o - R_o^2, t_o\right)$$

is all contained in E_T. Set

$$\begin{cases} \omega_o = \operatorname*{ess\,osc}_{Q_{R_o}} u = \operatorname*{ess\,sup}_{Q_{R_o}} u - \operatorname*{ess\,inf}_{Q_{R_o}} u \\ \omega_1 = \operatorname*{ess\,osc}_{Q_{R_o/2}} u \end{cases}$$

Lemma 4.7 *There exist constants $\theta \in (0,1)$ and $\gamma > 1$ depending only upon λ_o, Λ, B_o, N and independent of R_o, u, f such that*

$$\omega_1 \le \theta\omega_o + \gamma f_o R_o^{\frac{N}{N+1}}, \qquad f_o = \|f\|_{N+1;E_T}.$$

Proof. Together with Q_{R_o} consider the cylinders

$$Q^+ \equiv Q_{R_o/2}, \quad Q^- \equiv B_{\frac{R_o}{2}}(x_o) \times \left(t_o - \frac{7}{8}R_o^2, t_o - \frac{5}{8}R_o^2\right)$$

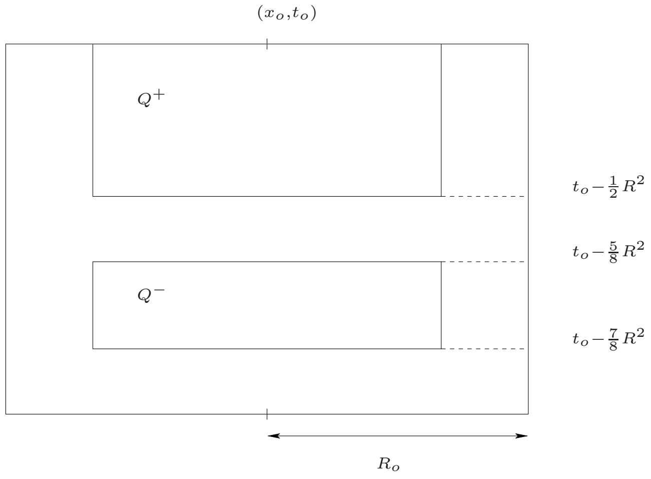

Fig. 4.9

(see Figure 4.9). Define the functions

$$U = \operatorname*{ess\,sup}_{Q_{R_o}} u - u; \quad V = u - \operatorname*{ess\,inf}_{Q_{R_o}} u.$$

Then, obviously, $U \ge 0$, $V \ge 0$ and they both satisfy (4.40) (with f replaced by $-f$ for the V). By the Harnack inequality (see Remark 4.7)

$$\begin{aligned}\operatorname*{ess\,sup}_{Q_{R_o}} u - \operatorname*{ess\,sup}_{Q_{R_o/2}} u &\ge \gamma_o \left(\operatorname*{ess\,sup}_{Q_{R_o}} u - \operatorname*{ess\,inf}_{Q^-} u \right) - \bar{\gamma} f_o R_o^{\frac{N}{N+1}} \\ \operatorname*{ess\,inf}_{Q_{R_o/2}} u - \operatorname*{ess\,inf}_{Q_{R_o}} u &\ge \gamma_o \left(\operatorname*{ess\,sup}_{Q^-} u - \operatorname*{ess\,inf}_{Q_{R_o}} u \right) - \bar{\gamma} f_o R_o^{\frac{N}{N+1}}\end{aligned}$$

where $\gamma_o \in (0,1)$ and $\bar{\gamma}$ depend only upon the data.

By addition

$$\omega_o - \omega_1 \ge \gamma_o \omega_o - 2\bar{\gamma} f_o R_o^{\frac{N}{N+1}},$$

and the lemma follows with $\theta = 1 - \gamma_o$, $\gamma = 2\bar{\gamma}$. ■

Next consider the sequence of radii

$$R_n = \frac{R_o}{2^n}, \quad n = 0, 1, 2, \dots \tag{4.48}$$

and set

$$\omega_n = \operatorname*{ess\,osc}_{Q_{R_n}} u.$$

Iteration of Lemma 4.7 yields $\omega_{n+1} \le \theta \omega_n + \gamma f_o R_n^{\frac{N}{N+1}}$ and

$$\omega_n \le \omega_o \theta^n + \gamma f_o \sum_{i=0}^{n} \theta^i R_{n-i}^{\frac{N}{N+1}}. \tag{4.49}$$

From (4.48) by taking logarithms in the basis $\frac{1}{\theta}$

$$n = \ln_{1/\theta} \left(\frac{R_o}{R_n} \right)^{1/\ln_{\frac{1}{\theta}} 2},$$

and

$$\theta^i = \left(\frac{R_i}{R_o} \right)^{1/\ln_{\frac{1}{\theta}} 2} = \left(\frac{R_i}{R_o} \right)^{\eta},$$

$$\eta = |\ln \theta| / \ln 2.$$

It follows from (4.49) that

$$\omega_n \le \omega_o \left(\frac{R_n}{R_o} \right)^{\eta} + \gamma f_o \sum_{i=0}^{n} \left(\frac{R_i}{R_o} \right)^{\eta} R_{n-i}^{\frac{N}{N+1}}. \tag{4.50}$$

If we let

$$\alpha = \min\{\eta; N/N+1\}, \tag{4.51}$$

$$\begin{aligned} \sum_{i=0}^{n} \left(\frac{R_i}{R_o} \right)^{n} R_{n-i}^{\frac{N}{N+1}} &\le \sum_{i=0}^{n} 2^{-i\alpha} 2^{-(n-i)\alpha} R_o^{\alpha} \\ &\le \gamma(\alpha) 2^{-n\alpha} R_o^{\alpha} = \gamma(\alpha) R_n^{\alpha}, \end{aligned}$$

so that (4.50) implies

$$\omega_n \le A \left(\frac{R_n}{R_o} \right)^{\alpha}, \quad n = 0, 1, 2, \dots, \tag{4.52}$$

where $\alpha > 0$ is defined by (4.51) and

$$A = (\omega_o + \gamma \|f\|_{N+1; E_T} R_o^{\alpha}).$$

Now if $0 < \rho < R_o$, there exists $n \in \mathbb{N}$ such that $R_{n+1} \le \rho < R_n$. Since

$$R_n = 2R_{n+1} \le 2\rho$$

and

$$\operatorname*{ess\,osc}_{Q_\rho} u \le \operatorname*{ess\,osc}_{Q_{R_n}} u \le \omega_n,$$

it follows from (4.52)

$$\operatorname*{ess\,osc}_{Q_\rho} u \le 2^{\alpha} A \left(\frac{\rho}{R_o} \right)^{\alpha}, \qquad \forall\, 0 < \rho < R_o. \tag{4.53}$$

Let $\mathcal{K}$ be a compact subset of E_T and set

$$d = \min \{1; \operatorname{dist}(\mathcal{K}, \partial_p E_T)\},$$

where $\partial_p E_T$ is the parabolic boundary of E_T.

Lemma 4.8 *For every pair of points* (x,t), $(y,\tau)\in\mathcal{K}$

$$|u(x,t)-u(y,\tau)|\le C\left\{|x-y|^\alpha+|t-\tau|^{\alpha/2}\right\}$$

where

$$C=\gamma^*\left\{\|u\|_{\infty;E_T}+\gamma\|f\|_{N+1;E_T}d^\alpha\right\}d^{-\alpha}, \tag{4.54}$$

where γ *depends only upon* N, α.

Proof. Suppose (x,t), $(y,\tau)\in\mathcal{K}$ are such that

$$|x-y|<d,\quad |t-\tau|<d^2. \tag{4.55}$$

Then by (4.53)

$$\begin{aligned}|u(x,t)-u(y,\tau)|\le&|u(x,t)-u(y,t)|+|u(y,t)-u(y,\tau)|\\ \le&2^{\alpha+1}Ad^{-\alpha}\left(|x-y|^\alpha+|t-\tau|^{\alpha/2}\right)\\ \le&2^{\alpha+1}\left(2\|u\|_{\infty;E_T}+\gamma\|f\|_{N+1;E_T}d^\alpha\right)\left(|x-y|^\alpha+|t-\tau|^{\alpha/2}\right)\\ \le&C\left\{|x-y|^\alpha+|t-\tau|^{\alpha/2}\right\}.\end{aligned}$$

If either one of (4.55) is violated, then obviously

$$|u(x,t)-u(y,\tau)|\le 2\|u\|_{\infty;E_T}\left(\frac{|x-y|^\alpha+|t-\tau|^{\alpha/2}}{d^\alpha}\right)$$

and the lemma follows. ∎

4.7 Hölder Continuity of Solutions of Quasi-Linear Equations

We assume here that $u\in W^{2,1}_{N+1,\mathrm{loc}}(E_T)$ is a local solution of

$$u_t-Q[u]=0\quad\text{ in }\ E_T, \tag{4.56}$$

where

$$Q(u]=A_{ij}(x,t,u,Du)u_{x_ix_j}-B(x,t,u,Du),$$

the matrix $(A_{ij}(x,t,u,Du))$ satisfies (4.41), and B satisfies

$$|B(x,t,u,Du)|\le\gamma_*(|u|+|Du|)+|f(x,t)|,\quad\text{with}\quad f\in L^{N+1}(E_T).$$

If $(x_o,t_o)\in E_T$ and

$$Q_{R_o}=B_{R_o}(x_o)\times\left(t_o-R_o^2,t_o\right)\subset E_T,$$

we let

$$\mu^+=\operatorname*{ess\,sup}_{Q_{R_o}}u,\qquad \mu^-=\operatorname*{ess\,inf}_{Q_{R_o}}u.$$

The functions

$$U(x,t) = \mu^+ - u(x,t), \qquad V = u(x,t) - \mu^-$$

are non-negative and satisfy an equation such as (4.56). To be specific, U satisfies

$$U_t - A_{ij}(x,t,u,Du)U_{x_i x_j} = B(x,t,u,Du)$$

and

$$|B(x,t,u,Du)| \le \gamma_* \left(|U| + |DU|\right) + \gamma_* \mu^+ + |f|.$$

Therefore, the Harnack inequality for U and V holds with $f_o = \|f\|_{N+1;E_T}$ replaced by

$$\|f\|_{N+1;E_T} + \gamma_* \mu^+ |E_T|.$$

Consequently, the proof of the Hölder continuity of solutions, given in Lemma 4.8, also remains unchanged, except for substituting C in (4.54) with

$$C' = \gamma^* \|u\|_{\infty;\mathcal{K}} \left(1 + \gamma \left(\|f\|_{N+1;\mathcal{K}} + 1\right) d^\alpha\right) d^{-\alpha}.$$

Problems and Complements

1c Introductory Material

1.1c Introduction

1.1.1c Linear Equations

We show how Lemma 1.1 implies that $(1.2)_{(i)}$ and $(1.2)_{(ii)}$ are equivalent. Indeed, employing the row by column product of matrices, we have

$$\begin{aligned} a_{ij}(x,t)\xi_i\xi_j &= (\xi_1, \dots, \xi_N)(a_{ij}(x,t)) \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_N \end{pmatrix} \\ &= \boldsymbol{\xi} \mathbb{X}\mathbb{X}^{-1}(a_{ij}(x,t))\mathbb{X}\mathbb{X}^{-1}\boldsymbol{\xi}^t. \end{aligned}$$

Since $\mathbb{X}^{-1} = \mathbb{X}^t$, setting $\boldsymbol{\eta} = (\eta_1, \dots, \eta_N) = (\xi_1, \dots, \xi_N)\mathbb{X}$, we have $\mathbb{X}^{-1}\boldsymbol{\xi}^t = \boldsymbol{\eta}^t$, and hence,

$$a_{ij}(x,t)\xi_i\xi_j = \boldsymbol{\eta} \begin{pmatrix} \lambda_1(x,t) & \dots & 0 \\ 0 & \lambda_2(x,t) & \dots \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_N(x,t) \end{pmatrix} \boldsymbol{\eta}^t = \sum_{i=1}^N \lambda_i(x,t)\eta_i^2$$

Now, $\eta_i = \sum_{j=1}^{N} \xi_j x_{ij}$ and

$$|\boldsymbol{\eta}|^2 = \sum_{i=1}^{N} \eta_i^2 = \sum_{i,j,l=1}^{N} \xi_j \xi_l x_{ij} x_{il} = \sum_{j,l=1}^{N} \xi_j \xi_l \sum_{i=1}^{N} x_{ij} x_{il}.$$

However, $\sum_{i=1}^{N} x_{ij} x_{il}$ is the $(j,l)^{th}$ element of the row by row product of $\mathbb{X} \cdot \mathbb{X}$. Since $\mathbb{X}^{-1} = \mathbb{X}^t$, this is the same as the $(l,j)^{th}$ element of the row by column product $\mathbb{X} \cdot \mathbb{X} = \mathbb{I}_N$; therefore,

$$|\boldsymbol{\eta}|^2 = \sum_{j,l=1}^{N} \xi_j \xi_l \delta_{il} = |\boldsymbol{\xi}|^2,$$

and it follows that $(1.2)_{(i)}$ and $(1.2)_{(ii)}$ are equivalent.

1.3c The Bellman–Dirichlet Equation

The parabolic Bellman–Dirichlet equation naturally arises when finding the minimal cost in a stochastic control problem (or equivalently, finding the optimal strategy).

Let $t \to \mathbb{X}_t \equiv \{\mathbb{X}_1(t), \mathbb{X}_2(t), \dots, \mathbb{X}_N(t)\} \in \mathbb{R}^N$ be a random process subject to the dynamics

$$\mathbb{X}_t = \mathbb{X} + \int_0^t b(\alpha_s, \mathbb{X}_s)\, ds + \int_0^t \sigma(\alpha_s, \mathbb{X}_s)\, dW_s \tag{1.1c}$$

where

- $\mathbb{X}$ is the initial value of the process taken, for example, in an open set $E \subset \mathbb{R}^N$;
- W_t is a $\bar{N}$-dimensional Wiener process ($\bar{N} \in \mathbb{N}$);
- $t \to \alpha_t$ is a control parameter;
- $(\alpha, y) \to b(\alpha, y)$ is a given measurable function with values in $\mathbb{R}^N$;
- $(\alpha, y) \to \sigma(\alpha, y)$ is a $N \times \bar{N}$ matrix $(\sigma_{ij}(\alpha, y))$.

Let A be the set of all admissible controls. Choosing appropriately $\alpha \in A$, we can determine or "pilot" the evolution of the process $\mathbb{X}_t$. The choice of α_s must depend on the value of the process up to time s, i.e.,

$$\alpha_s = \alpha_s(\mathbb{X}_{[0,s]}), \quad \mathbb{X}_{[0,s]} \equiv \{(t, \mathbb{X}_t) : 0 \le t \le s\}.$$

Suppose that for the optimal choice of the control to yield a desired $\mathbb{X}_t$ there is a cost functional

$$\rho^\alpha = \int_0^\infty f^{\alpha_t}(\mathbb{X}_t)\, dt$$

for each individual trajectory $\mathbb{X}_t$ starting at $\mathbb{X}$. The cost ρ^α is itself a random variable depending on the initial value $\mathbb{X}$ of the random trajectory $\mathbb{X}_t$. The "average" cost of the selected and fixed strategy $\alpha_s \in A$ is given by the expectation of ρ^α, i.e.,

$$v^\alpha(\mathbb{X}) = E\left[\int_0^\infty f^{\alpha_t}(\mathbb{X}_t)\, dt\right].$$

We seek to minimize the average cost, i.e., we seek to find

$$v(x) = \inf_{\alpha \in A} v^\alpha(x) = \inf_{\alpha \in A} E\left[\int_0^\infty f^{\alpha_t}(\mathbb{X}_t)\, dt\right].$$

The Bellman–Dirichlet principle states that

$$v(x) = \inf_{\alpha \in A} E\left[\int_0^t f^{\alpha_s}(\mathbb{X}_s)\, ds + v(\mathbb{X}_t)\right].$$

If $x \to u(x) \in C^2(E)$ by Ito's formula, the stochastic differential $du(\mathbb{X}_t)$ is given by

$$\begin{aligned} du(\mathbb{X}_t) &= \sum_{i=1}^N u_{x_i}(\mathbb{X}_t)\, d\mathbb{X}_t + \frac{1}{2}\sum_{i,j=1}^N u_{x_i x_j}(\mathbb{X}_t)\, d\mathbb{X}_t^i d\mathbb{X}_t^j \\ &= J^{(1)}(t) + J^{(2)}(t). \end{aligned} \tag{1.2c}$$

By using (1.1c)

$$J^{(1)}(t) = u_{x_i} b_i\, dt + \sum_{i,j=1}^N u_{x_i}\sigma_{ij}\, dW_t^j.$$

Moreover, by using the standard rules of stochastic differentiation, i.e.,

$$dW_t^i dW_t^j = 0 \text{ if } i \neq j;\ (dW_t^i)^2 = dt;\ dW_t^i dt = 0;\ (dt)^2 = 0,$$

we find from (1.1c)–(1.2c)

$$J^{(2)} = (\sigma_{ik}(\alpha_t, \mathbb{X}_t)\sigma_{jk}(\alpha_t, \mathbb{X}_t))\, dt = \sigma(\alpha_t, \mathbb{X}_t) \cdot \sigma^*(\alpha_t, \mathbb{X}_t) dt.$$

Carrying this in (1.2c) we find

$$du(\mathbb{X}_t) = \left[L^{\sigma(\alpha_t, \mathbb{X}_t), b(\alpha_t, \mathbb{X}_t)} u(\mathbb{X}_t)\right] dt + u_{x_i}(\mathbb{X}_t)\sigma_{ij}(\alpha_t, \mathbb{X}_t) dW_t^i, \tag{1.3c}$$

where

$$L^{\sigma(\alpha_t, \mathbb{X}_t), b(\alpha_t, \mathbb{X}_t)} u(\mathbb{X}_t) = a_{ij}(\alpha_t, \mathbb{X}_t) u_{x_i x_j}(\mathbb{X}_t) + b_i(\alpha_t, \mathbb{X}_t) u_{x_i}(\mathbb{X}_t), \tag{1.4c}$$

and

$$a_{ij}(\alpha_t, \mathbb{X}_t) = \sum_{k=1}^{N} \sigma_{ik}(\alpha_t, \mathbb{X}_t) \cdot \sigma_{jk}(\alpha_t, \mathbb{X}_t) = (\sigma\sigma^*)_{ij}.$$

We integrate (1.3c) over $(0, t)$ and take the expectation to obtain

$$u(x) = E\left[u(\mathbb{X}_t) - \int_0^t L^{\sigma_s, b_s} u(\mathbb{X}_s)\, ds\right], \tag{1.5c}$$

where we have used the fact (see Krylov [146], pages 293–297) that

$$E\left[\int_0^t u_{x_i}(\mathbb{X}_s)\sigma_{ij}(\alpha_s, \mathbb{X}_s)\, dW_s^j\right] = 0.$$

We now combine (1.5c) and the definition of $x \to v(x)$ given before, to obtain

$$\begin{aligned} 0 &= \inf_{\alpha \in A} E\left\{\left[\int_0^t f^\alpha(\mathbb{X}_s)\, ds + v(\mathbb{X}_t)\right] - v(x)\right\} \\ &= \inf_{\alpha \in A} E\left[\int_0^t f^\alpha(\mathbb{X}_s)\, ds + \int_0^t L^{\sigma_s, b_s} u(\mathbb{X}_s)\, ds\right]. \end{aligned}$$

Next, we divide by t and let $t \to 0$ to have

$$\inf_{\alpha \in A} [L^\alpha v + f^\alpha] = 0, \quad \text{where} \quad L^\alpha = L^{\sigma(\alpha,x), b(\alpha,x)}.$$

Let $(\mathcal{G}, \mathcal{F})$ be a measure space and $\{\mathcal{F}_t, t \geq 0\}$ an increasing family of σ-algebras (i.e., a flow of σ-algebras satisfying $\mathcal{F}_t \subset \mathcal{F}$ for any $t \geq 0$).

Now suppose we start controlling the process $\mathbb{X}_t$ given by (1.1c) only after the time t. Then the cost function is

$$\rho_t^\alpha = \int_t^\infty f^{\alpha_s}(\mathbb{X}_s)\, ds,$$

and the "average" cost is the conditional expectation of the random variable $\omega \to \rho_t^\alpha(\omega)$ given $\mathcal{F}_t$, i.e.,

$$v^\alpha(t, \mathbb{X}_t) = E\left[\int_t^\infty f^{\alpha_s}(\mathbb{X}_s)\, ds \,\middle|\, \mathcal{F}_t\right].$$

We seek to minimize the average cost, i.e., we seek to find

$$v(x, t) = \inf_{\alpha \in A} v^\alpha(x, t) = \inf_{\alpha \in A} E\left[\int_t^\infty f^{\alpha_s}(\mathbb{X}_s)\, ds \,\middle|\, \mathcal{F}_t\right].$$

By the Bellman–Dirichlet principle

$$v(x, t) = \inf_{\alpha \in A} E\left\{\int_t^\tau f^{\alpha_s}(\mathbb{X}_s)\, ds + v(\tau, \mathbb{X}_\tau) \,\middle|\, \mathcal{F}_t\right\}. \tag{1.6c}$$

Take now $(x,t) \to u(x,t) \in C^{2,1}(E_T)$. Then for all the random processes $(t, \mathbb{X}_t)$ that remain in E_T for any T almost surely, by Ito's formula

$$u(t,\mathbb{X}_t) = u_{x_i}(t,\mathbb{X}_t)d\mathbb{X}_t^i + u_t(t,\mathbb{X}_t)dt + \frac{1}{2}\sum_{i,j=1}^N u_{x_ix_j}(t,\mathbb{X}_t)d\mathbb{X}_t^i d\mathbb{X}_t^j.$$

Proceeding as before, we have

$$\begin{aligned} u(t,\mathbb{X}_t) = \Big[\frac{\partial}{\partial t}u(t,\mathbb{X}_t) + L^{\sigma(\alpha_t,\mathbb{X}_t),b(\alpha_t,\mathbb{X}_t)}u(t,\mathbb{X}_t)\Big]\, dt \\ + u_{x_i}(t,\mathbb{X}_t)\sigma_{ij}(\alpha_t,\mathbb{X}_t)dW_t^i, \end{aligned}$$

where $L^{\sigma,b}$ is defined as in (1.4c), now with

$$a_{ij} = a_{ij}(t,\mathbb{X}_t), \quad b_i = b_i(\alpha_t,\mathbb{X}_t,t).$$

If $c_t = c(t,\mathbb{X}_t)$ is a progressively measurable process, such that $\int_0^\infty c_t\, dt < \infty$, setting $\varphi_t = \int_0^t c_s\, ds$, we have

$$\begin{aligned} d[e^{-\varphi_t}u(t,\mathbb{X}_t)] =& u(t,\mathbb{X}_t)de^{-\varphi_t} + e^{-\varphi_t}du(t,\mathbb{X}_t) + (de^{-\varphi_t})(du(t,\mathbb{X}_t)) \\ =& e^{-\varphi_t}u_{x_i}(t,\mathbb{X}_t)\sigma_{ij}(\alpha_t,\mathbb{X}_t)dW_t^i \\ &+ e^{-\varphi_t}\left(\frac{\partial}{\partial t} + L^{\sigma,b} - c_t\right)u(t,\mathbb{X}_t)dt. \end{aligned}$$

We integrate over (t,τ), $\forall \tau \ge t$ and take the conditional expectation given $\mathcal{F}_t$ to find

$$\begin{aligned} e^{-\varphi_\tau}u(\tau,\mathbb{X}_\tau) =& e^{-\varphi_t}u(t,\mathbb{X}_t) \\ &+ E\left\{\int_t^\tau e^{-\varphi_s}\left(\frac{\partial}{\partial s} + L^{\sigma_s,b_s} - c_s\right)u(s,\mathbb{X}_s)ds \,\Big|\, \mathcal{F}_t\right\}. \end{aligned}$$

This formula holds for all $u \in C^{2,1}(E_T)$ for all the random processes $(t,\mathbb{X}_t) \in E_T$ almost surely. Also,

$$\begin{aligned} u(t,\mathbb{X}_t) =& e^{-\int_t^\tau c_s\, ds}u(\tau,\mathbb{X}_\tau) \\ &- E\left\{\int_t^\tau e^{-\int_t^s c_\eta\, d\eta}\left(\frac{\partial}{\partial s} + L^{\sigma_s,b_s} - c_s\right)u(s,\mathbb{X}_s)ds \,\Big|\, \mathcal{F}_t\right\}. \end{aligned} \tag{1.7c}$$

Next, in (1.6c), redefine the minimum $v(x,t)$ by setting $\forall \tau \ge t$

$$v(\tau,\mathbb{X}_\tau) = e^{-\int_t^\tau c_s\, ds}u(\tau,\mathbb{X}_\tau). \tag{1.8c}$$

Suppose that the processes involved are so smooth that $u \in C^{2,1}(E_T)$. Then, writing (1.6c) with the change of variable (1.8c), and using (1.7c), we find

$$\begin{aligned}
&\inf_{\alpha\in A} E\left\{\int_t^\tau f^{\alpha_s}(\mathbb{X}_s)ds + e^{-\int_t^\tau c_s\, ds}u(\tau,\mathbb{X}_\tau) - u(t,\mathbb{X}_t)\,\Big|\,\mathcal{F}_t\right\} = 0\\
&= \inf_{\alpha\in A} E\left\{\int_t^\tau f^{\alpha_s}(\mathbb{X}_s)ds\right.\\
&\quad\left. + \int_t^\tau e^{-\int_t^s c_\eta\, d\eta}\left(\frac{\partial}{\partial s} + L^{\sigma_s,b_s} - c_s\right)u(s,\mathbb{X}_s)ds\,\Big|\,\mathcal{F}_t\right\}.
\end{aligned}$$

Divide by $(\tau - t)$ and let $\tau \searrow t$ to obtain

$$\frac{\partial u}{\partial t} + \inf_{\alpha\in A}\left[(L^\alpha - c_\alpha + f^\alpha)u\right] = 0 \quad \text{in } E_T. \tag{1.9c}$$

Final value problems associated with (1.9c) give rise to the parabolic equation (1.12).

3c The Aleksandrov Maximum Principle

3.5c Estimates of the Supremum of a Function

The result of Section 3.5 generalize to the case when the matrix (a_{ij}) is mildly degenerate. Rather than dwelling on the best possible cases, we limit ourselves to the following situation

$$u \in C^{2,1}(E_T), \tag{3.1c}$$

$$(x,t) \to [\det(a_{ij}(x,t))]^{-1} \in L^1(E_T), \tag{3.2c}$$

$$u_t - \mathcal{L}_o(u) = u_t - a_{ij}(x,t)u_{x_ix_j}. \tag{3.3c}$$

We state and prove here a version of Lemma 3.9, since all the other results follow from it in a rather simple fashion.

Lemma 3.1c *Let* $(a_{ij}(x,t))$ *be a positive semi-definite,* $N \times N$ *symmetric matrix satisfying* (3.2c)*, and let* E *be convex.*

Then, for every $u \in C^{2,1}(E_T)$ *such that* $u\Big|_{\Gamma_T} \le 0$*, we have*

$$\sup_{E_T} u_+ \le \gamma_N(\operatorname{diam}(E))^{\frac{N}{N+1}}\left\{\iint_{[u=\xi_u^+]\cap E_T}\frac{(u_t - \mathcal{L}_o(u))_+^{N+1}}{\det(a_{ij}(x,t))}dxdt\right\}^{\frac{1}{N+1}},$$

where $\gamma_N = \frac{2}{N+1}\left(\frac{N+1}{\omega_N}\right)^{N+1}$.

For the proof of Lemma 3.1c, we need the following facts.

Lemma 3.2c *Let* $d \in \mathbb{N}$ *and let* $\mathbb{C}$ *be a* $d\times d$ *positive semi-definite, symmetric matrix. Then,*

$$\det(\mathbb{C}) \le \left(\frac{\operatorname{tr}(\mathbb{C})}{d}\right)^d$$

Proof. If $\{\lambda_i\}_{i=1}^d$ are the eigenvalues of $\mathbb{C}$, we have $\lambda_i \geq 0$, $i = 1, \dots, d$ and

$$\det(\mathbb{C}) = \prod_{i=1}^d \lambda_i \leq \left(\frac{\sum_{i=1}^d \lambda_i}{d} \right)^d = \left(\frac{\operatorname{tr}(\mathbb{C})}{d} \right)^d .$$

■

Corollary 3.1c *Let* $\mathbb{A}$ *and* $\mathbb{B}$ *be two positive semi-definite, symmetric matrices. Then,*

$$\det(\mathbb{A}\mathbb{B}) \leq \left(\frac{\operatorname{tr}(\mathbb{A}\mathbb{B})}{d} \right)^d$$

Proof. Consider the matrix $\mathbb{A}^{\frac{1}{2}}\mathbb{B}\mathbb{A}^{\frac{1}{2}}$; it is obviously symmetric, and it is also positive semidefinite. Indeed, for any $x \in \mathbb{R}^d$ we have $(\mathbb{A}^{\frac{1}{2}}\mathbb{B}\mathbb{A}^{\frac{1}{2}}x, x) = (\mathbb{B}\mathbb{A}^{\frac{1}{2}}x, \mathbb{A}^{\frac{1}{2}}x) \geq 0$. The result follows from Lemma 3.2c.

Proof of Lemma 3.1c. We start from Theorem 3.4 with $c_N = 2\left[\frac{N+1}{\omega_N}\right]^{\frac{1}{N+1}}$ and estimate the integrand in (3.19) by making use of Corollary 3.1c as follows.

$$\begin{aligned} u_t \det(-D^2u) &= [\det(a_{ij}(x,t))]^{-1}[\det(a_{ij}(x,t))]u_t \det(-D^2u) \\ &= [\det(a_{ij}(x,t))]^{-1} \det \begin{pmatrix} u_t & 0 \\ 0 & -D^2u \end{pmatrix} \det \begin{pmatrix} 1 & 0 \\ 0 & a_{ij}(x,t) \end{pmatrix} \\ &\leq [\det(a_{ij}(x,t))]^{-1} \left\{ \frac{1}{N+1} \operatorname{tr} \begin{pmatrix} u_t & 0 \\ 0 & -D^2u \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & a_{ij}(x,t) \end{pmatrix} \right\}^{N+1} \\ &= \frac{1}{(N+1)^{N+1}}[\det(a_{ij}(x,t))]^{-1}(u_t - \mathcal{L}_o(u))_+^{N+1}. \end{aligned}$$

■

We leave as an exercise the task of stating the facts analogous to the ones in Theorems 3.5 and 3.6.

14

NAVIER–STOKES EQUATIONS

1 Navier–Stokes Equations in Dimensionless Form

Let E be a physical open set in $\mathbb{R}^3$ filled with a fluid of dynamic viscosity μ and constant density ρ, whose infinitesimal ideal particles at $x \in E$ at time t move with velocity $\mathbf{v} = (v_1, v_2, v_3)$ function of (x,t), and are acted upon by the pressure $(x,t) \to p(x,t)$, and by possible external force densities $\mathbf{f}_e(x,t)$, per unit volume. Enforcing the local, pointwise conservation of momentum along each of the ideal Lagrangian paths $t \to x(t)$, yields the Navier–Stokes system as in (7.1)–(7.2) of the Preliminaries,

$$\left.\begin{aligned} \rho\big[\mathbf{v}_t + (\mathbf{v}\cdot\nabla)\mathbf{v}\big] - \mu\Delta_x\mathbf{v} + \nabla p = \mathbf{f}_e \\ \operatorname{div}_x \mathbf{v} = 0 \end{aligned}\right. \qquad \text{in } \ E\times(0,\infty). \tag{1.1}$$

Here, Δ_x, ∇ and div_x denote the corresponding differential operation with respect to the physical space variables x. If $\mathbf{f}_e$ is conservative, such as for example gravity, then $\mathbf{f}_e = \nabla F$ for some given potential F. In such a case (1.1) can be written in homogeneous form by redefining p as $(p - F)$.

The various terms in (1.1) are written in terms of pre-chosen physical unit length $[L]$ and time $[T]$ and corresponding unit velocity $[V] = [L][T]^{-1}$, unit pressure $[P] = \rho[V]^2$, unit force density $[F] = \rho[V][T]^{-1}$ and unit dynamic viscosity $[\mu] = \rho[V][L]$. They can be written in dimensionless form by introducing dimensionless space variables $y = x[L]^{-1}$ and time $\tau = t[T]^{-1}$ and corresponding dimensionless velocities, pressures, and force densities

$$\tilde{\mathbf{v}}(y,\tau) = \frac{\mathbf{v}(y[L],\tau[T])}{[V]}, \quad \tilde{p}(y,\tau) = \frac{p(y[L],\tau[T])}{[P]}, \quad \tilde{\mathbf{f}} = \frac{\mathbf{f}_e(y[L],\tau[T])}{[F]}.$$

Denote by $\tilde{E}$ the rescaled physical domain E expressed in terms of dimensionless coordinates. Then, dividing (1.1) by ρ and formally by $[V][T]^{-1}$, yields

$$\left.\begin{aligned} \tilde{\mathbf{v}}_\tau - \frac{1}{\mathcal{R}e}\Delta_y\tilde{\mathbf{v}} + (\tilde{\mathbf{v}}\cdot\nabla_y)\tilde{\mathbf{v}} + \nabla_y\tilde{p} = \tilde{f} \\ \operatorname{div}_y \tilde{\mathbf{v}} = 0 \end{aligned}\right. \qquad \text{in } \ \tilde{E}, \tag{1.2}$$

© Springer Nature Switzerland AG 2023

E. DiBenedetto, U. Gianazza, *Partial Differential Equations*,
Cornerstones, https://doi.org/10.1007/978-3-031-46618-2_15

where $\mathcal{R}e$ is the Reynolds number[1] of the system corresponding to the units $[L]$ and $[T]$ and defined by

$$\mathcal{R}e \overset{\text{def}}{=} \frac{\rho[V][L]}{\mu}.$$

Although $\mathcal{R}e$ is dimensionless, its numerical value depends on the choice of $[L]$ and $[T]$. Indeed, the dynamic viscosity μ for a fluid of density ρ, is experimentally determined in terms of some given units, say, for example, $\text{cm}^2\text{sec}^{-1}$. Expressing them in terms of new units $[L]$ and $[T]$ changes the numerical value of $\mathcal{R}e$. The coefficient of Δ_y in (1.2) is the dimensionless *kinematic* viscosity ν of the rescaled fluid.

This rescaling procedure is at the basis of predicting experimentally non-accessible fluid flows in large-scale domains, such as air past an airfoil or water past a vessel. The physical domains are rescaled to experimentally accessible dimensions, such as laboratory water channels or wind tunnels, with a properly redefined Reynolds number. Information provided by the dimensionless system (1.2) is then rescaled back to the physical domain.

To simplify the symbolism we continue to denote by x, t, $\mathbf{v}$, p, and $\mathbf{f}$ the rescaled, dimensionless quantities and rewrite the Navier–Stokes system (1.1) in the dimensionless domain E, for dimensionless times $t > 0$ in the form

$$\begin{array}{r} \mathbf{v}_t - \nu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p = \mathbf{f} \\ \operatorname{div}\mathbf{v} = 0 \end{array} \quad \text{in } \; E \times (0,\infty), \tag{1.3}$$

with $\nu = \mathcal{R}e^{-1}$. Typically, one prescribes the velocity field $\mathbf{v}_o = \mathbf{v}(\cdot,0)$ at time $t = 0$ and $\mathbf{v}(\cdot,t) = \mathbf{g}(\cdot,t)$ on ∂E for $t > 0$ and seeks to solve (1.3) subject to these data.

If E is a rigid container at rest with respect to an inertial system, then ∂E acts as a rigid wall and $\mathbf{g} = 0$, by viscosity. This is the so-called *no-slip* condition. The case $\mathbf{g} \neq 0$ may occur when ∂E is itself in motion with respect to an inertial system. In the applications, other types of boundary conditions have been considered. We talk of a *kinematic* condition, when the normal component of the velocity vanishes at the boundary, that is, the velocity $\mathbf{v}$ is tangent to the boundary:

$$\mathbf{v}(\cdot,t)\cdot\mathbf{n} = 0 \quad \text{ on } \; \partial E$$

for $t > 0$, where $\mathbf{n}$ is the outward unit normal to the boundary ∂E. In 1823 Navier proposed a more general condition, namely the so-called *Navier boundary* condition, which, roughly speaking, states that the tangential component of the velocity is proportional to the tangential stress at the boundary. We do not consider these different boundary conditions in the following.

The system (1.3) is formal since, even by prescribing smooth initial and boundary data $\mathbf{v}_o$ and $\mathbf{g}$ and the forcing term $\mathbf{f}$, one cannot a priori guarantee that $\mathbf{v}$ and p are so regular as to give pointwise meaning to its various terms.

---

[1] Osborne Reynolds, 1842–1912, Irish-born physicist, made important contributions to the understanding of fluid dynamics.

2 Steady-State Flow with Homogeneous Boundary Data

Let E be a bounded domain in $\mathbb{R}^3$ with boundary ∂E, and consider, formally, the steady-state flow in E,

$$\left.\begin{aligned} -\nu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p &= \mathbf{f}, \\ \operatorname{div}\mathbf{v} &= 0, \\ \mathbf{v}\,\big|_{\partial E} &= 0 \end{aligned}\right. \qquad \text{in } \ E. \tag{2.1}$$

Introduce the space of functions

$$\begin{aligned} \mathcal{V} &= \big\{\boldsymbol{\varphi}\in C_o^\infty(E;\mathbb{R}^3) \text{ such that } \operatorname{div}\boldsymbol{\varphi}=0 \ \text{ in } \ E\big\}\,; \\ H &= \big\{\text{closure of } \ \mathcal{V} \ \text{ in the norm of } \ L^2(E;\mathbb{R}^3)\big\}\,; \\ V &= \big\{\text{closure of } \ \mathcal{V} \text{ in the norm of } \ W_o^{1,2}(E;\mathbb{R}^3)\big\}\,. \end{aligned} \tag{2.2}$$

Formally inner-multiply the first of (2.1) by $\boldsymbol{\varphi}\in\mathcal{V}$ and integrate by parts in E. Since $\mathbf{v}$ and $\boldsymbol{\varphi}$ are both divergence-free, obtain formally

$$\int_E \big[\nu\nabla\mathbf{v}:\nabla\boldsymbol{\varphi} - \mathbf{v}\cdot(\mathbf{v}\cdot\nabla)\boldsymbol{\varphi} - \mathbf{f}\cdot\boldsymbol{\varphi}\big]dx = 0, \tag{2.3}$$

where

$$\nabla\mathbf{v}:\nabla\boldsymbol{\varphi} = \textstyle\sum_{j=1}^3 \nabla v_j\cdot\nabla\varphi_j.$$

Here, we have used the relation

$$\int_E \boldsymbol{\psi}\cdot(\boldsymbol{\zeta}\cdot\nabla)\boldsymbol{\varphi}dx = -\int_E \boldsymbol{\varphi}\cdot(\boldsymbol{\zeta}\cdot\nabla)\boldsymbol{\psi}dx$$

valid for any triple of solenoidal vectors $\boldsymbol{\varphi},\boldsymbol{\psi},\boldsymbol{\zeta}\in W^{1,2}(E;\mathbb{R}^3)$ such that at least one of them is in V. As a consequence,

$$\int_E \boldsymbol{\varphi}\cdot(\boldsymbol{\zeta}\cdot\nabla)\boldsymbol{\varphi}dx = 0.$$

These calculus operations are repeatedly used without specific mention.

By the embedding Theorem 2.1 of Chapter 9 there exists a constant γ independent of E and $\mathbf{v}$, such that

$$\|\mathbf{v}\|_6 \le \gamma\|\nabla\mathbf{v}\|_2 \qquad \text{for all } \ \mathbf{v}\in V. \tag{2.4}$$

Therefore, for all such $\mathbf{v}$

$$\|\mathbf{v}\|_2 \le \gamma|E|^{\frac13}\|\nabla\mathbf{v}\|_2 \qquad \text{and} \qquad \|\mathbf{v}\|_4 \le \gamma|E|^{\frac1{12}}\|\nabla\mathbf{v}\|_2. \tag{2.5}$$

As a consequence,

$$\gamma_o\|\mathbf{v}\|_V \le \|\nabla\mathbf{v}\|_2 \le \|\mathbf{v}\|_V, \qquad \text{where } \ \gamma_o = \frac{1}{\gamma|E|^{\frac13}+1} \tag{2.6}$$

for all $\mathbf{v} \in V$, where the rigorous definition of norm in V is given in (3.1). By these inequalities, all terms in (2.3) are well defined for all $\boldsymbol{\varphi} \in V$ and $\mathbf{f} \in L^{\frac{6}{5}}(E;\mathbb{R}^3)$. Thus, having prescribed one such $\mathbf{f}$, we define a weak solution of (2.1) as a function $\mathbf{v} \in V$ satisfying (2.3) for all $\boldsymbol{\varphi} \in V$. The homogeneous boundary data on ∂E are taken in the sense of the membership $\mathbf{v} \in V$. The same membership guarantees that $\operatorname{div} \mathbf{v} = 0$ in the weak form

$$\int_E \mathbf{v} \cdot \nabla \varphi dx = 0 \qquad \text{for all } \ \varphi \in C^\infty(E).$$

By this definition of solution, the choice $\boldsymbol{\varphi} = \mathbf{v}$ is admissible in (2.3) yielding the basic energy estimate

$$\nu \|\nabla \mathbf{v}\|_2 \le \gamma \|\mathbf{f}\|_{\frac{6}{5}}, \tag{2.7}$$

to be satisfied by any weak solution to (2.1), where γ is the constant of the embedding of V into $L^6(E;\mathbb{R}^3)$. Thus, if $\mathbf{f} = 0$, then $\mathbf{v} = 0$ is the only weak solution of (2.1).

2.1 Uniqueness of Solutions to (2.1)

Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be weak solutions to (2.1) corresponding to the choice of $\mathbf{f} \in L^{\frac{6}{5}}(E;\mathbb{R}^3)$. Write (2.3) for $\mathbf{v}_1$ and $\mathbf{v}_2$, subtract the expression thus obtained and in the resulting integral identity choose $\boldsymbol{\varphi} = \mathbf{w} \overset{\text{def}}{=} (\mathbf{v}_1 - \mathbf{v}_2)$, to obtain

$$\begin{aligned} \nu \|\nabla \mathbf{w}\|_2^2 &= \int_E \big[\mathbf{v}_1 \cdot (\mathbf{w} \cdot \nabla)\mathbf{w} + \mathbf{w} \cdot (\mathbf{v}_2 \cdot \nabla)\mathbf{w}\big] dx \\ &\le \big(\|\mathbf{v}_1\|_4 + \|\mathbf{v}_2\|_4\big) \|\mathbf{w}\|_4 \|\nabla \mathbf{w}\|_2. \end{aligned}$$

Since $\mathbf{v}_j$ are solutions, combining (2.5) and (2.7) gives

$$\|\mathbf{w}\|_4 \le \gamma |E|^{\frac{1}{12}} \|\nabla \mathbf{w}\|_2 \qquad \text{and} \qquad \|\mathbf{v}_j\|_4 \le \frac{\gamma^2}{\nu} |E|^{\frac{1}{3}} \|\mathbf{f}\|_{\frac{6}{5}}.$$

Therefore,

$$\|\nabla \mathbf{w}\|_2 \le 2\gamma \Big(\frac{\gamma}{\nu}\Big)^2 |E|^{\frac{5}{12}} \|\mathbf{f}\|_{\frac{6}{5}} \|\nabla \mathbf{w}\|_2.$$

If the coefficient on the right-hand side is less than 1, then $\mathbf{w} = 0$ and the solution is unique. Such a coefficient depends on the absolute constant γ of the embedding $V \subset L^6(E;\mathbb{R}^3)$, on the size of E, the viscosity ν, and the nature of the forcing term $\mathbf{f}$. Given E and $\mathbf{f}$ uniqueness holds if the Reynolds number of the system is sufficiently small or equivalently if the fluid is sufficiently viscous.

It should be noted that the definition of a weak solution does not depend on the pressure p, which itself is an unknown to be found from (2.1).

3 Existence of Solutions to (2.1)

The spaces H and V introduced in (2.2) are separable Hilbert spaces by the inner products

$$\begin{aligned} &H \ni (\mathbf{u},\mathbf{v}) \to \langle \mathbf{u},\mathbf{v}\rangle_H \overset{\text{def}}{=} \int_E \mathbf{u}\cdot\mathbf{v}dx \\ &V \ni (\mathbf{u},\mathbf{v}) \to \big(\mathbf{u},\mathbf{v}\big)_V \overset{\text{def}}{=} \langle \mathbf{u},\mathbf{v}\rangle_H + \int_E \nabla\mathbf{u} : \nabla\mathbf{v}dx. \end{aligned} \tag{3.1}$$

By (2.6) the inner product $(\cdot,\cdot)_V$ is equivalent to

$$V \ni (\mathbf{u},\mathbf{v}) \to \langle \mathbf{u},\mathbf{v}\rangle_V = \int_E \nabla\mathbf{u} : \nabla\mathbf{v}dx = \textstyle\sum_{j=1}^3 \langle \nabla u_j, \nabla v_j\rangle_H,$$

which from now on we adopt. Having fixed $\mathbf{f} \in L^{\frac{6}{5}}(E;\mathbb{R}^3)$ and $\mathbf{v} \in V$, return to (2.3) and consider the two linear maps

$$\begin{aligned} &V \ni \boldsymbol{\varphi} \to \overset{\text{def}}{=} \int_E \mathbf{f}\cdot\boldsymbol{\varphi}dx \\ &V \ni \boldsymbol{\varphi} \to \overset{\text{def}}{=} \int_E \mathbf{v}\cdot(\mathbf{v}\cdot\nabla)\boldsymbol{\varphi}dx. \end{aligned}$$

By Hölder's inequality and the embedding $V \subset L^6(E;\mathbb{R}^3)$

$$\Big|\int_E \mathbf{f}\cdot\boldsymbol{\varphi}dx\Big| \le \gamma\|\mathbf{f}\|_{\frac{6}{5}}\|\nabla\boldsymbol{\varphi}\|_2.$$

Therefore, the first is a bounded linear functional on V. By the Riesz representation theorem, there exists a unique $\mathbf{F} \in V$ such that[2]

$$V \ni \boldsymbol{\varphi} \to \int_E \mathbf{f}\cdot\boldsymbol{\varphi}dx = \langle \mathbf{F},\boldsymbol{\varphi}\rangle_V.$$

Likewise, by the same embedding and (2.5)

$$\Big|\int_E \mathbf{v}\cdot(\mathbf{v}\cdot\nabla)\boldsymbol{\varphi}dx\Big| \le \gamma^2|E|^{\frac{1}{6}}\|\nabla\mathbf{v}\|_2^2\|\nabla\boldsymbol{\varphi}\|_2.$$

Therefore, also the second map, for every fixed $\mathbf{v} \in V$, is a bounded linear functional in V. By the Riesz representation theorem, there exists a unique $B(\mathbf{v}) \in V$ such that

$$V \ni \boldsymbol{\varphi} \to \int_E \mathbf{v}\cdot(\mathbf{v}\cdot\nabla)\boldsymbol{\varphi}dx = \langle B(\mathbf{v}),\boldsymbol{\varphi}\rangle_V.$$

With these identifications, the weak formulation (2.3) can be recast in the form

---

[2] Dunford and Schwartz [62], Chap. IV, § 4

$$V \ni \boldsymbol{\varphi} \to \nu\langle \mathbf{v}, \boldsymbol{\varphi}\rangle_V = \langle B(\mathbf{v}) + \mathbf{F}, \boldsymbol{\varphi}\rangle_V.$$

Equivalently, in functional form

$$\mathbf{v} = \mathcal{B}(\mathbf{v}) \quad \text{in } V^* \quad \text{where} \quad \mathcal{B}(\mathbf{v}) = \frac{1}{\nu}\big(B(\mathbf{v}) + \mathbf{F}\big), \tag{3.2}$$

and V^* denotes the dual of V identified with V itself up to an isometric isomorphism. Thus, existence of a weak solution to (2.1) in the sense of (2.3) is equivalent to finding a fixed point of the map $V \ni \mathbf{v} \to \mathcal{B}(\mathbf{v}) \in V^*$.

Lemma 3.1 *The map $\mathcal{B}(\cdot) : V \to V^*$ is compact.*

Proof. Since V and V^* are separable metric spaces, compactness is equivalent to sequential compactness. Let K be a bounded subset of V, i.e., there exists a constant C such that $\|\mathbf{v}\|_V \le C$ for all $\mathbf{v} \in K$. The image $\mathcal{B}(K)$ is pre-compact in V^* if for every sequence $\{\mathbf{v}_n\} \subset K$ there exists a subsequence $\{\mathbf{v}_{n'}\} \subset \{\mathbf{v}_n\}$ such that $\{\mathcal{B}(\mathbf{v}_{n'})\}$ is a Cauchy sequence in the operator topology of V^*. By the Rellich–Kondrachov compact embedding theorem (Theorem 2.2 of Chapter 9), the embedding $V \supset K \hookrightarrow L^p(E;\mathbb{R}^3)$ is compact for all $1 \le p < 6$. Therefore, having fixed $1 \le p < 6$, from every sequence $\{\mathbf{v}_n\} \subset K$ one can extract a subsequence $\{\mathbf{v}_{n'}\} \subset \{\mathbf{v}_n\}$, which is Cauchy in the topology of $L^p(E;\mathbb{R}^3)$. Hence, to show that $\mathcal{B}(K)$ is pre-compact in V^* it suffices to show that for every sequence $\{\mathbf{v}_n\} \subset K$, Cauchy in $L^4(E;\mathbb{R}^3)$ the corresponding sequence $\{\mathcal{B}(\mathbf{v}_n)\}$ is Cauchy in the operator topology of V^*. Having fixed one such sequence $\{\mathbf{v}_n\} \subset K$, the action of $\nu[\mathcal{B}(\mathbf{v}_n) - \mathcal{B}(\mathbf{v}_m)]$ on elements $\boldsymbol{\varphi} \in V$, is computed from

$$\begin{aligned}
\langle \nu[\mathcal{B}(\mathbf{v}_n) - \mathcal{B}(\mathbf{v}_m)], \boldsymbol{\varphi}\rangle_V &= \int_E [\mathbf{v}_n \cdot (\mathbf{v}_n \cdot \nabla) - \mathbf{v}_m \cdot (\mathbf{v}_m \cdot \nabla)]\boldsymbol{\varphi} dx \\
&= \int_E (\mathbf{v}_n - \mathbf{v}_m) \cdot (\mathbf{v}_n \cdot \nabla)\boldsymbol{\varphi} dx \\
&\qquad + \int_E \mathbf{v}_m \cdot ((\mathbf{v}_n - \mathbf{v}_m) \cdot \nabla)\boldsymbol{\varphi} dx \\
&\le \big(\|\mathbf{v}_n\|_4 + \|\mathbf{v}_m\|_4\big)\|\mathbf{v}_n - \mathbf{v}_m\|_4 \|\boldsymbol{\varphi}\|_V \\
&\le \gamma |E|^{\frac{1}{12}} \big(\|\mathbf{v}_n\|_V + \|\mathbf{v}_m\|_V\big)\|\mathbf{v}_n - \mathbf{v}_m\|_4 \|\boldsymbol{\varphi}\|_V.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|\mathcal{B}(\mathbf{v}_n) - \mathcal{B}(\mathbf{v}_m)\|_{V^*} &= \sup_{\|\boldsymbol{\varphi}\|_V = 1} \langle [\mathcal{B}(\mathbf{v}_n) - \mathcal{B}(\mathbf{v}_m)], \boldsymbol{\varphi}\rangle_V \\
&\le 2C\gamma |E|^{\frac{1}{12}} \|\mathbf{v}_n - \mathbf{v}_m\|_4.
\end{aligned}$$ ■

Consider next the family of variants of (3.2)

$$\mathbf{v} = \lambda\mathcal{B}(\mathbf{v}) \quad \text{in } V \quad \text{for} \quad \lambda \in (0,1). \tag{3.3}$$

If $\mathbf{v}_\lambda$ is a solution of (3.3), it is also a solution of (2.3) with ν replaced by ν/λ. As such, the a priori estimates (2.6) and (2.7) remain in force with $\mathbf{v}$ replaced by $\mathbf{v}_\lambda$ and ν replaced by ν/λ, i.e.,

$$\|\mathbf{v}_\lambda\|_V \le \frac{1}{\gamma_o}\|\nabla\mathbf{v}_\lambda\|_2 \le \frac{\lambda}{\nu}\frac{\gamma}{\gamma_o}\|\mathbf{f}\|_{\frac{6}{5}}.$$

Therefore, all possible solutions of (3.3) are uniformly bounded in λ. Existence of solutions of (3.2), and hence of (2.1), now follows from the Schauder–Leray Fixed Point Theorem.

Theorem 3.1 (Schauder–Leray [169]). *Let T be a continuous, compact mapping from a Banach space $\{X; \|\cdot\|_X\}$ into itself, such that all possible solutions of $x = \lambda T(x)$ are equi-bounded uniformly in $\lambda \in (0,1)$. Then, T has a fixed point.*

4 Nonhomogeneous Boundary Data

Let E be a simply connected, bounded domain in $\mathbb{R}^3$ with boundary ∂E of class C^1 and satisfying the segment property[3] and consider, formally, the steady-state flow in E,

$$\begin{aligned} -\nu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p &= \mathbf{f}, \\ \operatorname{div}\mathbf{v} &= 0, \qquad \text{in } E, \\ \mathbf{v}\Big|_{\partial E} &= \mathbf{a} \end{aligned} \tag{4.1}$$

where $\mathbf{a}$ is a vector-valued function defined on ∂E, whose regularity will be specified as we proceed. If $\mathbf{v}$ is a solution of (4.1), then, formally, by Green's theorem,

$$0 = \int_E \operatorname{div}\mathbf{v}dx = \int_{\partial E}\mathbf{a}\cdot\mathbf{n}\,d\sigma, \tag{4.2}$$

where $\mathbf{n}$ is the outward unit normal to ∂E. This is then a necessary condition to be imposed on $\mathbf{a}$ for the solvability of (4.1). The solvability of (4.1) hinges on extending $\mathbf{a}$ with a divergence-free vector-valued function $\mathbf{b}$ defined in E. The smoothness of $\mathbf{b}$ and the meaning of $\mathbf{b} = \mathbf{a}$ on ∂E will be made precise as we proceed. Assuming that such an extension can be found, seek a solution to (4.1) in the form $\mathbf{v} = \mathbf{b} + \mathbf{u}$, where formally

$$\begin{aligned} -\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + (\mathbf{b}\cdot\nabla)\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{b} + \nabla p &= \mathbf{g}, \\ \operatorname{div}\mathbf{u} &= 0, \qquad \text{in } E, \\ \mathbf{u}\Big|_{\partial E} &= \mathbf{0} \end{aligned} \tag{4.3}$$

and

---

[3] Section 8.1 of Chapter 9.

$$\mathbf{g} = \mathbf{f} + \nu\Delta\mathbf{b} - (\mathbf{b}\cdot\nabla)\mathbf{b}. \tag{4.4}$$

Solutions of (4.3) are sought in V with the equation being interpreted in its weak form

$$\begin{aligned}\nu\int_E \nabla\mathbf{u} : \nabla\boldsymbol{\varphi}dx \\ = \int_E \{[\mathbf{u}\cdot(\mathbf{u}\cdot\nabla) + \mathbf{u}\cdot(\mathbf{b}\cdot\nabla) + \mathbf{b}\cdot(\mathbf{u}\cdot\nabla)]\,\boldsymbol{\varphi} + \mathbf{g}\cdot\boldsymbol{\varphi}\}\,dx\end{aligned} \tag{4.5}$$

for all $\boldsymbol{\varphi}\in V$. Taking $\boldsymbol{\varphi} = \mathbf{u}$ gives the a priori estimate

$$\left(\nu - \gamma|E|^{\frac{1}{12}}\|\mathbf{b}\|_4\right)\|\nabla\mathbf{u}\|_2^2 \le \left|\int_E \mathbf{g}\cdot\mathbf{u}\,dx\right|, \tag{4.6}$$

where γ is the constant of the embedding of V into $L^6(E;\mathbb{R}^3)$. The right-hand is finite if $\mathbf{f}\in L^{\frac{6}{5}}(E;\mathbb{R}^3)$ and $\mathbf{b}\in W^{1,2}(E;\mathbb{R}^3)$, since by the Sobolev–Nikolskii embedding theorem [4] this implies $\mathbf{b}\in L^6(E;\mathbb{R}^3)$. Indeed,

$$\left|\int_E \mathbf{g}\cdot\mathbf{u}\,dx\right| \le \left[\gamma\|\mathbf{f}\|_{\frac{6}{5}} + \nu\|\nabla\mathbf{b}\|_2 + \|\mathbf{b}\|_4^2\right]\|\nabla\mathbf{u}\|_2, \tag{4.7}$$

where again γ is the constant of the embedding of V into $L^6(E;\mathbb{R}^3)$.

If the domain E has boundary ∂E of class C^1 and satisfies in addition the segment property, functions $\mathbf{b}\in W^{1,2}(E;\mathbb{R}^3)$ have traces on ∂E in the fractional Sobolev space[5]

$$\mathbf{b}\,\Big|_{\partial E} = \mathbf{a}\in W^{\frac{1}{2},2}(\partial E;\mathbb{R}^3). \tag{4.8}$$

Henceforth, given a boundary datum $\mathbf{a}\in W^{\frac{1}{2},2}(\partial E;\mathbb{R}^3)$, we assume that it can be extended into a solenoidal vector field $\mathbf{b}\in W^{1,2}(E;\mathbb{R}^3)$. A compatibility condition for such an extension to exist is that $\mathbf{a}$ has zero flux across ∂E as indicated by (4.2). We also assume that such an extension can be constructed to satisfy

$$\gamma|E|^{\frac{1}{12}}\|\mathbf{b}\|_4 \le \tfrac{1}{2}\nu. \tag{4.9}$$

The actual construction of an extension $\mathbf{b}$ satisfying (4.8) is carried out in Section 4.2c of the Complements. Moreover, we assume that (4.9) can be derived from (4.7c). Accepting it for the moment, this last requirement combined with (4.6)–(4.7) yields the a priori estimate

$$\|\nabla\mathbf{u}\|_2 \le \frac{2\gamma}{\nu}\left[\|\mathbf{f}\|_{\frac{6}{5}} + \nu\|\nabla\mathbf{b}\|_2 + \|\mathbf{b}\|_4^2\right] \tag{4.10}$$

to be satisfied by any weak solution to (4.1).

---

[4] Theorem 2.1 of Chapter 9.

[5] Theorem 8.1 of Chapter 9.

4.1 Uniqueness of Solutions to (4.1)

If $\mathbf{u}_1$ and $\mathbf{u}_2$ in V solve (4.1) write their weak formulations (4.5), subtract them out and in the integral identity thus obtained take the testing function $\boldsymbol{\varphi} = (\mathbf{u}_1 - \mathbf{u}_2) \overset{\text{def}}{=} \mathbf{w}$, and make use of the embedding (2.5) and the upper bound (4.10) to be satisfied by all weak solutions to (4.1), to obtain

$$\begin{aligned}
\|\nabla\mathbf{w}\|_2^2 &= \frac{1}{\nu}\int_E \big[\mathbf{u}_1\cdot(\mathbf{w}\cdot\nabla)\mathbf{w} + \mathbf{w}\cdot(\mathbf{u}_2\cdot\nabla)\mathbf{w} + \mathbf{b}\cdot(\mathbf{w}\cdot\nabla)\mathbf{w}\big]\,dx \\
&\le \frac{1}{\nu}\big(\|\mathbf{u}_1\|_4 + \|\mathbf{u}_2\|_4 + \|\mathbf{b}\|_4\big)\|\mathbf{w}\|_4\|\nabla\mathbf{w}\|_2 \\
&\le \frac{\gamma}{\nu}|E|^{\frac{1}{12}}\left(4\frac{\gamma^2}{\nu}|E|^{\frac{1}{12}}\Big[\|\mathbf{f}\|_{\frac{6}{5}} + \nu\|\nabla\mathbf{b}\|_2 + \|\mathbf{b}\|_4^2\Big] + \|\mathbf{b}\|_4\right)\|\nabla\mathbf{w}\|_2^2.
\end{aligned}$$

If the coefficient of $\|\nabla\mathbf{w}\|_2^2$ on the right-hand side does not exceed 1 then $\mathbf{w} = 0$ and the problem admits at most one solution. The uniqueness condition hinges on several factors including $|E|$ and the size of the extension $\mathbf{b}$ through the norms $\|\nabla\mathbf{b}\|_2$ and $\|\mathbf{b}\|_4$. The key condition, however, is expressed by the smallness of the Reynolds number $\mathcal{R}e = \nu^{-1}$. Thus, uniqueness holds if the Reynolds number is sufficiently small or equivalently if the fluid is sufficiently viscous.

4.2 Existence of Solutions to (4.1)

Consider the linear maps

$$\begin{aligned}
&V \ni \boldsymbol{\varphi} \to \overset{\text{def}}{=} \int_E \mathbf{g}\cdot\boldsymbol{\varphi}\,dx \\
&V \ni \boldsymbol{\varphi} \to \overset{\text{def}}{=} \int_E \big[\mathbf{u}\cdot(\mathbf{u}\cdot\nabla) + \mathbf{u}\cdot(\mathbf{b}\cdot\nabla) + \mathbf{b}\cdot(\mathbf{u}\cdot\nabla)\big]\boldsymbol{\varphi}\,dx.
\end{aligned}$$

Estimate

$$\Big|\int_E \mathbf{g}\cdot\boldsymbol{\varphi}\,dx\Big| \le \gamma\Big[\|\mathbf{f}\|_{\frac{6}{5}} + \nu\|\nabla\mathbf{b}\|_2 + \|\mathbf{b}\|_4^2\Big]\|\nabla\boldsymbol{\varphi}\|_2.$$

Therefore, the first is a bounded linear functional in V. By the Riesz representation theorem there exists a unique $\mathbf{G}\in V$ such that

$$V \ni \boldsymbol{\varphi} \to \int_E \mathbf{g}\cdot\boldsymbol{\varphi}\,dx = \langle\mathbf{G},\boldsymbol{\varphi}\rangle_V.$$

Likewise, estimate

$$\begin{aligned}
&\Big|\int_E \big[\mathbf{u}\cdot(\mathbf{u}\cdot\nabla) + \mathbf{u}\cdot(\mathbf{b}\cdot\nabla) + \mathbf{b}\cdot(\mathbf{u}\cdot\nabla)\big]\boldsymbol{\varphi}\,dx\Big| \\
&\qquad \le \gamma|E|^{\frac{1}{12}}\Big(\gamma|E|^{\frac{1}{12}}\|\nabla\mathbf{u}\|_2 + 2\|\mathbf{b}\|_4\Big)\|\nabla\mathbf{u}\|_2\|\nabla\boldsymbol{\varphi}\|_2,
\end{aligned}$$

where γ is the constant of the embedding $L^{\frac{6}{5}}(E;\mathbb{R}^3) \subset V$. Therefore, the second map is also a bounded linear functional in V. By the Riesz representation theorem[6] there exists $\bar{B}(\mathbf{u}) \in V$, such that

$$\int_E \big[\mathbf{u}\cdot(\mathbf{u}\cdot\nabla) + \mathbf{u}\cdot(\mathbf{b}\cdot\nabla) + \mathbf{b}\cdot(\mathbf{u}\cdot\nabla)\big]\varphi\,dx = \langle \bar{B}(\mathbf{u}), \varphi\rangle_V .$$

With these identifications the weak form (4.5) reads

$$V \ni \varphi \to \nu\langle \mathbf{u}, \varphi\rangle_V = \langle \bar{B}(\mathbf{u}) + \mathbf{G}, \varphi\rangle_V .$$

Equivalently, in functional form

$$\mathbf{u} = \bar{\mathcal{B}}(\mathbf{u}) \quad \text{in } V^*, \qquad \text{where} \qquad \bar{\mathcal{B}}(\mathbf{u}) = \frac{1}{\nu}\big(\bar{B}(\mathbf{u}) + \mathbf{G}\big), \tag{4.11}$$

where, as before, V^* denotes the dual of V identified with V itself up to an isometric isomorphism. Thus, the existence of a weak solution to (4.1) in the sense of (4.5) is equivalent to finding a fixed point of the map $V \ni \mathbf{u} \to \bar{\mathcal{B}}(\mathbf{u}) \in V^*$.

Lemma 4.1 *The map $\bar{\mathcal{B}}(\cdot) : V \to V^*$ is compact.*

The proof is analogous to that of Lemma 3.1 with minor changes. Consider next the family of variants of (4.11)

$$\mathbf{u} = \lambda\mathcal{B}(\mathbf{u}) \quad \text{in } V \quad \text{for} \quad \lambda \in (0,1). \tag{4.12}$$

If $\mathbf{u}_\lambda$ is a solution of (4.12), it is also a solution of (4.5) with ν replaced by ν/λ. As such, the a priori estimate (4.10) remains in force with $\mathbf{u}$ replaced by $\mathbf{u}_\lambda$ and ν replaced by ν/λ, i.e.,

$$\|\mathbf{u}_\lambda\|_V \le \frac{1}{\gamma_o}\|\nabla\mathbf{u}_\lambda\|_2 \le 2\frac{\lambda}{\nu}\frac{\gamma}{\gamma_o}\Big[\|\mathbf{f}\|_{\frac{6}{5}} + \nu\|\nabla\mathbf{b}\|_2 + \|\mathbf{b}\|_4^2\Big] .$$

Therefore, all possible solutions of (4.12) are uniformly bounded in λ. The existence of the solutions of (4.11), and hence of (4.1), now follows from the Schauder–Leray Fixed Point Theorem 3.1.

5 Recovering the Pressure

Return to the steady-state Navier–Stokes system (2.1) in its weak form (2.3). The existence of solutions to such a system has been established in Section 3 irrespective of the pressure p appearing in the formal pointwise form (2.1). Assume momentarily that

$$\mathbf{v} \in W^{2,2}(E;\mathbb{R}^3) \qquad \text{and} \qquad \mathbf{f} \in L^2(E;\mathbb{R}^3). \tag{5.1}$$

---

[6] Dunford and Schwartz [62], Chap. IV, § 4

Then, (2.3) by back-integration by parts yields

$$\int_E (\mathbf{NS}) \cdot \boldsymbol{\varphi} dx = 0, \qquad \text{where} \qquad (\mathbf{NS}) = -\nu \Delta \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} - \mathbf{f}$$

for all $\boldsymbol{\varphi} \in \mathcal{V}$. Since $(\mathbf{NS}) \in L^2(E;\mathbb{R}^3)$ this continues to hold for all $\boldsymbol{\varphi} \in H$. Therefore, $(\mathbf{NS}) \in H^\perp$. Introduce the space of functions

$$G = \left\{ \begin{array}{c} \text{collection of } \ \boldsymbol{\varphi} \in L^2(E;\mathbb{R}^3) \ \text{ of the form} \\ \boldsymbol{\varphi} = \nabla p \ \text{ for some } \ p \in W^{1,2}(E) \end{array} \right\}$$

Proposition 5.1 (Helmholtz–Weyl Decomposition [274]) *Let $E \subset \mathbb{R}^3$ be open, bounded, and convex. Then, $G = H^\perp$ or equivalently*

$$L^2(E;\mathbb{R}^3) = H \oplus G.$$

Indeed, Proposition 5.1 is a special case of the Helmholtz–Weyl decomposition; its proof will be given in Section 5c of the Complements.

The system (2.1), as such, does not provide sufficient information to determine the pressure p. However, its weak formulation (2.3) permits one to assert that the principal part $(\mathbf{NS})$ of the Navier–Stokes system has, at least under the regularity assumptions (5.1) on $\mathbf{v}$ and $\mathbf{f}$, and locally in E, the form of a gradient of some pressure $p \in W^{1,2}_{\text{loc}}(E)$. This follows by applying Proposition 5.1 to open, convex subsets of E.

6 Steady-State Flows in Unbounded Domains

Let E be an unbounded, open set in $\mathbb{R}^3$ filled with a fluid of dynamic viscosity μ. The problem is particularly interesting from the physical point of view if E is an *exterior domain*, that is, the complement of a bounded set; such a situation can then be used to model the motion of a rigid body through a viscous fluid, or the flow past an obstacle (see also Galdi [91], Chapter 1, § 2 for more details).

The domain E will be assumed to be open and simply connected, with boundary ∂E of class C^1, and satisfying the segment property. The fluid velocity $\mathbf{v}$ is assumed to take the value $\mathbf{a}$ on ∂E, for a vector field $\mathbf{a}$ whose regularity will be specified as we proceed, and to approach a constant vector $\mathbf{a}_\infty$ as $|x| \to \infty$. The fluid is stirred in its interior by a forcing term $\mathbf{f}$ whose properties are to be defined. Consider formally the steady-state flow in E,

$$\left. \begin{aligned} -\nu \Delta \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} + \nabla p &= \mathbf{f}, \\ \operatorname{div} \mathbf{v} &= 0, \\ \mathbf{v} \big|_{\partial E} &= \mathbf{a}, \\ \lim_{|x|\to\infty} \mathbf{v}(x) &= \mathbf{a}_\infty. \end{aligned} \right. \qquad \text{in } \ E \tag{6.1}$$

Notice that, in general, (4.2) is no longer a necessary condition on $\mathbf{a}$ for the solvability of (6.1), even if $\mathbf{a}_\infty = 0$.

6.1 Assumptions on a and f

It is assumed that the boundary datum $\mathbf{a} \in W^{\frac{1}{2},2}(\partial E; \mathbb{R}^3)$ can be extended into a solenoidal $\mathbf{b} \in W^{1,2}_{\mathrm{loc}}(E; \mathbb{R}^3)$, satisfying

$$\begin{aligned} &\mathbf{b} = \mathbf{a} \quad \text{ on } \partial E \text{ as traces of functions in } W^{1,2}(E; \mathbb{R}^3), \\ &\mathbf{b} - \mathbf{a}_\infty \in L^2(E; \mathbb{R}^3), \\ &|\mathbf{b}(x) - \mathbf{a}_\infty| \le \frac{M_o}{\sqrt{1+|x|^2}}, \qquad \text{and } \ |\nabla \mathbf{b}| \le \frac{M_1}{1+|x|^2} \quad \text{ in } E, \end{aligned} \tag{6.2}$$

for two given constants M_o and M_1. For exterior domains and smooth $\mathbf{a}$ with zero flux on ∂E, such an extension can always be realized. Indeed, we have the following.

Proposition 6.1 *Let E be an exterior domain, complement of a bounded, simply connected domain $E^c = \mathbb{R}^3 \backslash \bar{E}$. Then, any $\mathbf{a} \in C^2(\partial E; \mathbb{R}^3)$ satisfying (4.2) admits a solenoidal extension $\mathbf{b} \in C^2(\mathbb{R}^3; \mathbb{R}^3)$ satisfying (6.2).*

Proof. For $\delta > 0$, consider the set $E_\delta = [\mathrm{dist}(\cdot, \partial E) < \delta]$ and construct the vector field $\boldsymbol{\psi}_{\mathbf{a}} \in C^3(\mathbb{R}^3; \mathbb{R}^3)$ corresponding to $\mathbf{a}$, compactly supported in E_δ, such that the solenoidal extension $\mathbf{b}_{\mathbf{a}}$ of $\mathbf{a}$ is realized by $\mathbf{b}_{\mathbf{a}} = \mathrm{curl}\, \boldsymbol{\psi}$. Such a construction is guaranteed by Proposition 4.3c of the Complements.

Let $R > 1$ be sufficiently large, such that $B_{R-1} \supset E^c$, and set

$$\mathbf{b}' = \mathrm{curl}\, \big[\big(x_3 a_{\infty,2}, x_1 a_{\infty,3}, x_2 a_{\infty,1} \big) \zeta \big], \tag{6.3}$$

where

$$\zeta = \begin{cases} 1 \ \text{ outside a ball of radius } \ R, \\ 0 \ \text{ inside a ball of radius } \ R-1, \\ \ \text{smooth, } \ 0 \le \zeta \le 1 \ \text{ otherwise.} \end{cases}$$

Finally, let $\mathbf{b}(x) = \mathbf{b}_{\mathbf{a}}(x) + \mathbf{b}'(x)$. One verifies that such a $\mathbf{b}$ is solenoidal, and satisfies the requirements (6.2). ■

For general vector fields with the regularity assumed in (6.2), again one relies on Proposition 4.3c of the Complements for the construction of $\mathbf{b}_{\mathbf{a}}$, whereas $\mathbf{b}'$ is built as in (6.3).

By the previous construction, it is also apparent that $\mathrm{supp}\, \nabla \mathbf{b}$ is a compact set in $\mathbb{R}^3$.

The forcing term $\mathbf{f}$ is taken in $L^2_{\mathrm{loc}}(E; \mathbb{R}^3)$ and decreasing sufficiently fast as $|x| \to \infty$, in the sense

$$|x|\, \mathbf{f} \in L^2(E; \mathbb{R}^3). \tag{6.4}$$

6.2 Toward a Notion of a Solution to (6.1)

Proceeding as in the case of bounded domains, solutions are sought of the form $\mathbf{v} = \mathbf{b} + \mathbf{u}$, for some $\mathbf{u} \in V$, where formally $\mathbf{u}$ satisfies (4.3)–(4.5), the latter holding for all $\boldsymbol{\varphi} \in \mathcal{V}$. The membership $\mathbf{u} \in V$ provides weak forms of the last two conditions in (6.1), whereas (4.5) interprets weakly the Navier–Stokes system. The next step is in deriving a priori estimates for $\mathbf{u}$, by taking $\boldsymbol{\varphi} = \mathbf{u}$ in (4.5).

For bounded domains E, the inner product $(\cdot,\cdot)_V$ introduced in (3.1) is equivalent to the inner product $\langle\nabla\cdot,\nabla\cdot\rangle_H$. This follows from the embedding inequalities (2.5)–(2.6). If E is unbounded this is, in general, no longer the case and the topology generated by $(\cdot,\cdot)_V$ cannot be related to the norm $\|\nabla\cdot\|_2$. Nevertheless, the first of (2.5) has a weaker counterpart in V.

Proposition 6.2 *Let* $\mathbf{u} \in V$ *and* $0 \in \mathbb{R}^3 \backslash \operatorname{supp}\mathbf{u}$. *Then,*

$$\int_E \frac{|\mathbf{u}|^2}{|x|^2} dx \le 4 \int_E |\nabla \mathbf{u}|^2 dx \tag{6.5}$$

Proof. Since $\mathbf{u}$ has a vanishing trace on ∂E, by extending it with zero outside E, regard $\mathbf{u}$ as an element of V in $\mathbb{R}^3$. Assume momentarily that $\mathbf{u} \in \mathcal{V}$, and for $\varepsilon \in (0,1)$ compute and estimate

$$\begin{aligned}\int_{\varepsilon<|x|<\varepsilon^{-1}} \frac{|\mathbf{u}|^2}{|x|^2} dx &= \int_{\varepsilon<|x|<\varepsilon^{-1}} (\Delta \ln|x|)|\mathbf{u}|^2 dx \\ &= \int_{|x|=\varepsilon^{-1}} \nabla \ln|x| \cdot \frac{x}{|x|}|\mathbf{u}|^2 d\sigma - \int_{|x|=\varepsilon} \nabla \ln|x| \cdot \frac{x}{|x|}|\mathbf{u}|^2 d\sigma \\ &\quad - 2\int_{\varepsilon<|x|<\varepsilon^{-1}} \frac{\mathbf{u}}{|x|} \cdot \Big(x \cdot \frac{\nabla \mathbf{u}}{|x|}\Big) dx.\end{aligned}$$

Letting $\varepsilon \to 0$ gives

$$\int_E \frac{|\mathbf{u}|^2}{|x|^2} dx \le 2\int_E \frac{|\mathbf{u}|}{|x|}|\nabla \mathbf{u}| dx \le \Big(4\int_E \frac{|\mathbf{u}|^2}{|x|^2} dx\Big)^{\frac{1}{2}} \Big(\int_E |\nabla \mathbf{u}|^2 dx\Big)^{\frac{1}{2}},$$

which yields (6.5). The proof is then concluded by density. ■

Notice that in the proof $\mathbf{u}$ being solenoidal is not used.

7 Existence of Solutions to (6.1)

7.1 Approximating Solutions and A Priori Estimates

For $n > \operatorname{diam}(E^c)$ let B_n be the ball of radius n about the origin of $\mathbb{R}^3$, set $E_n = E \cap B_n$, and let $\mathcal{V}_n$ and V_n be the spaces introduced in (2.2) for E_n.

In each E_n we consider the problem

$$\begin{aligned}-\nu\Delta\mathbf{v}_n + (\mathbf{v}_n\cdot\nabla)\mathbf{v}_n + \nabla p &= \mathbf{f},\\ \operatorname{div}\mathbf{v}_n &= 0,\\ \mathbf{v}_n\,\Big|_{\partial E} &= \mathbf{a},\\ \mathbf{v}_n\,\Big|_{\partial B_n} &= \mathbf{b}\,\Big|_{\partial B_n}.\end{aligned}$$

Let $\mathbf{v}_n = \mathbf{b} + \mathbf{u}_n$, where $\mathbf{u}_n \in V_n$ is a weak solution in E_n of

$$\begin{aligned}-\nu\Delta\mathbf{u}_n + (\mathbf{u}_n\cdot\nabla)\mathbf{u}_n + (\mathbf{b}\cdot\nabla)\mathbf{u}_n + (\mathbf{u}_n\cdot\nabla)\mathbf{b} + \nabla p &= \mathbf{g},\\ \operatorname{div}\mathbf{u}_n &= 0,\\ \mathbf{u}_n\,\Big|_{\partial E} &= \mathbf{0},\\ \mathbf{u}_n\,\Big|_{\partial B_n} &= \mathbf{0},\end{aligned}\tag{7.1}$$

where $\mathbf{g} = \mathbf{f} + \nu\Delta\mathbf{b} - (\mathbf{b}\cdot\nabla)\mathbf{b}$. Since E_n is bounded, such a $\mathbf{u}_n$ exists, by the construction in Section 4.

Proposition 7.1 *Let* $\mathbf{u}_n \in V_n$ *be a solution of* (7.1) *in* E_n*, with* $\mathbf{b}$ *and* $\mathbf{f}$ *satisfying (6.2) and (6.4) respectively. Then, either of these a priori estimates holds, uniformly in n*

$$\begin{matrix}\big(\nu - 2M_o\big)\|\nabla\mathbf{u}_n\|_2\\ \big(\nu - 4M_1\big)\|\nabla\mathbf{u}_n\|_2\end{matrix} \le \gamma_{\mathbf{g}} \overset{def}{=} \||x|\mathbf{f}\|_2 + \pi(C\pi M_1^2 + \nu M_1),\tag{7.2}$$

where $C > 0$ *is a constant that depends on* δ *and* R*, introduced in the proof of Proposition 6.1.*

Proof. Since $\mathbf{u}_n \in V_n$, it can be regarded as $\mathbf{u}_n \in V$, by extending it to be zero outside E_n. Likewise, the test function $\boldsymbol{\varphi} \in V_n$ is regarded as in V. Insert $\boldsymbol{\varphi} = \mathbf{u}$ in the weak formulation of (7.1), and transform and estimate the various terms by using the assumptions (6.4) on $\mathbf{f}$ and (6.2) on $\mathbf{b}$. In this process we use the elementary calculation[7]

$$\int_{\mathbb{R}^3}\frac{dx}{(1+|x|^2)^2} = \pi^2.$$

We have

$$\begin{aligned}\nu\int_{E_n}\nabla\mathbf{u}_n : \nabla\mathbf{u}_n\,dx \le &\Big|\int_{E_n}\mathbf{u}_n\cdot(\mathbf{b}\cdot\nabla)\mathbf{u}_n\,dx\Big|\\ &+\Big|\int_{E_n}\mathbf{u}_n\cdot(\mathbf{u}_n\cdot\nabla)\mathbf{b}\,dx\Big|\\ &+\Big|\int_{E_n}\mathbf{f}\cdot\mathbf{u}_n\,dx\Big|\\ &+\Big|\nu\int_{E_n}\Delta\mathbf{b}\cdot\mathbf{u}_n\,dx\Big|\\ &+\Big|\int_{E_n}(\mathbf{b}\cdot\nabla)\mathbf{b}\cdot\mathbf{u}_n\,dx\Big|.\end{aligned}$$

---

[7] Combine (8.5) of Chapter 2 and Problem **2.3** of the Complements of the same chapter.

The various terms above are transformed and estimated as follows:

$$\Big|\int_{E_n} \mathbf{f}\cdot\mathbf{u}_n\,dx\Big| \le \||x|\mathbf{f}\|_2\||x|^{-1}\mathbf{u}_n\|_2 \le 2\||x|\mathbf{f}\|_2\|\nabla\mathbf{u}_n\|_2;$$

$$\Big|\int_{E_n} \nu\Delta\mathbf{b}\cdot\mathbf{u}_n\,dx\Big| \le \nu\|\nabla\mathbf{b}\|_2\|\nabla\mathbf{u}_n\|_2 \le \nu M_1\pi\|\nabla\mathbf{u}_n\|_2;$$

$$\begin{aligned}
\Big|\int_{E_n} \mathbf{u}_n\cdot(\mathbf{b}\cdot\nabla)\mathbf{u}_n\,dx\Big| &= 0 \quad \text{since } \mathbf{b} \text{ is solenoidal};\\
\Big|\int_{E_n} \mathbf{u}_n\cdot(\mathbf{u}_n\cdot\nabla)\mathbf{b}\,dx\Big| &\le M_1\int_{E_n}|x|^{-2}|\mathbf{u}_n|^2dx \le 4M_1\|\nabla\mathbf{u}_n\|_2^2;\\
\Big|\int_{E_n} \mathbf{u}_n\cdot(\mathbf{u}_n\cdot\nabla)\mathbf{b}\,dx\Big| &= \Big|\int_{E_n} \mathbf{u}_n\cdot(\mathbf{u}_n\cdot\nabla)(\mathbf{b}-\mathbf{a}_\infty+\mathbf{a}_\infty)\,dx\Big|\\
&\le \Big|\int_{E_n} \mathbf{u}_n\cdot(\mathbf{u}_n\cdot\nabla)(\mathbf{b}-\mathbf{a}_\infty)\,dx\Big|\\
&\le \Big|\int_{\mathbb{R}^3} \mathbf{u}_n\cdot(\mathbf{u}_n\cdot\nabla)(\mathbf{b}-\mathbf{a}_\infty)\,dx\Big|\\
&= \Big|\int_{\mathbb{R}^3} (\mathbf{b}-\mathbf{a}_\infty)\cdot(\mathbf{u}_n\cdot\nabla)\mathbf{u}_n\,dx\Big|\\
&\le M_o\int_E |x|^{-1}|\mathbf{u}_n||\nabla\mathbf{u}_n|dx \le 2M_o\|\nabla\mathbf{u}_n\|_2^2;\\
\Big|\int_{E_n} \mathbf{u}_n\cdot(\mathbf{b}\cdot\nabla)\mathbf{b}\,dx\Big| &= \Big|\int_{E_n\cap\operatorname{supp}\nabla\mathbf{b}} \mathbf{u}_n\cdot(\mathbf{b}\cdot\nabla)\mathbf{b}\,dx\Big|\\
&= \Big|\int_{E_n\cap\operatorname{supp}\nabla\mathbf{b}} \mathbf{b}\cdot(\mathbf{b}\cdot\nabla)\mathbf{u}_n\,dx\Big|\\
&\le \|\mathbf{b}\|^2_{4;E_n\cap\operatorname{supp}\nabla\mathbf{b}}\|\nabla\mathbf{u}_n\|_{2,;E_n\cap\operatorname{supp}\nabla\mathbf{b}}\\
&\le C(\delta,R)\|\nabla\mathbf{b}\|_2^2\|\nabla\mathbf{u}_n\|_{2;E_n\cap\operatorname{supp}\nabla\mathbf{b}}\\
&\le C(\delta,R)\|\nabla\mathbf{b}\|_2^2\|\nabla\mathbf{u}_n\|_2\\
&= C(\delta,R)M_1^2\pi^2\|\nabla\mathbf{u}_n\|_2.
\end{aligned}$$

Combining these calculations proves (7.2). ■

The proposition provides an a priori estimate for $\|\nabla\mathbf{u}_n\|_2$, independent of n if either M_o or M_1 is sufficiently small. In what follows, assume

$$\max\{2M_o;4M_1\}<\nu \quad \text{and set} \quad \nu-\max\{2M_o;4M_1\}=\alpha>0. \tag{7.3}$$

For unbounded E, introduce the space

$$\mathcal{H}=\{\text{completion of } \mathcal{V} \text{ in the norm } \|\cdot\|_{\mathcal{H}}=\|\nabla\cdot\|_2\}.$$

This is a separable Hilbert space by the inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}}=\langle\nabla\cdot,\nabla\cdot\rangle_H$. By construction $V\subset\mathcal{H}$, as elements in $\mathcal{H}$ are not required to be in $L^2(E;\mathbb{R}^3)$.

7.2 The Limiting Process

If (7.3) holds, then $\{\mathbf{u}_n\}$ is a sequence bounded in $\mathcal{H}$, and hence weakly precompact in the same space. Every element $\mathbf{u}$ in the weak closure of $\{\mathbf{u}_n\}$ is a weak solution of (4.5) in the following sense. First, $\mathbf{u} \in \mathcal{H}^*$ and hence $\mathbf{u} \in \mathcal{H}$ by the Riesz identification map. Next, having fixed $\boldsymbol{\varphi} \in \mathcal{V}$, let F be its support and consider the sequence $\{\mathbf{u}_n|_F\}$ of restrictions of $\mathbf{u}_n$ to F. Since F is bounded, by the embedding inequalities (2.5)–(2.6), the norm $\|\nabla \cdot\|_{2;F}$ is equivalent to the norm of $W^{1,2}(F;\mathbb{R}^3)$. Therefore, there exists a constant C, depending on F, such that

$$\|\mathbf{u}_n|_F\|_{W^{1,2}(F;\mathbb{R}^3)} \le C \qquad \text{uniformly in } n.$$

By the Rellich–Kondrachov compact embedding theorem (Theorem 2.2 of Chapter 9), the embedding $W^{1,2}(F;\mathbb{R}^3) \hookrightarrow L^p(F;\mathbb{R}^3)$ is compact for all $1 \le p < 6$. Therefore, a subsequence $\{\mathbf{u}_{n'}|_F\} \subset \{\mathbf{u}_n|_F\}$ can be selected so that

$$\begin{aligned} \{\mathbf{u}_{n'}\} &\to \mathbf{u} \quad \text{weakly in } W^{1,2}(F;\mathbb{R}^3), \text{ and} \\ \{\mathbf{u}_{n'}\} &\to \mathbf{u} \quad \text{strongly in } L^r(F;\mathbb{R}^3). \end{aligned} \tag{7.4}$$

Theorem 7.1. *Let (6.2), (6.3), and (7.3) hold. Then, (4.5) admits a solution* $\mathbf{u} \in \mathcal{H}$ *satisfying*

$$\nu \int_E \nabla \mathbf{u} : \nabla \boldsymbol{\varphi} dx = \int_E \{[\mathbf{u} \cdot (\mathbf{u} \cdot \nabla) + \mathbf{u} \cdot (\mathbf{b} \cdot \nabla) + \mathbf{b} \cdot (\mathbf{u} \cdot \nabla)]\, \boldsymbol{\varphi} + \mathbf{g} \cdot \boldsymbol{\varphi}\}\, dx \tag{7.5}$$

for all $\boldsymbol{\varphi} \in \mathcal{V}$.

Proof. Let $\mathbf{u} \in \mathcal{H}$ be in the closure of $\{\mathbf{u}_n\}$ in the weak topology of $\mathcal{H}$. Write down (4.5) for $\mathbf{u}_n$ over E_n for $\boldsymbol{\varphi}_n \in \mathcal{V}_n$. Having fixed $\boldsymbol{\varphi} \in \mathcal{V}$, let F be its support and let n_F be so large that $F \subset\subset B_n$ for all $n \ge n_F$. Then, for such $\boldsymbol{\varphi}$ fixed, (4.5) will hold for all $n \ge n_F$. Letting $n \to \infty$ along proper subsequences depending of $\boldsymbol{\varphi}$ satisfying (7.4) establishes (7.5) ■

Remark 7.1 Notice that the indicated limiting process can be carried out for a fixed $\boldsymbol{\varphi}$ of compact support and not for $\boldsymbol{\varphi} \in V$. Thus, (7.5) holds only for $\boldsymbol{\varphi} \in \mathcal{V}$ and, in general, not for $\boldsymbol{\varphi} \in V$. Once $\mathbf{u}$ in the weak closure of $\{\mathbf{u}_n\}$ has been identified, the choice of subsequences for which (7.4) holds depends on the selected testing function $\boldsymbol{\varphi}$. However, the limiting identity (7.5) continues to hold for all $\boldsymbol{\varphi} \in \mathcal{V}$. Also, for unbounded E, solutions are found in $\mathcal{H}$ and in general not in V.

8 Time-Dependent Navier–Stokes Equations in Bounded Domains

Continue to denote by $E \subset \mathbb{R}^3$ an open, bounded set with boundary ∂E of class C^1 and satisfying the segment property. For $0 < T < \infty$ let $E_T = E \times (0,T)$, and introduce the spaces

$$L^2(0,T;V) = \{\mathbf{v}(\cdot,t) \in V \text{ for a.e. } t \in (0,T) \text{ with finite norm } \|\nabla\mathbf{v}\|_{2;E_T}\};$$

$$W = \left\{ \begin{array}{c} \mathbf{v}(\cdot,t) \in V \text{ for a.e. } t \in (0,T) \text{ with finite norm} \\ \|\mathbf{v}\|_W^2 = \operatorname{ess\,sup}_{(0,T)} \|\mathbf{v}(\cdot,t)\|_{2;E}^2 + \|\nabla\mathbf{v}\|_{2;E_T}^2 \end{array} \right\};$$

$$C^\infty(0,T;\mathcal{V}) = \big\{\boldsymbol{\varphi} \in C^\infty(E_T;\mathbb{R}^3) \text{ with } \boldsymbol{\varphi}(\cdot,t) \in \mathcal{V} \text{ for all } t \in (0,T)\big\}.$$

For these spaces the operations of ∇ and div are meant weakly and with respect to the space variables only. Functions $\boldsymbol{\varphi} \in C^\infty(0,T;\mathcal{V})$ are divergence-free and of compact support in E, in the space variables, but are permitted not to vanish for $t = 0$ or for $t = T$.

Lemma 8.1 *Let* $\mathbf{v} \in W$. *Then,* $\mathbf{v} \in L^{\frac{10}{3}}(E_T;\mathbb{R}^3)$ *and*

$$\|\mathbf{v}\|_{\frac{10}{3};E_T} \le \gamma^{\frac{3}{5}} \|\mathbf{v}\|_W,$$

where γ *is the constant of the embedding of* V *into* $L^6(E;\mathbb{R}^3)$.

Proof.

$$\begin{aligned} \int_0^T \int_E |\mathbf{v}|^{\frac{10}{3}} dxdt &= \int_0^T \int_E |\mathbf{v}|^{\frac{4}{3}} |\mathbf{v}|^2 dxdt \\ &\le \int_0^T \Big(\int_E |\mathbf{v}|^2 dx\Big)^{\frac{2}{3}} \Big(\int_E |\mathbf{v}|^6 dx\Big)^{\frac{1}{3}} dt \\ &\le \Big(\operatorname*{ess\,sup}_{(0,T)} \|\mathbf{v}(\cdot,t)\|_{2;E}\Big)^{\frac{4}{3}} \int_0^T \|\mathbf{v}(\cdot,t)\|_{6;E}^2\, dt \\ &\le \gamma^2 \|\mathbf{v}\|_W^{\frac{10}{3}} \end{aligned}$$

The last inequality follows from the embedding (2.4). ■

Consider a viscous fluid of Reynolds number ν^{-1} filling a rigid, still container E and stirred by a forcing term $\mathbf{f}$. Its time evolution over $(0,T)$ is modeled, formally, by the system

$$\begin{aligned} \mathbf{v}_t - \nu\Delta\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p = \mathbf{f} &\quad \text{in } E_T; \\ \operatorname{div}\mathbf{v} = 0&; \\ \mathbf{v}(\cdot,t)\,\big|_{\partial E} = 0&; \\ \mathbf{v}(\cdot,0) = \mathbf{v}_o &\quad \text{in } E. \end{aligned} \tag{8.1}$$

The homogeneous boundary condition for the velocity $\mathbf{v}$, also called *no-slip condition*, says that at the boundary, the fluid will have zero velocity with respect to the same boundary.

Multiply the first of these, formally, by $\boldsymbol{\varphi} \in C^\infty(0,T;\mathcal{V})$ and integrate by parts over $E_t = E \times (0,t)$ for $t \in (0,T]$. Using that $\operatorname{div}\mathbf{v} = 0$ gives

$$\begin{aligned}\int_E \mathbf{v}(t)\cdot\boldsymbol{\varphi}(t)dx &- \int_0^t\int_E \mathbf{v}\cdot\boldsymbol{\varphi}_\tau\,dxd\tau \\ &+ \int_0^t\int_E \big(\nu\nabla\mathbf{v}:\nabla\boldsymbol{\varphi} + (\mathbf{v}\cdot\nabla)\mathbf{v}\cdot\boldsymbol{\varphi}\big)dxd\tau \\ &= \int_E \mathbf{v}_o\cdot\boldsymbol{\varphi}(0)dx + \int_0^t\int_E \mathbf{f}\cdot\boldsymbol{\varphi}\,dxd\tau.\end{aligned}\tag{8.2}$$

Assuming momentarily that $\mathbf{v}\in C^\infty(0,T;\mathcal{V})$ take $\boldsymbol{\varphi}=\mathbf{v}$ and observe that the nonlinear term gives, formally, zero contribution. Using Lemma 8.1 also yields the formal energy inequality

$$\begin{aligned}\operatorname*{ess\,sup}_{(0,T)}&\|\mathbf{v}(t)\|_{2;E}^2 + 2\nu\|\nabla\mathbf{v}\|_{2;E_T}^2 \\ &\le \|\mathbf{v}_o\|_{2;E}^2 + 2\int_0^T\int_E \mathbf{f}\cdot\mathbf{v}\,dxdt \\ &\le \|\mathbf{v}_o\|_{2;E}^2 + 2\|\mathbf{f}\|_{2;E_T}\|\mathbf{v}\|_{2;E_T} \\ &\le \|\mathbf{v}_o\|_{2;E}^2 + 2\sqrt{T}\|\mathbf{f}\|_{2;E_T}\operatorname*{ess\,sup}_{(0,T)}\|\mathbf{v}(\cdot,t)\|_{2;E}.\end{aligned}\tag{8.3}$$

In what follows, the set of parameters $\{\nu, T, |E|, \|\mathbf{v}_o\|_{2;E}, \|\mathbf{f}\|_{2;E_T}\}$ constitutes the given data and we denote by γ a generic positive constant that can be determined quantitatively, a priori only in these terms. With this notation, by a standard application of the Cauchy–Schwarz inequality, (8.3) implies

$$\|\mathbf{v}\|_W \le \gamma\big(\|\mathbf{v}_o\|_{2;E} + \|\mathbf{f}\|_{2;E_T}\big).\tag{8.4}$$

These formal remarks suggest that we define a *weak solution* to (8.1) as an element of W satisfying (8.2) for all $\boldsymbol{\varphi}\in C^\infty(0,T;\mathcal{V})$, and the energy estimate (8.4). The membership $\mathbf{v}(\cdot,t)\in V$ for a.e. $t\in(0,T)$ gives meaning, in the sense of traces, to the homogeneous boundary data on ∂E. The same membership ensures that $\operatorname{div}\mathbf{v}=0$ weakly in E_T. As for the initial data, observe that, for solutions in this class, all integrals in (8.2) are well defined. As a consequence, by Vitali's absolute continuity of the integral, all integrals extended over E_t tend to zero as $t\to 0$. Therefore,

$$\lim_{t\to 0}\int_E \mathbf{v}(t)\cdot\boldsymbol{\varphi}(t)dx = \int_E \mathbf{v}_o\cdot\boldsymbol{\varphi}(0)dx \qquad \text{for all } \boldsymbol{\varphi}\in C^\infty(0,T;\mathcal{V}).$$

Thus, the initial datum $\mathbf{v}_o$ is taken in the sense of such a weak continuity of $\mathbf{v}(\cdot,t)$ in $L^2(E;\mathbb{R}^3)$. The same continuity also implies that $\operatorname{div}\mathbf{v}_o=0$ weakly in E. The latter emerges then as a compatibility condition to be imposed on the initial datum $\mathbf{v}_o$ for a solution to exist.

Theorem 8.1 (Hopf [123]). *Let* $\mathbf{f}\in L^2(E_T;\mathbb{R}^3)$ *and let* $\mathbf{v}_o\in L^2(E;\mathbb{R}^3)$ *be weakly divergence-free in* E. *Then there exists a weak solution to (8.1).*

Remark 8.1 In the following we refer to such a solution as *Hopf's solution.*

9 The Galerkin Approximations

Let $\mathbf{e} = (\mathbf{e}_1, \dots, \mathbf{e}_n, \dots)$ be a complete system for V. Since $\mathcal{V}$ is dense in V, by sequential selection and Zorn's lemma, the elements $\mathbf{e}_j$ can be chosen in $\mathcal{V}$. Also, by sequential orthonormalization, although not necessarily orthonormal with respect to the inner product $\langle \cdot, \cdot \rangle_V$ defined in (3.1), they can be chosen to be orthonormal in $L^2(E; \mathbb{R}^3)$, i.e., $\langle \mathbf{e}_i, \mathbf{e}_j \rangle_H = \delta_{ij}$. Write a possible solution in the form

$$\mathbf{v} = \mathbf{v}_n + \mathbf{v}_{r,n}, \quad \text{where} \quad \mathbf{v}_n = \sum_{j=1}^{n} c_j(t)\mathbf{e}_j \quad \text{and} \quad \mathbf{v}_{r,n} = \sum_{j>n} c_j(t)\mathbf{e}_j \tag{9.1}$$

for scalar functions $(0,T) \ni t \to c_j(t)$. The remainder $\mathbf{v}_{r,n}$ of the series satisfies

$$\|\mathbf{v}_{r,n}\|_V^2 = \|\mathbf{v}_{r,n}\|_2^2 + \|\nabla \mathbf{v}_{r,n}\|_2^2 \to 0 \quad \text{as } n \to \infty.$$

Since $(\mathbf{e}_1, \dots, \mathbf{e}_n, \dots)$ is complete in V, it is also complete in $L^2(E; \mathbb{R}^3)$. Therefore, by the indicated orthonormalization in $L^2(E; \mathbb{R}^3)$ and Parseval's identity

$$\|\mathbf{v}\|_{2;E}^2 = \sum_{j \geq 1} c_j^2.$$

Write $\mathbf{v}$ in (8.2) in the form (9.1) and observe that the terms involving $\mathbf{v}_{r,n}$ tend to zero as $n \to \infty$. This suggests defining an approximate solution to (8.1) as a function $\mathbf{v}_n \in C^\infty(0,T; \mathcal{V})$, with $\mathbf{v}_n = \sum_{i=1}^{n} c_{n,i}\mathbf{e}_i$, satisfying (8.2) for $\boldsymbol{\varphi} = \mathbf{e}_i$, for all $i = 1, \dots, n$, i.e.,

$$\begin{aligned} \int_0^T \Big[c_{n,i}' + \sum_{j=1}^{n} c_{n,j} \Big\{ \int_E \nu \nabla \mathbf{e}_j : \nabla \mathbf{e}_i dx \Big\}_{ij}^{\text{sym}} \\ + \sum_{j=1}^{n} c_{n,j} \Big\{ \int_E \mathbf{e}_i \cdot (\mathbf{v}_n \cdot \nabla)\mathbf{e}_j dx \Big\}_{ij}^{\text{skew}} d\tau - \int_E \mathbf{f} \cdot \mathbf{e}_i dx \Big] d\tau = 0. \end{aligned} \tag{9.2}$$

For fixed $n \in \mathbb{N}$ the terms $A_{ij}^{\text{sym}} = \{\cdots\}_{ij}^{\text{sym}}$ define the entries of a $n \times n$ time independent symmetric matrix $\mathbf{A}_n^{\text{sym}}$, whereas the terms $A_{ij}^{\text{skew}} = \{\cdots\}_{ij}^{\text{skew}}$ define the entries of a $n \times n$ skew symmetric matrix $\mathbf{A}_n^{\text{skew}}$ linearly dependent on the time-dependent vector $\mathbf{c}_n = (c_{n,1}, \dots, c_{n,n})$. The last term defines a vector $\mathbf{f}_n = (f_1, \dots, f_n)$ dependent on t. Set also

$$c_{o,i} = \int_E \mathbf{v}_o \cdot \mathbf{e}_i \, dx, \qquad \mathbf{c}_o = (c_{o,1}, \dots, c_{o,n}), \qquad \mathbf{c}_n(0) = \mathbf{c}_o.$$

Requiring the integrand over $(0,T)$ in (9.2) to vanish identically gives the differential system in $\mathbf{c}_n$

$$c_{n,i}' + \sum_{j=1}^{n} \big(A_{ij}^{\text{sym}} + A_{ij}^{\text{skew}}\big) c_{n,j} = f_i \quad \text{with } c_{n,i}(0) = c_{o,i}. \tag{9.3}$$

The unique solvability of this system hinges upon some a priori estimates, which we derive next.

Proposition 9.1 *Let* $\mathbf{c}_n = (c_{n,1}, \dots, c_{n,n})$ *be a solution to (9.3) and set* $\mathbf{v}_n = \sum_{i=1}^n c_{n,i}\mathbf{e}_i$*. There is a constant* γ *depending only on the data and independent of* n *and* i*, such that*

$$\begin{aligned} \|\mathbf{v}_n\|_W &\le \gamma; \\ \operatorname*{ess\,sup}_{(0,T)} |c_{n,i}(t)| &\le \gamma; \\ |c_{n,i}(t_2) - c_{n,i}(t_1)| &\le \gamma(1 + \|\nabla \mathbf{e}_i\|_{\infty;E})\sqrt{t_2 - t_1} \end{aligned} \tag{9.4}$$

for all $(t_1, t_2) \subset (0, T)$*.*

Proof. Multiply (9.3) by $c_{n,i}$, add over $i = 1, \dots, n$, and observe that $\mathbf{c}_n^t \mathbf{A}_n^{\text{skew}} \mathbf{c}_n = 0$, where $\mathbf{c}_n^t$ denotes the transpose of the vector $\mathbf{c}_n$. This gives

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\sum_{i=1}^n c_{n,i}^2 + \nu \int_E \nabla \sum_{j=1}^n c_{n,j}\mathbf{e}_j : \nabla \sum_{i=1}^n c_{n,i}\mathbf{e}_i dx \\ &\qquad = \sum_{i=1}^n f_i c_{n,i} \le \Big(\sum_{i=1}^n f_i^2\Big)^{\frac{1}{2}} \Big(\sum_{i=1}^n c_{n,i}^2\Big)^{\frac{1}{2}} = \|\mathbf{f}_n(t)\|_{2;E} \|\mathbf{v}_n(t)\|_{2;E}. \end{aligned}$$

Equivalently,

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{v}_n(t)\|_{2;E}^2 + \nu \|\nabla \mathbf{v}_n(t)\|_{2;E}^2 \le \|\mathbf{f}_n(t)\|_{2;E} \|\mathbf{v}_n(t)\|_{2;E}.$$

To prove the first of (9.4), integrate this over $(0, t) \subset (0, T)$ to get

$$\begin{aligned} &\operatorname*{ess\,sup}_{(0,T)} \|\mathbf{v}_n(t)\|_{2;E}^2 + 2\nu \|\nabla \mathbf{v}_n\|_{2;E_T}^2 \\ &\qquad\qquad \le \|\mathbf{v}_o\|_{2;E}^2 + 2\sqrt{T}\|\mathbf{f}_n\|_{2;E_T} \operatorname*{ess\,sup}_{(0,T)} \|\mathbf{v}_n(t)\|_{2;E}. \end{aligned}$$

The proof is concluded by a standard application of Cauchy–Schwarz inequality in the last term. The second of (9.4) follows from this and Parseval's identity. To prove the last of (9.4), return to (9.3) and, for fixed $i \in \{1, \dots, n\}$, estimate

$$\begin{aligned} |c_{n,i}'| \le{}& \nu \Big| \int_E \nabla \mathbf{v}_n : \nabla \mathbf{e}_i \, dx \Big| + \Big| \int_E (\mathbf{v}_n \cdot \nabla)\mathbf{e}_i \cdot \mathbf{v}_n \, dx \Big| + \int_E |\mathbf{f}| \, dx \\ \le{}& \nu \int_E |\nabla \mathbf{v}_n| |\nabla \mathbf{e}_i| \, dx + \int_E |\mathbf{v}_n|^2 |\nabla \mathbf{e}_i| \, dx + \int_E |\mathbf{f}| \, dx \\ \le{}& \nu \|\nabla \mathbf{e}_i\|_{\infty;E} \int_E |\nabla \mathbf{v}_n| \, dx + \|\nabla \mathbf{e}_i\|_{\infty;E} \int_E |\mathbf{v}_n|^2 \, dx + \int_E |\mathbf{f}| \, dx \\ \le{}& \nu \|\nabla \mathbf{e}_i\|_{\infty;E} \|\nabla \mathbf{v}_n(t)\|_{2;E} |E|^{\frac{1}{2}} + \|\nabla \mathbf{e}_i\|_{\infty;E} \Big(\operatorname*{ess\,sup}_{(0,T)} \|\mathbf{v}_n(t)\|_{2;E}^2 \Big) \\ &+ |E|^{\frac{1}{2}} \|\mathbf{f}(t)\|_{2;E} \\ ={}& \|\nabla \mathbf{e}_i\|_{\infty;E} \Big(\nu |E|^{\frac{1}{2}} \|\nabla \mathbf{v}_n(t)\|_{2;E} + \operatorname*{ess\,sup}_{(0,T)} \|\mathbf{v}_n(t)\|_{2;E}^2 \Big) + |E|^{\frac{1}{2}} \|\mathbf{f}(t)\|_{2;E}. \end{aligned}$$

Integrating over $(t_1,t_2)\subset(0,T)$ and using the first of (9.4) gives

$$\begin{aligned}|c_{n,i}(t_2)-c_{n,i}(t_1)|\le&\Big|\int_{t_1}^{t_2}c'_{n,i}(t)dt\Big|\le\int_{t_1}^{t_2}|c'_{n,i}(t)|dt\\ \le&\nu\|\nabla\mathbf{e}_i\|_{\infty;E}|E|^{\frac12}\int_{t_1}^{t_2}\Big(\int_E|\nabla\mathbf{v}_n|^2\,dx\Big)^{\frac12}dt\\ &+\|\nabla\mathbf{e}_i\|_{\infty;E}\operatorname*{ess\,sup}_{(0,T)}\|\mathbf{v}_n(t)\|^2_{2;E}(t_2-t_1)\\ &+|E|^{\frac12}\int_{t_1}^{t_2}\Big(\int_E|\mathbf{f}|^2\,dx\Big)^{\frac12}dt\\ &\le\gamma(\nu,T,|E|,\|\mathbf{v}_o\|_2,\|\mathbf{f}\|_{2;E_T})(1+\|\nabla\mathbf{e}_i\|_\infty)\sqrt{t_2-t_1}.\qquad\blacksquare\end{aligned}$$

The existence and uniqueness of the solutions to (9.3) can be established in the small, for example, by a contraction fixed point argument. Then the solution can be continued in the whole $(0,T)$, so long as it remains bounded. Such a bound, independent of t, n, and i, is ensured by the second of (9.4).

10 Selecting Subsequences Strongly Convergent in $L^2(E_T;\mathbb{R}^3)$

It follows from Proposition 9.1 that for fixed $j\in\mathbb{N}$ the sequences $\{c_{n,j}\}_{n=1}^\infty$ are equibounded and equicontinuous, so that by the Ascoli–Arzelà theorem a subsequence $\{c_{n_j,j}\}\subset\{c_{n,j}\}_{n=1}^\infty$ can be selected converging to some c_j uniformly in $(0,T)$. By the Cantor diagonalization procedure a further subsequence can be selected and relabelled with n, such that $\{c_{n,j}\}\to c_j$ uniformly in $[0,T]$. However, it should be noted that, because of the last of (9.4), the rate of convergence depends on the index j. Set formally

$$\mathbf{v}=\sum_{j=1}^\infty c_j\mathbf{e}_j.$$

Proposition 10.1 *For the same constant γ as in the first of (9.4) there holds*

$$\operatorname*{ess\,sup}_{(0,T)}\|\mathbf{v}(\cdot,t)\|_{2;E}+\|\nabla\mathbf{v}\|_{2;E_T}\le\gamma.$$

Moreover, $\{\mathbf{v}_n(\cdot,t)\}\to\mathbf{v}(\cdot,t)$ weakly in $L^2(E;\mathbb{R}^3)$, uniformly in $t\in(0,T)$.

Proof. For a fixed positive integer k and all n

$$\begin{aligned}\sum_{j=1}^k c_j^2(t)\le&\Big|\sum_{j=1}^k c_j^2(t)-\sum_{j=1}^k c_{n,j}^2(t)\Big|+\sum_{j=1}^k c_{n,j}^2(t)\\ \le&\sum_{j=1}^k\big|c_j^2(t)-c_{n,j}^2(t)\big|+\|\mathbf{v}_n(\cdot,t)\|^2_{2;E}.\end{aligned}$$

By the first of (9.4) the last term is bounded by a constant γ depending only upon the data and independent of t and n. Letting $n \to \infty$ the first term in the right-hand side tends to zero by the uniform convergence of $\{c_{n,j}\} \to c_j$ for $j = 1, \dots, k$. Thus, $\sum_{j=1}^k c_j^2(t) \le \gamma$. Since k is arbitrary, the series $\sum_{j=1}^k c_j^2$ converges to $\|\mathbf{v}(\cdot, t)\|_{2;E}^2$ and $\operatorname{ess\,sup}_{(0,T)} \|\mathbf{v}(\cdot, t)\|_{2;E} \le \gamma$. To prove the second statement, fix $k \in \mathbb{N}$ and take first a function of the form $\boldsymbol{\varphi} = \sum_{j=1}^k \varphi_j \mathbf{e}_j$. For such a function, by the othonormality of $(\mathbf{e}_1, \dots, \mathbf{e}_n, \dots)$

$$\int_E (\mathbf{v}_n - \mathbf{v}) \cdot \boldsymbol{\varphi}\, dx = \sum_{j=1}^k (c_{n,j} - c_j)\varphi_j \ \to \ 0 \ \text{ as } \ n \to \infty$$

by the uniform convergence of $\{c_{n,j}\} \to c_j$ for $j = 1, \dots, k$. For a general $\boldsymbol{\varphi} = \sum \varphi_j \mathbf{e}_j \in L^2(E; \mathbb{R}^3)$, having fixed $\varepsilon > 0$, there exists k_ε, depending on ε and $\boldsymbol{\varphi}$, such that $\sum_{j > k_\varepsilon} \varphi_j^2 < \varepsilon$. Then estimate

$$\begin{aligned} \Big| \int_E [\mathbf{v}_n(t) - \mathbf{v}(t)] \cdot \boldsymbol{\varphi}\, dx \Big| \le \sum_{j=1}^{k_\varepsilon} & |c_{n,j}(t) - c_j(t)|\, |\varphi_j| \\ & + \operatorname*{ess\,sup}_{(0,T)} \|\mathbf{v}_n(t) - \mathbf{v}(t)\|_{2;E} \bigg(\sum_{j > k_\varepsilon} \varphi_j^2 \bigg)^{\frac{1}{2}}. \end{aligned}$$

By the first of (9.4) a further subsequence out of $\{\mathbf{v}_n\}$ can be selected and relabeled with n, such that $\{\mathbf{v}_n\} \to \mathbf{v}'$ and $\{\nabla \mathbf{v}_n\} \to \nabla \mathbf{w}$ weakly in $L^2(E_T; \mathbb{R}^3)$. By the uniqueness of the weak limit $\mathbf{v}' = \mathbf{v}$ and $\nabla \mathbf{w} = \nabla \mathbf{v}$. By the weak lower semi-continuity of the norm and the first of (9.4)

$$\|\nabla \mathbf{v}\|_{2;E_T} \le \liminf \|\nabla \mathbf{v}_n\|_{2;E_T} \le \gamma. \qquad \blacksquare$$

Proposition 10.2 $\{\mathbf{v}_n\} \to \mathbf{v}$ *strongly in* $L^2(E_T; \mathbb{R}^3)$.

The proof uses the following lemma.

Lemma 10.1 (Friedrichs [86]) *For every* $\varepsilon > 0$ *there exist a positive integer* N_ε *depending only on* ε *and* $|E|$, *and independent of* $\mathbf{v}_n$, *and* N_ε *linearly independent functions* $\{\boldsymbol{\psi}_\ell\}_{\ell=1}^{N_\varepsilon} \subset L^2(E; \mathbb{R}^3)$ *such that*

$$\|\mathbf{v}_n - \mathbf{v}\|_{2;E_T}^2 \le \sum_{\ell=1}^{N_\varepsilon} \int_0^T \Big| \int_E (\mathbf{v}_n - \mathbf{v}) \cdot \boldsymbol{\psi}_\ell dx \Big|^2 dt + \varepsilon \|\nabla(\mathbf{v}_n - \mathbf{v})\|_{2;E_T}^2. \tag{10.1}$$

Inequality (10.1) is a special case, applied to $(\mathbf{v}_n - \mathbf{v})$ of a more general Friedrichs' Lemma, which we prove in Section 10c of the Complements.

Proof (of Proposition 10.2). Fix $\varepsilon > 0$ and determine N_ε and the system $\{\boldsymbol{\psi}_\ell\}_{\ell=1}^{N_\varepsilon} \subset L^2(E; \mathbb{R}^3)$. Let now $n \to \infty$ in (10.1). The first term goes to zero because of the weak uniform convergence of $(\mathbf{v}_n - \mathbf{v})$ in $L^2(E; \mathbb{R}^3)$. The last term is majorized by $2\gamma^2\varepsilon$, where γ is the constant in the first of (9.4). $\blacksquare$

11 The Limiting Process and Proof of Theorem 8.1

Let $\boldsymbol{\varphi}_k = \sum_{\ell=1}^{k} \varphi_\ell \mathbf{e}_\ell$ for fixed $k \in \mathbb{N}$. Multiply (9.3) by φ_i, add for $i = 1, \dots, k$ and integrate over $(0, t) \subset (0, T)$ to obtain for $n \ge k$

$$\begin{aligned}\int_E \mathbf{v}_n(t) \cdot \boldsymbol{\varphi}_k(t) dx - \int_0^t\!\!\int_E \mathbf{v}_n \cdot \boldsymbol{\varphi}_{k,\tau} dx d\tau \\ + \nu \int_0^t\!\!\int_E \nabla \mathbf{v}_n : \nabla \boldsymbol{\varphi}_k dx d\tau \\ + \int_0^t\!\!\int_E (\mathbf{v}_n \cdot \nabla) \mathbf{v}_n \cdot \boldsymbol{\varphi}_k dx d\tau \\ = \int_E \mathbf{v}_o \cdot \boldsymbol{\varphi}_k(0) dx + \int_0^t\!\!\int_E \mathbf{f} \cdot \boldsymbol{\varphi}_k dx d\tau.\end{aligned}$$

In turn, this is averaged in time over $(t, t+h) \subset (0, T)$, for a fixed $h > 0$, sufficiently small so that $0 < t + h < T$. Denoting by

$$\fint_t^{t+h} \{\cdots\} d\tau = \frac{1}{h} \int_t^{t+h} \{\cdots\} d\tau$$

such averages gives

$$\begin{aligned}\fint_t^{t+h}\!\!\int_E \mathbf{v}_n(\tau) \cdot \boldsymbol{\varphi}_k(\tau)\, dx d\tau - \fint_t^{t+h}\!\!\int_0^\tau\!\!\int_E \mathbf{v}_n(s) \cdot \boldsymbol{\varphi}_{k,s}(s)\, dx ds d\tau \\ + \nu \fint_t^{t+h}\!\!\int_0^\tau\!\!\int_E \nabla \mathbf{v}_n(s) : \nabla \boldsymbol{\varphi}_k(s)\, dx ds d\tau \\ + \fint_t^{t+h}\!\!\int_0^\tau\!\!\int_E (\mathbf{v}_n(s) \cdot \nabla) \mathbf{v}_n(s) \cdot \boldsymbol{\varphi}_k(s)\, dx ds d\tau \\ = \int_E \mathbf{v}_o \cdot \boldsymbol{\varphi}_k(0)\, dx + \fint_t^{t+h}\!\!\int_0^\tau\!\!\int_E \mathbf{f}(s) \cdot \boldsymbol{\varphi}_k(s)\, dx ds d\tau.\end{aligned}$$

Let $n \to \infty$ by keeping k fixed, to get

$$\begin{aligned}\fint_t^{t+h}\!\!\int_E \mathbf{v}(\tau) \cdot \boldsymbol{\varphi}_k(\tau)\, dx d\tau - \fint_t^{t+h}\!\!\int_0^\tau\!\!\int_E \mathbf{v}(s) \cdot \boldsymbol{\varphi}_{k,s}(s)\, dx ds d\tau \\ + \nu \fint_t^{t+h}\!\!\int_0^\tau\!\!\int_E \nabla \mathbf{v}(s) : \nabla \boldsymbol{\varphi}_k(s)\, dx ds d\tau \\ + \fint_t^{t+h}\!\!\int_0^\tau\!\!\int_E (\mathbf{v}(s) \cdot \nabla) \mathbf{v}(s) \cdot \boldsymbol{\varphi}_k(s)\, dx ds d\tau \\ = \int_E \mathbf{v}_o \cdot \boldsymbol{\varphi}_k(0)\, dx + \fint_t^{t+h}\!\!\int_0^\tau\!\!\int_E \mathbf{f}(s) \cdot \boldsymbol{\varphi}_k(s)\, dx ds d\tau.\end{aligned}$$

The various limits are justified by the weak convergence $\{\nabla \mathbf{v}_n\} \to \nabla \mathbf{v}$ and the strong convergence $\{\mathbf{v}_n\} \to \mathbf{v}$. In particular, such a strong convergence

permits one to pass to the limit in the nonlinear term. In Section 11c of the Complements we discuss a counterexample to show that in general, having weak convergence does not suffice to pass to the limit in such a term.
Next, take $\boldsymbol{\varphi} \in C^\infty(0,T;\mathcal{V})$, write it as $\boldsymbol{\varphi} = \sum \varphi_j \mathbf{e}_j$, and let $\boldsymbol{\varphi}_k$ be its truncated series. Because of the predicated smoothness of $\boldsymbol{\varphi}$

$$\{\boldsymbol{\varphi}_k\},\ \{\nabla\boldsymbol{\varphi}_k\},\ \{\boldsymbol{\varphi}_{k,t}\} \to \boldsymbol{\varphi},\ \nabla\boldsymbol{\varphi},\ \boldsymbol{\varphi}_t \ \text{ in } L^2(E_T),$$

and also $\{\boldsymbol{\varphi}_k\} \to \boldsymbol{\varphi}$ in $L^5(E_T;\mathbb{R}^3)$. Compute and estimate

$$\begin{aligned}\limsup_{k\to\infty}\Big|\iint_{E_t} \nabla\mathbf{v} : \nabla\boldsymbol{\varphi}_k\,dxd\tau - \iint_{E_t} \nabla\mathbf{v} : \nabla\boldsymbol{\varphi}\,dxd\tau\Big| \\ \le \|\nabla\mathbf{v}\|_{2;E_T} \lim_{k\to\infty} \|\nabla(\boldsymbol{\varphi}_k - \boldsymbol{\varphi})\|_{2;E_T} = 0.\end{aligned}$$

The limits in all the other terms but the nonlinear one are treated similarly. For the nonlinear term

$$\begin{aligned}\limsup_{k\to\infty}\Big|\iint_{E_t} (\mathbf{v}\cdot\nabla)\mathbf{v}\cdot\boldsymbol{\varphi}_k\,dxd\tau - \iint_{E_t} (\mathbf{v}\cdot\nabla)\mathbf{v}\cdot\boldsymbol{\varphi}_k\,dxd\tau\Big| \\ \le \|\nabla\mathbf{v}\|_{2;E_T}\|\mathbf{v}\|_{\frac{10}{3};E_T} \lim_{k\to\infty} \|\boldsymbol{\varphi}_k - \boldsymbol{\varphi}\|_{5;E_T}.\end{aligned}$$

Letting $k \to \infty$ yields, for all $\boldsymbol{\varphi} \in C^\infty(0,T;\mathcal{V})$

$$\begin{aligned}\fint_t^{t+h}\int_E \mathbf{v}(\tau)\cdot\boldsymbol{\varphi}(\tau)\,dxd\tau &- \fint_t^{t+h}\int_0^\tau\int_E \mathbf{v}(s)\cdot\boldsymbol{\varphi}_s(s)\,dxdsd\tau \\ &+ \nu\fint_t^{t+h}\int_0^\tau\int_E \nabla\mathbf{v}(s) : \nabla\boldsymbol{\varphi}(s)\,dxdsd\tau \\ &+ \fint_t^{t+h}\int_0^\tau\int_E (\mathbf{v}(s)\cdot\nabla)\mathbf{v}(s)\cdot\boldsymbol{\varphi}(s)\,dxdsd\tau \\ &= \int_E \mathbf{v}_o\cdot\boldsymbol{\varphi}(0)\,dx + \fint_t^{t+h}\int_0^\tau\int_E \mathbf{f}(s)\cdot\boldsymbol{\varphi}(s)\,dxdsd\tau.\end{aligned}$$

Finally, let $h \to 0$ and notice that

$$\lim_{h\to0}\fint_t^{t+h}\int_E \mathbf{v}(\tau)\cdot\boldsymbol{\varphi}(\tau)\,dxd\tau = \int_E \mathbf{v}(t)\cdot\boldsymbol{\varphi}(t)\,dx \qquad \text{for a.e. } t \in (0,T),$$

since, for integrable functions in $(0,T)$, a.e. t is a Lebesgue point. Thus, the function $\mathbf{v}$ thus constructed satisfies the definition (8.2) of a weak solution. It should be stressed that the testing functions $\boldsymbol{\varphi}$ cannot, in general, be taken out of $C^1(0,T;V)$ as the limiting process for $k \to \infty$ requires a further smoothness, guaranteed in general by taking $\boldsymbol{\varphi} \in C^\infty(0,T;\mathcal{V})$.

12 Higher Integrability and Some Consequences

The Hopf solution has a limited degree of regularity due to the nonlinear term $(\mathbf{v}\cdot\nabla)\mathbf{v}\cdot\boldsymbol{\varphi}$. The weak formulation (8.2) holds for all $\boldsymbol{\varphi}\in C^\infty(0,T;\mathcal{V})\subset W$, whereas the solution $\mathbf{v}$ is required to be in W. If in (8.2) one could take $\boldsymbol{\varphi}=\mathbf{v}$, then, since $\operatorname{div}\mathbf{v}=0$, the nonlinear term would vanish and further regularity could be inferred on $\mathbf{v}$. Optimal local and global regularity of the Hopf solutions is unknown and it is currently a major topic of investigation. To underscore this point, here we indicate some consequences of assuming higher integrability on $\mathbf{v}$ and on the various terms of (8.1), including the pressure term ∇p.

Lemma 12.1 *Let* $\mathbf{v}$ *be a Hopf solution of (8.1). Then,* $(\mathbf{v}\cdot\nabla)\mathbf{v}\in L^{\frac{5}{4}}(E_T;\mathbb{R}^3)$, *and*

$$\|(\mathbf{v}\cdot\nabla)\mathbf{v}\|_{\frac{5}{4};E_T}\le\|\mathbf{v}\|_{\frac{10}{3};E_T}\|\nabla\mathbf{v}\|_{2;E_T}.$$

Proof. Let $q,q'>1$ be Hölder conjugate and for $p>1$ to be chosen, compute and estimate

$$\iint_{E_T}|(\mathbf{v}\cdot\nabla)\mathbf{v}|^p dxdt\le\Big(\iint_{E_T}|\nabla\mathbf{v}|^{pq}dxdt\Big)^{\frac{1}{q}}\Big(\iint_{E_T}|\mathbf{v}|^{pq'}dxdt\Big)^{\frac{1}{q'}}. \tag{12.1}$$

Choose $pq=2$ and $pq'=\frac{10}{3}$, which yields $p=\frac{5}{4}$. ■

Assume momentarily that $\nabla p\in L^{\frac{5}{4}}_{\rm loc}(E_T;\mathbb{R}^3)$ and set

$$\boldsymbol{\Phi}=\mathbf{f}-\nabla p-(\mathbf{v}\cdot\nabla)\mathbf{v}\in L^{\frac{5}{4}}_{\rm loc}(E_T;\mathbb{R}^3).$$

Then the weak formulation (8.2) yields[8]

$$\mathbf{v}_t-\nu\Delta\mathbf{v}=\boldsymbol{\Phi}\quad\text{weakly in } E_T \text{ for all } \boldsymbol{\varphi}\in C_o^\infty(E_T;\mathbb{R}^3). \tag{12.2}$$

This is a linear parabolic system with the forcing term $\boldsymbol{\Phi}\in L^{\frac{5}{4}}_{\rm loc}(E_T;\mathbb{R}^3)$. Then, by classical parabolic theory [67], the weak derivatives $\mathbf{v}_{x_ix_j}$ and $\mathbf{v}_t$ are in $L^{\frac{5}{4}}_{\rm loc}(E_T;\mathbb{R}^3)$. The argument can be repeated to yield further regularity on $\mathbf{v}$. Therefore, assuming a moderate degree of integrability of ∇p yields a considerably higher regularity on $\mathbf{v}$.

In § 20, we get back to the regularity of the pressure for Hopf solutions.

12.1 The $L^{p,q}(E_T;\mathbb{R}^N)$ Spaces

For $p,q>1$ let

$$L^{p,q}(E_T;\mathbb{R}^N)=\left\{\begin{array}{c}\text{Lebesgue measurable functions } \mathbf{f}:E_T\to\mathbb{R}^N \text{ with}\\ \text{finite norm } \|\mathbf{f}\|_{p,q;E_T}=\Big(\int_0^T\|\mathbf{f}(\cdot,t)\|_{p;E}^q\,dt\Big)^{\frac{1}{q}}\end{array}\right\}.$$

---

[8] See **12.1.** of the Complements.

In the scalar case, we have already introduced these spaces in Section 1 of Chapter 11. In what follows we let $p > N$ and $q > 2$ be linked by

$$\frac{N}{p} + \frac{2}{q} = 1. \tag{12.3}$$

Condition (12.3) is usually known as the *Ladyzhenskaya–Prodi–Serrin condition.*

Recall also the following special case of the Gagliardo–Nirenberg embedding inequality[9]

$$\|\mathbf{v}\|_{r;E} \le \gamma(N,p)\|\nabla\mathbf{v}\|_{2;E}^{\frac{N}{p}}\|\mathbf{v}\|_{2;E}^{\frac{2}{q}}, \qquad \text{where} \qquad r = \frac{2p}{p-2}.$$

Lemma 12.2 *There exists a constant $\gamma(N,p)$ depending only on N and p, such that for any triple $(\mathbf{u},\mathbf{v},\mathbf{w})$ with $\mathbf{u} \in L^{p,q}(E_T;\mathbb{R}^N)$, $\mathbf{v} \in W$, and $\mathbf{w} \in W$, there holds*

$$\begin{gathered}\int_0^T\!\!\int_E \big|(\mathbf{v}\cdot\nabla)\mathbf{w}\cdot\mathbf{u}\big|dxdt \le \gamma\|\mathbf{u}\|_{p,q;E_T}\|\mathbf{v}\|_W\|\nabla\mathbf{w}\|_{2;E_T};\\ \int_0^T\!\!\int_E \big|(\mathbf{w}\cdot\nabla)\mathbf{w}\cdot\mathbf{u}\big|dxdt\\ \le \gamma\Big(\int_0^T \|\mathbf{u}(\cdot,t)\|_{p;E}^q\|\mathbf{w}(\cdot,t)\|_{2;E}^2\,dt\Big)^{\frac{1}{q}}\|\nabla\mathbf{w}\|_{2;E_T}^{1+\frac{N}{p}}.\end{gathered} \tag{12.4}$$

Proof. By Hölder's inequality with conjugate exponents

$$\frac{1}{r} + \frac{1}{p} = \frac{1}{2}, \qquad \text{i.e.,} \qquad \frac{1}{r} + \frac{1}{p} + \frac{1}{2} = 1,$$

also using the indicated special case of Gagliardo–Nirenberg inequality we have

$$\begin{aligned}\int_E \big|(\mathbf{v}\cdot\nabla)\mathbf{w}\cdot\mathbf{u}\big|dx &\le \|\mathbf{v}\|_{r;E}\|\nabla\mathbf{w}\|_{2;E}\|\mathbf{u}\|_{p;E}\\ &\le \gamma\|\mathbf{v}\|_{2;E}^{\frac{2}{q}}\|\nabla\mathbf{v}\|_{2;E}^{\frac{N}{p}}\|\nabla\mathbf{w}\|_{2;E}\|\mathbf{u}\|_{p;E}.\end{aligned}$$

Next, integrate over $(0,T)$ and use Hölder's inequality with conjugate exponents

$$\frac{N}{2p} + \frac{1}{q} + \frac{1}{2} = 1,$$

$$\begin{aligned}\int_0^T\!\!\int_E \big|(\mathbf{v}\cdot\nabla)\mathbf{w}\cdot\mathbf{u}\big|dxdt &\le \gamma\Big(\int_0^T \|\mathbf{v}(\cdot,t)\|_{2;E}^2\|\mathbf{u}(\cdot,t)\|_{p;E}^q\,dt\Big)^{\frac{1}{q}}\\ &\quad\times\Big(\int_0^T \|\nabla\mathbf{v}(\cdot,t)\|_{2;E}^2\,dt\Big)^{\frac{N}{2p}}\Big(\int_0^T \|\nabla\mathbf{w}(\cdot,t)\|_{2;E}^2\,dt\Big)^{\frac{1}{2}}\\ &\le \gamma\Big(\operatorname*{ess\,sup}_{(0,T)}\|\mathbf{v}(\cdot,t)\|_{2;E}\Big)^{\frac{2}{q}}\|\nabla\mathbf{v}\|_{2;E_T}^{\frac{N}{p}}\|\nabla\mathbf{w}\|_{2;E_T}\|\mathbf{u}\|_{p,q;E_T}\\ &\le \gamma\|\mathbf{v}\|_W\|\nabla\mathbf{w}\|_{2;E_T}\|\mathbf{u}\|_{p,q;E_T}.\end{aligned}$$

---

[9] See DiBenedetto [50], Chap. 10, Theorem 1.1.

This proves the first of (12.4). The proof of the second is the same by interchanging the roles of $\mathbf{v}$ and $\mathbf{w}$. ■

12.2 The Case $N = 2$

Lemma 12.3 *Let $N = 2$. Then, for all $\mathbf{v} \in W$ (12.3) holds with $p = q = 4$, and*

$$\|\mathbf{v}\|_{4;E_T} \le \pi^{-\frac{1}{4}} \|\mathbf{v}\|_W.$$

Proof. The Gagliardo–Nirenberg multiplicative inequality for $u \in W_o^{1,p}(E)$ reads[10]

$$\|u\|_{p^*;E} \le \gamma(N,p)\|\nabla u\|_{p;E}, \qquad \text{where} \qquad p^* = \frac{Np}{N-p} \qquad \text{and} \qquad 1 \le p < N$$

for a constant $\gamma(N,p)$ depending only on N and p. When $p = 1$ the optimal constant is $\gamma(N,1) = \frac{1}{N}\Big(\frac{N}{\omega_N}\Big)^{\frac{1}{N}}$, where ω_N is the measure of the unit sphere in $\mathbb{R}^N$. Apply the inequality for $N = 2$, with $u = |\mathbf{v}|^2$ and $p = 1$ to get

$$\begin{aligned}\Big(\int_E |\mathbf{v}|^4 dx\Big)^{\frac{1}{2}} &\le \frac{1}{2}\frac{1}{\sqrt{\pi}}\int_E \big|\nabla|\mathbf{v}|^2\big| dx \le \frac{1}{\sqrt{\pi}}\int_E |\mathbf{v}|\big|\nabla\mathbf{v}\big| dx \\ &\le \frac{1}{\sqrt{\pi}}\Big(\int_E |\mathbf{v}|^2 dx\Big)^{\frac{1}{2}}\Big(\int_E |\nabla\mathbf{v}|^2 dx\Big)^{\frac{1}{2}} \\ &\le \frac{1}{\sqrt{\pi}}\Big(\operatorname*{ess\,sup}_{(0,T)}\int_E |\mathbf{v}|^2 dx\Big)^{\frac{1}{2}}\Big(\int_E |\nabla\mathbf{v}|^2 dx\Big)^{\frac{1}{2}}.\end{aligned}$$

From this

$$\int_E |\mathbf{v}(\cdot,t)|^4 dx \le \frac{1}{\pi}\|\mathbf{v}\|_W^2 \int_E |\nabla\mathbf{v}(\cdot,t)|^2 dx.$$

Integrating over $(0,T)$ yields

$$\|\mathbf{v}\|_{4;E_T}^4 \le \frac{1}{\pi}\|\mathbf{v}\|_W^2 \|\nabla\mathbf{v}\|_{2;E_T}^2.$$ ■

Corollary 12.1 *Any Hopf solution to (8.1) for $N = 2$ satisfies (12.3) for $p = q = 4$.*

13 Energy Identity for the Homogeneous Boundary Value Problem with Higher Integrability

We get back to (8.1) with $\mathbf{f} = 0$ to which we refer as the *homogeneous problem* and label it as $(8.1)_o$. A weak solution is meant in the sense of $(8.2)_o$, with $\mathbf{f} = 0$, for all $\boldsymbol{\varphi} \in C^\infty(0,T;\mathcal{V})$. Although a weak solution has been constructed by Hopf's procedure we assume here that one is given and meant weakly.

---

[10] See DiBenedetto [50], Chap. 10, Corollary 1.1.

Proposition 13.1 (Prodi [207]) *Let* $\mathbf{v}$ *be a weak solution to* $(8.1)_o$*. Moreover, assume that* $\mathbf{v} \in L^{p,q}(E_T; \mathbb{R}^N)$ *with* $p > N$ *and* $q > 2$ *satisfying (12.3). Then,*

$$\|\mathbf{v}(\cdot,t)\|_{2;E}^2 + 2\nu\|\nabla\mathbf{v}\|_{2;E_T}^2 = \|\mathbf{v}_o\|_{2;E}^2 \quad \textit{for a.e. } t \in (0,T). \tag{13.1}$$

Proof. The proof consists in taking formally $\boldsymbol{\varphi} = \mathbf{v}$ in $(8.2)_o$. The assumption (12.3) makes this possible by a series of approximations. First, since $\mathbf{v} \in L^2(0,T;V)$ there exists a sequence $\{\mathbf{v}_k\} \subset C^\infty(0,T;\mathcal{V})$ such that $\{\mathbf{v}_k\} \to \mathbf{v}$ in $L^2(0,T;V)$. Next, let $J(\cdot)$ be the Friedrichs' mollifying kernel in $\mathbb{R}$ and denote by $J_\varepsilon(\cdot)$ its rescaling by a parameter $\varepsilon \in (0,1)$,

$$J(\tau) = C\begin{cases} \exp\left(\frac{\tau^2}{\tau^2-1}\right) & \text{for } |\tau| < 1, \\ 0 & \text{for } |\tau| \ge 1, \end{cases} \qquad J_\varepsilon(\tau) = \frac{1}{\varepsilon}J\Big(\frac{\tau}{\varepsilon}\Big),$$

where $C > 0$ is a constant that normalizes the kernel J. Notice that

$$J(-t) = J(t), \quad J'(-t) = -J'(t).$$

Then, for a.e. $t \in (0,T]$ fixed, set

$$\mathbf{v}_{\varepsilon,k}(\tau) = \int_0^t J_\varepsilon(\tau - s)\mathbf{v}_k(s)\,ds; \qquad \mathbf{v}_\varepsilon(\tau) = \int_0^t J_\varepsilon(\tau - s)\mathbf{v}(s)\,ds. \tag{13.2}$$

One verifies that $\mathbf{v}_{k,\varepsilon} \in C^\infty(0,T;\mathcal{V})$ and therefore, it is an admissible test function in the weak formulation $(8.2)_o$. Such a choice gives

$$\begin{aligned} \int_E \mathbf{v}(t)\cdot\mathbf{v}_{\varepsilon,k}(t)\,dx - \int_0^t\int_E \mathbf{v}\cdot\mathbf{v}_{\varepsilon,k;\tau}\,dxd\tau \\ + \int_0^t\int_E \big(\nu\nabla\mathbf{v} : \nabla\mathbf{v}_{\varepsilon,k} + (\mathbf{v}\cdot\nabla)\mathbf{v}\cdot\mathbf{v}_{\varepsilon,k}\big)dxd\tau \\ = \int_E \mathbf{v}_o\cdot\mathbf{v}_{\varepsilon,k}(0)\,dx. \end{aligned}$$

Letting $k \to \infty$ now gives

$$\begin{aligned} \int_E \mathbf{v}(t)\cdot\mathbf{v}_\varepsilon(t)\,dx - \int_0^t\int_E \mathbf{v}\cdot\mathbf{v}_{\varepsilon;\tau}\,dxd\tau \\ + \int_0^t\int_E \big(\nu\nabla\mathbf{v} : \nabla\mathbf{v}_\varepsilon + (\mathbf{v}\cdot\nabla)\mathbf{v}\cdot\mathbf{v}_\varepsilon\big)\,dxd\tau \\ = \int_E \mathbf{v}_o\cdot\mathbf{v}_\varepsilon(0)\,dx. \end{aligned} \tag{13.3}$$

The various limits, but the first one and the one regarding the nonlinear term, are justified since $\{\mathbf{v}_{\varepsilon,k}\} \to \mathbf{v}_\varepsilon$ in $L^2(0,T;V)$.

The limit of the first term is justified, for fixed $\varepsilon > 0$ since $\{\mathbf{v}_k\} \to \mathbf{v}$ in $L^2(E_T; \mathbb{R}^N)$ and the definition of $\mathbf{v}_\varepsilon$. Indeed,

$$\begin{aligned}
\Big| \int_E \mathbf{v}(t) \cdot [\mathbf{v}_{\varepsilon,k}(t) - \mathbf{v}_\varepsilon(t)]\, dx \Big| \\
&\le \int_E |\mathbf{v}(t)| \int_0^t J_\varepsilon(t-s) |\mathbf{v}_k(s) - \mathbf{v}(s)|\, dsdx \\
&= \int_0^t J_\varepsilon(t-s) \Big(\int_E |\mathbf{v}(t)| |\mathbf{v}_k(s) - \mathbf{v}(s)|\, dx \Big) ds \\
&\le \int_0^t J_\varepsilon(t-s) \|\mathbf{v}(t)\|_{2;E} \|\mathbf{v}_k(s) - \mathbf{v}(s)\|_{2;E}\, ds \\
&\le \operatorname*{ess\,sup}_{(0,T)} \|\mathbf{v}(t)\|_{2;E} \int_0^T J_\varepsilon(t-s) \|\mathbf{v}_k(s) - \mathbf{v}(s)\|_{2;E}\, ds \\
&\le \|\mathbf{v}\|_W \Big(\int_{\mathbb{R}} J_\varepsilon^2(t)\, dt \Big)^{\frac{1}{2}} \|\mathbf{v}_k - \mathbf{v}\|_{2;E_T}.
\end{aligned}$$

The last term tends to zero as $k \to \infty$ since $\{\mathbf{v}_k\} \to \mathbf{v}$ in $L^2(E_T; \mathbb{R}^N)$. As for the nonlinear term, compute and estimate

$$\begin{aligned}
\Big| \int_0^t \int_E (\mathbf{v} \cdot \nabla)\mathbf{v} \cdot (\mathbf{v}_{\varepsilon,k} - \mathbf{v}_\varepsilon)\, dxd\tau \Big| &= \Big| \int_0^t \int_E (\mathbf{v} \cdot \nabla)(\mathbf{v}_{\varepsilon,k} - \mathbf{v}_\varepsilon) \cdot \mathbf{v}\, dxd\tau \Big| \\
&\le \gamma \|\mathbf{v}\|_W \|\mathbf{v}\|_{p,q;E_T} \|\nabla(\mathbf{v}_{\varepsilon,k} - \mathbf{v}_\varepsilon)\|_{2;E_T},
\end{aligned}$$

by virtue of Lemma 12.2. This is indeed the role of the assumption (12.3) and the ensuing Lemma. The last term tends to zero as $k \to \infty$ since $\{\mathbf{v}_{\varepsilon,k}\} \to \mathbf{v}_\varepsilon$ in $L^2(0, T; V)$.

Next, we let $\varepsilon \to 0$ in (13.3). For the first term we have

$$\begin{aligned}
\int_E \mathbf{v}(t) \cdot \mathbf{v}_\varepsilon(t) dx &= \int_E \mathbf{v}(t) \int_0^t J_\varepsilon(t-s) \mathbf{v}(s)\, dsdx \\
&= \int_E \int_0^t J_\varepsilon(\eta) \mathbf{v}(t-\eta) \cdot \mathbf{v}(t)\, d\eta dx \\
&= \int_E \int_0^t J_\varepsilon(\eta) |\mathbf{v}(t)|^2 d\eta dx \\
&\quad + \int_E \int_0^t J_\varepsilon(\eta) \mathbf{v}(t) \cdot [\mathbf{v}(t-\eta) - \mathbf{v}(t)]\, d\eta dx.
\end{aligned}$$

Since J_ε is even and it has been normalized, as $\varepsilon \to 0$,

$$\int_E \int_0^t J_\varepsilon(\eta) |\mathbf{v}(t)|^2 d\eta dx \ \to\ \frac{1}{2} \int_E |\mathbf{v}(t)|^2 dx.$$

On the other hand,

$$\Big|\int_0^t J_\varepsilon(\eta)\int_E \mathbf{v}(t)\cdot[\mathbf{v}(t-\eta)-\mathbf{v}(t)]\,dxd\eta\Big|$$
$$\le \int_0^t J_\varepsilon(\eta)\Big|\int_E \mathbf{v}(t)\cdot[\mathbf{v}(t-\eta)-\mathbf{v}(t)]\,dx\Big|d\eta$$

and the integral tends to zero as $|\eta|<\varepsilon\to 0$ by the weak continuity of $t\to\mathbf{v}(t)$ in $L^2(E)$. A similar result holds for the right-hand side of (13.3). The second term is identically zero in ε. Indeed, after interchanging the order of integration, it can be written as

$$\int_E\Big(\int_0^t\int_0^t J_\varepsilon'(\tau-s)\mathbf{v}(s)\cdot\mathbf{v}(\tau)\,dsd\tau\Big)dx.$$

Now the integral in $(\cdots)$, for a.e. fixed $x\in E$, is a double integral extended over the rectangle of vertices $\{(0,0),(t,0),(t,t),(0,t)\}$, which, in turn is the union of two disjoint, equal triangles of vertices $\{(0,0),(t,0),(t,t)\}$ and $\{(0,0),(t,t),(0,t)\}$. Now the argument $\mathbf{v}(s)\cdot\mathbf{v}(\tau)$ is even with respect to these triangles, whereas $J_\varepsilon'(\tau-s)$ is odd.

Next,

$$\Big|\int_0^t\int_E \nabla\mathbf{v}:\nabla(\mathbf{v}_\varepsilon-\mathbf{v})\,dxd\tau\Big|\le\int_E\int_0^t|\nabla\mathbf{v}|\int_{\mathbb{R}}J_\varepsilon(\tau-s)|\nabla[\mathbf{v}(s)-\mathbf{v}(\tau)]|\,dsd\tau dx$$

and this tends to zero as $\varepsilon\to 0$. Finally, for the nonlinear term compute and estimate, with the aid of Lemma 12.2,

$$\int_0^t\int_E(\mathbf{v}\cdot\nabla)\mathbf{v}\cdot\int_0^t J_\varepsilon(\tau-s)[\mathbf{v}(s)-\mathbf{v}(\tau)]\,dxdsd\tau$$
$$\le\gamma\|\mathbf{v}\|_W\|\mathbf{v}\|_{p,q;E_T}\Big(\int_E\int_0^t\Big(\int_0^t J_\varepsilon(\tau-s)[\nabla\mathbf{v}(s)-\nabla\mathbf{v}(\tau)]ds\Big)^2d\tau dx\Big)^{\frac{1}{2}},$$

which tends to zero as $\varepsilon\to 0$ by the property of the mollifiers. Observe that the limit of the nonlinear term

$$\lim_{\varepsilon\to 0}\int_0^t\int_E(\mathbf{v}\cdot\nabla)\mathbf{v}\cdot\mathbf{v}_\varepsilon\,dxd\tau=\int_0^t\int_E(\mathbf{v}\cdot\nabla)\mathbf{v}\cdot\mathbf{v}\,dxd\tau=0$$

gives zero contribution since $\operatorname{div}\mathbf{v}=0$. Collecting these calculations proves (13.1). ■

Remark 13.1 For $N=2$ condition (12.3) is redundant, as already stated in Lemma 12.3.

14 Stability and Uniqueness for the Homogeneous Boundary Value Problem with Higher Integrability

Proposition 14.1 [177] *Let* $\mathbf{v}$ *and* $\mathbf{u}$ *be two weak solutions of (8.1) with* $\mathbf{f}=0$*, originating from initial data* $\mathbf{v}_o$ *and* $\mathbf{u}_o$ *in* $L^2(E;\mathbb{R}^N)$*, meant in the* *sense of (8.2)$_o$, for all* $\boldsymbol{\varphi} \in C^\infty(0,T;\mathcal{V})$*. Moreover, assume that at least one* $\mathbf{v}$ *or* $\mathbf{u}$*, say, for example,* $\mathbf{u}$ *is in* $L^{p,q}(E_T;\mathbb{R}^N)$ *with* $p > N$ *and* $q > 2$ *satisfying (12.3). Assume finally that they both satisfy the energy estimates*

$$\begin{aligned} \|\mathbf{v}(\cdot,t)\|_{2;E}^2 + 2\nu\|\nabla\mathbf{v}\|_{2;E_t}^2 &\le \|\mathbf{v}_o\|_{2;E}^2 \\ \|\mathbf{u}(\cdot,t)\|_{2;E}^2 + 2\nu\|\nabla\mathbf{u}\|_{2;E_T}^2 &\le \|\mathbf{u}_o\|_{2;E}^2 \end{aligned} \qquad \textit{for a.e. } t \in (0,T). \tag{14.1}$$

Then, there exist a constant γ *depending only upon* N *and* ν *such that setting* $\mathbf{w} = \mathbf{v} - \mathbf{u}$ *there holds*

$$\|\mathbf{w}(\cdot,t)\|_{2;E}^2 \le \|\mathbf{w}_o\|_{2;E}^2 \exp\left\{\gamma \int_0^t \|\mathbf{u}(\cdot,\tau)\|_{p;E}^q \, d\tau\right\}$$

for a.e. $t \in (0,T)$*.*

Remark 14.1 If both $\mathbf{v}$ and $\mathbf{u}$ are in $L^{p,q}(E_T;\mathbb{R}^N)$ with $p > N$ and $q > 2$ satisfying (12.3) then by Proposition 13.1, the energy estimates (14.1) are satisfied. The Proposition is a statement of stability and uniqueness. If $N = 2$, $\mathbf{v}$ and $\mathbf{u}$ are both in $L^4(E_T;\mathbb{R}^2)$ and therefore, weak solutions are unique.

Proof. Let $\mathbf{v}$ and $\mathbf{u}$ be two weak solutions to (8.1) originating from initial data $\mathbf{v}_o$ and $\mathbf{u}_o$ in $L^2(E)$, meant in the sense of $(8.2)_o$, with $\mathbf{f} = 0$, for all $\boldsymbol{\varphi} \in C^\infty(0,T;\mathcal{V})$. In the weak formulation of $\mathbf{v}$ take the testing function $\mathbf{u}_{\varepsilon,k}$ defined as in (13.2) and in the weak formulation of $\mathbf{u}$ take the testing function $\mathbf{v}_{\varepsilon,k}$. Then let $k \to \infty$ by the same arguments as in the proof of Proposition 13.1, and add the resulting identities getting

$$\begin{aligned} &\int_E [\mathbf{v}(t)\cdot\mathbf{u}_\varepsilon(t) + \mathbf{v}_\varepsilon(t)\cdot\mathbf{u}(t)]\,dx \\ &\quad - \int_E \Big(\int_0^t \int_0^t J_\varepsilon'(\tau - s)[\mathbf{v}(\tau)\cdot\mathbf{u}(s) + \mathbf{v}(s)\cdot\mathbf{u}(\tau)]ds d\tau \Big)\,dx \\ &\quad + \nu \int_0^t \int_E \big(\nabla\mathbf{v} : \nabla\mathbf{u}_\varepsilon + \nabla\mathbf{v}_\varepsilon : \nabla\mathbf{u}\big)\,dx d\tau \\ &\quad + \int_0^t \int_E \big[(\mathbf{v}\cdot\nabla)\mathbf{v}\cdot\mathbf{u}_\varepsilon + (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{v}_\varepsilon\big]\,dx d\tau \\ &\quad = \int_E [\mathbf{v}_o\cdot\mathbf{u}_\varepsilon(0) + \mathbf{v}_\varepsilon(0)\cdot\mathbf{u}_o]\,dx. \end{aligned}$$

Arguing as in the proof of Proposition 13.1, the second integral is identically zero in ε since the argument $[\mathbf{v}(\tau)\mathbf{u}(s) + \mathbf{v}(s)\mathbf{u}(\tau)]$ is even with respect to the two triangles of vertices $\{(0,0),(t,0),(t,t)\}$ and $\{(0,0),(t,t),(0,t)\}$ and J_ε' is odd with respect to the same triangles. We may now let $\varepsilon \to$ by the same arguments and get

$$\begin{aligned}\int_E \mathbf{v}(\cdot,t)\cdot\mathbf{u}(\cdot,t)\,dx + 2\nu\int_0^t\int_E \nabla\mathbf{v}:\nabla\mathbf{u}\,dxd\tau \\ +\int_0^t\int_E \big[(\mathbf{v}\cdot\nabla)\mathbf{v}\cdot\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{v}\big]\,dxd\tau \\ =\int_E \mathbf{v}_o\cdot\mathbf{u}_o\,dx.\end{aligned} \tag{14.2}$$

Next observe that since weak solutions are divergence-free

$$\begin{aligned}\int_0^t\int_E(\mathbf{v}\cdot\nabla)\mathbf{v}\cdot\mathbf{u}\,dxd\tau &= -\int_0^t\int_E(\mathbf{v}\cdot\nabla)\mathbf{v}\cdot\mathbf{w}\,dxd\tau\\ \int_0^t\int_E(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{v}\,dxd\tau &= \int_0^t\int_E(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{w}\,dxd\tau,\end{aligned}$$

where we have set $\mathbf{w}=\mathbf{v}-\mathbf{u}$. Using again that $\mathbf{w}$ is divergence-free, the sum of these terms equals

$$\int_0^t\int_E\big[(\mathbf{v}\cdot\nabla)\mathbf{v}\cdot\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{v}\big]\,dxd\tau = -\int_0^t\int_E(\mathbf{w}\cdot\nabla)\mathbf{w}\cdot\mathbf{u}\,dxd\tau.$$

Adding the energy inequalities (14.1) and subtracting (14.2) multiplied by 2 gives

$$\|\mathbf{w}(t)\|_{2;E}^2+2\nu\|\nabla\mathbf{w}\|_{2;E_t}^2\le\|\mathbf{w}_o\|_{2;E}^2+\Big|\int_0^t\int_E(\mathbf{w}\cdot\nabla)\mathbf{w}\cdot\mathbf{u}\,dxd\tau\Big|.$$

The right-hand side is estimated by the second of (12.4) of Lemma 12.2, and Young's inequality with conjugate exponents $\frac{1}{q}$ and $\frac{1}{2}+\frac{N}{2p}$, and gives

$$\Big|\int_0^t\int_E(\mathbf{w}\cdot\nabla)\mathbf{w}\cdot\mathbf{u}\,dxd\tau\Big|\le\gamma\int_0^t\|\mathbf{u}(\tau)\|_{p;E}^q\|\mathbf{w}(\tau)\|_{2;E}^2\,d\tau+2\nu\|\nabla\mathbf{w}\|_{2;E_t}^2,$$

for a constant γ depending only upon N and ν. Combining these estimates gives

$$\|\mathbf{w}(t)\|_{2;E}^2\le\|\mathbf{w}_o\|_{2;E}^2+\gamma\int_0^t\|\mathbf{u}(\tau)\|_{p;E}^q\|\mathbf{w}(\tau)\|_{2;E}^2\,d\tau.$$

The proof is concluded by an application of Gronwall's inequality. ■

15 Local Regularity of Solutions with Higher Integrability

We continue assuming the higher integrability (12.3), and we address the smoothness of weak solutions. Notice that there is a difference between studying the regularity of solutions to the initial-boundary value problem (8.1) or the interior regularity.

In this second case, we deal with the intrinsic properties of the Navier–Stokes equations; indeed, we consider a local, weak solution in E_T, namely $\mathbf{v}$, which is weakly divergence-free in E_T, and such that for any subset

$$\Omega_{t_1,t_2} \overset{\text{def}}{=} \Omega \times (t_1, t_2) \subset\subset E_T,$$

we have

$$\mathbf{v} \in L^2(t_1, t_2; W^{1,2}(\Omega)) \cap L^\infty(t_1, t_2; L^2(\Omega)),$$

and $\mathbf{v}$ satisfies (8.2) for all solenoidal test functions $\boldsymbol{\varphi} \in C_o^\infty(\Omega_{t_1,t_2})$.

If we consider a function $\psi = \psi(x)$ harmonic in Ω and an integrable function $a = a(t)$, it is a matter of straightforward computation to check that

$$\mathbf{v} = a(t)\nabla\psi(x)$$

is a local, weak solution of the Navier–Stokes equations for $\mathbf{f} = 0$. Hence, it is infinitely differentiable with respect to space, but it might have integrable singularities with respect to time.

This example, which is attributed to Serrin [233], suggests that the time differentiability of a weak solution is directly connected to the time regularity, which is assumed from the very beginning.

Moreover, as pointed out by Galdi (see [93], page 41]), these highly irregular solutions exist because possible singularities are absorbed by the pressure term. As summarized by Struwe [248], local regularity properties are not influenced by the nonlocal effects of the pressure, as long as we are interested only in boundedness and spatial regularity.

The situation is different if one considers the initial-boundary value problem (8.1) and its weak formulation (8.2), where one can hope to gain regularity in time from the assigned conditions. This has to do with the incompressibility of the fluids, since a sudden modification of the boundary value of the motion will be immediately felt throughout the whole flow region.

In this section we report a sufficient condition for the interior regularity, whereas in a subsequent section we get back to regularity for the initial-boundary value problem.

Theorem 15.1. [233] *Let* $\mathbf{v}$ *be a local, weak solution of the Navier–Stokes equations in* E_T *in the sense defined before.*

Assume that $\mathbf{f}$ *is conservative and at least in* $L^{1,1}(E_T; \mathbb{R}^N)$*, and that* $\mathbf{v} \in L^{p,q}(E_T; \mathbb{R}^N)$*, where* $p > N$*,* $q > 2$*, satisfy* (12.3)*. Then,* $\mathbf{v}$ *is of class* C^∞ *with respect to the space variable* x*, and each space derivative is bounded in compact subsets of* E_T*.*

If, in addition, $\mathbf{v}_t \in L^{2,s}(E_T; \mathbb{R}^N)$ *for some* $s \geq 1$*, then the space derivatives are absolutely continuous functions of time. Moreover, there exists a strongly differentiable function* $p = p(x,t)$ *such that*

$$\mathbf{v}_t - \nu\Delta\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} + \nabla p = \mathbf{f} \tag{15.1}$$

almost everywhere in E_T*.*

Remark 15.1 Owing to the local nature of the result, without loss of generality, one could more generally assume that $\mathbf{v} \in L^{p,q}(\mathcal{K};\mathbb{R}^N)$ for any $\mathcal{K} \subset\subset E_T$, with similar local integrability assumptions on $\mathbf{f}$ and $\mathbf{v}_t$.

Remark 15.2 If we limit ourselves to $N = 3$, using the Sobolev inequalities (see Chapters 9 and 11), one can show that a weak solution naturally belongs to $L^{p,q}(E_T;\mathbb{R}^N)$, where $\dfrac{3}{p}+\dfrac{2}{q} \ge \dfrac{3}{2}$; hence, there is a gap between the natural regularity of $\mathbf{v}$ and what is assumed in (12.3) in order to have differentiability in space for $\mathbf{v}$.

Theorem 15.1 is originally attributed to Serrin [233], who developed some of the methods introduced by Ohyama [196] a few years before. Moreover, he used the stronger condition

$$\frac{N}{p}+\frac{2}{q}<1. \tag{15.2}$$

The full (12.3) with $p > N$ was proved by Fabes et al., Sohr and von Wahl, and Struwe [70, 244, 248]; see also Giga [99]. The limiting case of $p = N$ was dealt with in Struwe [248] under a smallness condition; namely, Struwe assumes that $\mathbf{v} \in L^{N,\infty}(E_T;\mathbb{R}^N)$ and that there is a $\rho > 0$ such that

$$\int_{B_\rho \cap E} |\mathbf{v}(\cdot,t)|^N \, dx \le \epsilon \tag{15.3}$$

uniformly with respect to t in $(0,T)$ for some absolute constant ϵ (see also Sohr and won Wahl [244]).

For $N = 3$, condition (15.3) was fully removed in Escuriaza et al. [65]. The regularity approach to $L^{3,\infty}$-solutions developed in Escuriaza et al. [65] requires a completely different method with respect to Serrin's techniques and further developments, and the proof is based on the reduction of the regularity problem to a backward uniqueness problem.

For the sake of simplicity, here we present the original proof of Serrin [233], and therefore, we limit ourselves to (15.2).

At the end of Serrin [233], he conjectures that under the same assumptions on $\mathbf{v}$ and $\mathbf{f}$, it should be possible to prove that solutions are analytic in the space variables. This was indeed proved by Kahane [132].

Let V^2 be the closure of $\mathcal{V}$ in $W^{2,2}(E)$: for $N = 2$ and $N = 3$ and the initial datum $\mathbf{v}_o \in V^2$, Kiselev and Ladyzhenskaya [138] have proved the existence of a weak solution of the initial-boundary value problem (8.1) with

$$\mathbf{v} \in L^{4,\infty}(E_T;\mathbb{R}^N), \qquad \mathbf{v},\, \nabla\mathbf{v},\, \mathbf{v}_t \in L^{2,\infty}(E_T;\mathbb{R}^N);$$

hence, Theorem 15.1 contains as a special case that the Kiselev–Ladyzhenskaya solution is of class C^∞ in the space variable, and is Lipschitz continuous in time, at least if $\mathbf{f}$ is conservative.

Moreover, if $N = 2$ or $N = 3$ and the initial data are smooth enough for the Kiselev–Ladyzhenskaya solution to exist, then by Proposition 14.1, Hopf's solution must be the same and consequently has to be of class C^∞ in the space variables.

Remark 15.3 As pointed out by Serrin [234], p. 76, for the solutions in the sense of Kiselev–Ladyzhenskaya, the case $N = 4$ can be treated by methods similar to the ones employed by Kiselev and Ladyzhenskaya [138].

16 Proof of Theorem 15.1 – Introductory Results

In the following, $\Omega_{t_1,t_2} = \Omega \times (t_1, t_2)$ denotes an open set compactly contained in E_T; moreover, we frequently deal with convolution integrals of the type

$$h(x,t) = \iint_{\Omega_{t_1,t_2}} k(x-\xi, t-\tau) g(\xi,\tau)\, d\xi d\tau,$$

and we write $h(x,t) = (k * g)(x,t)$. A first fundamental result is given by the following.

Proposition 16.1 *Let* $k \in L^{p,p'}(\mathbb{R}^N \times \mathbb{R}; \mathbb{R})$ *and* $g \in L^{q,q'}(\Omega_{t_1,t_2}; \mathbb{R})$ *with* $N \geq 1$, *and*

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1, \qquad \frac{1}{p'} + \frac{1}{q'} = \frac{1}{r'} + 1. \tag{16.1}$$

Then, for the convolution

$$h(x,t) \overset{def}{=} \iint_{\Omega_{t_1,t_2}} k(x-\xi, t-\tau) g(\xi,\tau)\, d\xi d\tau, \quad (x,t) \in \Omega_{t_1,t_2}, \tag{16.2}$$

we have

$$\|h\|_{r,r'} \leq \|k\|_{p,p'} \|g\|_{q,q'}.$$

For the proof, we refer to Section 16c of the Complements.

We take as $k = k(x,t)$ a space derivative of the fundamental solution Γ of the heat equation

$$\frac{\partial u}{\partial t} - \nu \Delta u = 0. \tag{16.3}$$

The function Γ has already been introduced in (2.1) of Chapter 5, and here we extend it to $\mathbb{R}^N \times \mathbb{R}$, also taking into account a general diffusion coefficient $\nu > 0$, not necessarily equal to 1 (for the physical motivation of the coefficient ν see (3.2) of Chapter 0); we set

$$\Gamma(x,t) = \begin{cases} \dfrac{1}{(4\pi\nu t)^{\frac{N}{2}}} \exp\left(-\dfrac{|x|^2}{4\nu t}\right) & t > 0, \\ \hfill 0 & t \leq 0. \end{cases} \tag{16.4}$$

We have the following.

Lemma 16.1 *Let k be a space derivative of the function Γ defined in* (16.4)*. Then, for any $g \in L^{q,q'}(\Omega_{t_1,t_2};\mathbb{R})$, given $h = (k * g)(x,t)$ we have*

$$\|h\|_{r,r';\Omega_{t_1,t_2}} \leq \gamma \|g\|_{q,q';\Omega_{t_1,t_2}},$$

where $\gamma = \gamma(t_2 - t_1, \nu, N, q, q', r, r')$, provided that $1 \leq q \leq r$, $1 \leq q' \leq r'$, and $N\left(\frac{1}{q} - \frac{1}{r}\right) + 2\left(\frac{1}{q'} - \frac{1}{r'}\right) < 1$.

Proof. Since $k = \dfrac{\partial \Gamma}{\partial x_i}$ with $i = 1, \ldots, N$, we have

$$|k(x,t)| \leq \gamma_1 t^{-\frac{N}{2}-1} |x| \exp\left(-\frac{|x|^2}{4\nu t}\right), \tag{16.5}$$

where $\gamma_1 = \gamma_1(\nu, N)$. Moreover, since both t and τ belong to (t_1, t_2), taking into account the definition of Γ, we have

$$\|k\|_{p,p'} \leq \left(\int_0^{t_2 - t_1} \left(\int_{\mathbb{R}^N} |k|^p \, dx\right)^{p'/p} dt\right)^{1/p'}.$$

Taking (16.5) into account, we have

$$\begin{aligned}
\left(\int_{\mathbb{R}^N} |k|^p \, dx\right)^{1/p} &= \frac{\gamma_1}{t^{\frac{N}{2}+1}} \left(\int_{\mathbb{R}^N} |x|^p \exp\left(-p\frac{|x|^2}{4\nu t}\right) dx\right)^{1/p} \\
&= \frac{\gamma_2}{t^{\frac{N}{2}+1}} t^{\frac{N+p}{2p}} \left(\int_0^{+\infty} s^{N+p-1} \exp(-\frac{p}{4}s^2) \, ds\right)^{1/p} \\
&= \gamma_2 t^{-\alpha},
\end{aligned}$$

where $\gamma_2 = \gamma_2(\nu, N, p)$ and $\alpha = \dfrac{N}{2}\left(1 - \dfrac{1}{p}\right) + \dfrac{1}{2}$. Hence,

$$\|k\|_{p,p'} \leq \left(\int_0^{t_2 - t_1} \gamma_2^{p'} t^{-\alpha p'} \, dt\right)^{1/p'} = \gamma_3 (t_2 - t_1)^{-\alpha + \frac{1}{p'}},$$

provided that $\alpha p' < 1$, and where $\gamma_3 = \gamma_3(\nu, N, p, p')$. From (16.1) we have that

$$1 - \frac{1}{p} = \frac{1}{q} - \frac{1}{r}, \quad \frac{1}{p'} = \frac{1}{r'} - \frac{1}{q'} + 1;$$

hence, condition $\alpha < \frac{1}{p'}$ can be rewritten as

$$\frac{N}{2}\left(\frac{1}{q} - \frac{1}{r}\right) + \frac{1}{2} < \frac{1}{r'} - \frac{1}{q'} + 1 \quad \Rightarrow \quad N\left(\frac{1}{q} - \frac{1}{r}\right) + 2\left(\frac{1}{q'} - \frac{1}{r'}\right) < 1.$$

■

In the sequel, we work with $\boldsymbol{\omega}$, the so-called *vorticity* of the fluid. We have

$$N = 2 \qquad \boldsymbol{\omega} = \operatorname{curl} \mathbf{v} = (0, 0, \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}),$$

$$N = 3 \qquad \boldsymbol{\omega} = \operatorname{curl} \mathbf{v} = (\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}, \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}, \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}),$$

when $N > 3$, $\boldsymbol{\omega}$ is an $(N-2)$-skew symmetric tensor, whose components are

$$\omega_{kl} = \frac{\partial v_k}{\partial x_l} - \frac{\partial v_l}{\partial x_k}.$$

Remark 16.1 In order to streamline the presentation and avoid distinctions for the values of N, in the following we always write $\boldsymbol{\omega} = \operatorname{curl} \mathbf{v}$ and on the other hand, even when $N = 2$ or $N = 3$, we think of $\boldsymbol{\omega}$ as a skew symmetric tensor; for example, for $N = 3$, we have

$$\boldsymbol{\omega} = \begin{bmatrix} 0 & \frac{\partial v_1}{\partial x_2} - \frac{\partial v_2}{\partial x_1} & \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1} \\ \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} & 0 & \frac{\partial v_2}{\partial x_3} - \frac{\partial v_3}{\partial x_2} \\ \frac{\partial v_3}{\partial x_1} - \frac{\partial v_1}{\partial x_3} & \frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3} & 0 \end{bmatrix},$$

and similarly for $N = 2$.

Let $\boldsymbol{A} = (A_1, A_2, \ldots, A_N)$: we define

$$\boldsymbol{A} \wedge \boldsymbol{\omega} = \boldsymbol{B} \equiv (B_1, B_2, \cdots, B_N),$$

where

$$B_k = \sum_{l=1}^{N} A_l \, \omega_{kl} = \sum_{l=1}^{N} A_l \left(\frac{\partial v_k}{\partial x_l} - \frac{\partial v_l}{\partial x_k} \right). \tag{16.6}$$

17 Proof of Theorem 15.1 Continued

Let E be a region in $\mathbb{R}^N$, $N \geq 2$ and $\Omega \subset E$ an open set such that $\bar{\Omega}$ is compact in E. Let $\mathbf{v} \in V$ (i.e., $|\nabla \mathbf{v}| \in L^2(\Omega)$, $\operatorname{div} \mathbf{v} = 0$ weakly) and consider the vorticity $\boldsymbol{\omega} = \operatorname{curl} \mathbf{v}$, where we take into account the previous definition and Remark 16.1.

Theorem 17.1. *Let $y \in \Omega$; then there exists a vector $\boldsymbol{A} = \boldsymbol{A}(y)$, harmonic in Ω, such that*

$$\mathbf{v}(y) = \int_{\Omega} \nabla_x H(y - x) \wedge \boldsymbol{\omega}(x) \, dx + \boldsymbol{A}(y),$$

where $H(y - x)$ is the fundamental solution of the Laplacean in $\mathbb{R}^N$ centered at y.

Proof. Let $x \mapsto u(x)$ be a $C_o^\infty(\mathbb{R}^N)$ scalar function and $H(y-x)$ be the solution of

$$-\Delta H(y-x) = \delta_y \qquad (\delta_y \ \text{ Dirac mass at } \ y)$$

in $\mathcal{D}'(\mathbb{R}^N)$. Since $u \in C_o^\infty(\mathbb{R}^N)$, in the sense of distributions we have

$$\langle -\Delta H(y-x), u\rangle = \langle \delta_y, u(x)\rangle = u(y) \quad \Rightarrow \quad u(y) = \langle H(y-x), -\Delta u(x)\rangle.$$

Since $-\Delta u(x) \in C_o^\infty(\mathbb{R}^N)$ and $H(y-x)$ is summable,

$$\begin{aligned} u(y) =& \langle H(y-x), -\Delta u(x)\rangle = -\int_{\mathbb{R}^N} \Delta u(x) H(y-x)\,dx = \\ =& -\int_\Omega \Delta u(x) H(y-x)\,dx - \int_{\mathbb{R}^N\setminus\Omega} \Delta u(x) H(y-x)\,dx \\ =& I_1 + I_2. \end{aligned}$$

Let us start with the computation of I_2. Over $\mathbb{R}^N\setminus\Omega$ we have $H(y-x) \in C^\infty$. Hence,

$$\begin{aligned} -\int_{\mathbb{R}^N\setminus\Omega} \Delta u(x) H(y-x)\,dx = &\int_{\partial\Omega} H(y-x)\,\nabla u(x)\cdot \mathbf{n}\,d\sigma \\ &- \int_{\mathbb{R}^N\setminus\Omega} \nabla u(x)\cdot\nabla H(y-x)\,dx = J_1 + J_2. \end{aligned}$$

Moreover,

$$J_2 = \int_{\partial\Omega} u(x)\,\nabla H(y-x)\cdot\mathbf{n}\,d\sigma + \underbrace{\int_{\mathbb{R}^N\setminus\Omega} u(x)\,\Delta H(y-x)\,dx.}_{=0}$$

Hence,

$$I_2 = \int_{\partial\Omega} H(y-x)\,\nabla u(x)\cdot\mathbf{n}\,d\sigma + \int_{\partial\Omega} u(x)\,\nabla H(y-x)\cdot\mathbf{n}\,d\sigma.$$

Coming to the computation of I_1, since $u \in C_o^\infty(\mathbb{R}^N)$, we have

$$\begin{aligned} I_1 =& -\int_\Omega \Delta u(x)\,H(y-x)\,dx = -\int_{\partial\Omega} H(y-x)\,\nabla u(x)\cdot\mathbf{n}\,d\sigma + \\ &+ \int_\Omega \nabla u(x)\cdot\nabla H(y-x)\,dx. \end{aligned}$$

Finally, summing up

$$\begin{aligned} u(y) =& \int_\Omega \nabla u(x)\cdot\nabla H(y-x)\,dx \\ &+ \int_{\partial\Omega} u(x)\nabla H(y-x)\cdot\mathbf{n}\,d\sigma. \end{aligned} \tag{17.1}$$

Notice that up to now we have assumed $u \in C_o^\infty$. However, a careful inspection of the proof shows that in (17.1) the only requirement for the existence of the integrals is $u \in H^1(\Omega)$, so that u has L^2 trace over $\partial\Omega$, where $\partial\Omega$ is assumed smooth. Therefore, by a limiting process and a standard approximation procedure, we have that for any $u \in H^1(\Omega)$

$$u(y) = \int_\Omega \nabla H(y-x) \cdot \nabla u(x)\, dx + \int_{\partial\Omega} u(x) \nabla H(y-x) \cdot \mathbf{n}\, d\sigma$$

for a.e. $y \in \Omega$.

Let now $\mathbf{v} \in V$ with $\mathbf{v} = (v_1, v_2, \dots, v_N)$. For each $k = 1, 2, \dots, N$ and for a.e. $y \in \Omega$

$$v_k(y) = \int_\Omega \nabla H(y-x) \cdot \nabla v_k(x)\, dx + \int_{\partial\Omega} v_k(x) \nabla H(y-x) \cdot \mathbf{n}\, d\sigma.$$

We rewrite

$$\begin{aligned}\int_\Omega \nabla H(y-x) \cdot \nabla v_k(x)\, dx &= \int_\Omega \sum_{i=1}^N \frac{\partial H(y-x)}{\partial x_i} \left(\frac{\partial v_k(x)}{\partial x_i} - \frac{\partial v_i(x)}{\partial x_k} \right) dx \\ &\quad + \int_\Omega \sum_{i=1}^N \frac{\partial H(y-x)}{\partial x_i} \frac{\partial v_i(x)}{\partial x_k}\, dx = \Sigma_1 + \Sigma_2.\end{aligned}$$

By (16.6)

$$\Sigma_1 = \int_\Omega \left(\nabla H(y-x) \wedge \operatorname{curl} \mathbf{v}(x)\right)_k\, dx = \int_\Omega (\nabla H(y-x) \wedge \boldsymbol{\omega}(x))_k\, dx,$$

and

$$\Sigma_2 = \int_{\partial\Omega} H(y-x) \frac{\partial}{\partial x_k} \sum_{i=1}^N (v_i\, n_i)(x)\, d\sigma - \int_\Omega H(y-x) \frac{\partial}{\partial x_k} \sum_{i=1}^N \frac{\partial v_i(x)}{\partial x_i}\, dx.$$

Since $\displaystyle\sum_{i=1}^N \frac{\partial v_i}{\partial x_i} = \operatorname{div} \mathbf{v} = 0$, we finally conclude that

$$\begin{aligned}\mathbf{v}(y) &= \int_\Omega \nabla H(y-x) \wedge \boldsymbol{\omega}(x)\, dx \\ &\quad + \int_{\partial\Omega} \mathbf{v}(x) \nabla H(y-x) \cdot \mathbf{n}\, d\sigma - \int_{\partial\Omega} H(y-x) \nabla \sum_{i=1}^N (v_i\, n_i)(x)\, d\sigma.\end{aligned}$$

The last two integrals represent a harmonic vector $\mathbf{A}(y)$ in Ω, since $y \neq x \in \partial\Omega$ in the classical sense. ■

In the following, mainly for the sake of notational simplicity, we make use of tensors. For an introduction to these objects, see for example Abraham et al. [4].

Definition 17.2. *Let* $\mathbf{k}$ *be a* N*-vector defined in* $\mathbb{R}^N \times \mathbb{R}$ *and* $\mathbf{g}$ *an* M*-tensor defined in* Ω_{t_1,t_2}*. Then the convolution* $\mathbf{k} * \mathbf{g}$ *is a* $(M-1)$*-tensor defined in* Ω_{t_1,t_2} *with components*

$$(\mathbf{k} * \mathbf{g})_{lm} = \iint_{\Omega_{t_1,t_2}} \sum_{i=1}^{N} k_i(x-\xi, t-\tau) g_{ilm}(\xi,\tau)\, d\xi d\tau.$$

Moreover, we let

$$\operatorname{div} \mathbf{g} \overset{def}{=} \sum_{i=1}^{N} \frac{\partial g_{ilm}}{\partial x_i}.$$

We have the following.

Proposition 17.1 *Assume that* $\mathbf{v}$ *is a local, weak solution of the Navier–Stokes equations in* E_T*,* $\mathbf{v} \in L^{2,2}(E_T;\mathbb{R}^N)$*,* $\boldsymbol{\omega} \in L^{2,2}(E_T)$*, and that* $\mathbf{f}$ *is conservative with* $\mathbf{f} \in L^{1,1}(E_T;\mathbb{R}^N)$*. Then in any* $\Omega_{t_1,t_2} \subset\subset E_T$ *we have*

$$\boldsymbol{\omega} = \nabla\Gamma * \mathbf{g} + \mathbf{B}, \tag{17.2}$$

where Γ *is the function of* (16.4)*,* $\mathbf{g} = (N-1)\,\boldsymbol{\omega} \wedge \mathbf{v}$*, and* $\mathbf{B} = \mathbf{B}(x,t)$ *is a solution of the heat equation* (16.3) *in* Ω_{t_1,t_2}*.*

Proof. We initially assume that $\mathbf{v}$ and $\boldsymbol{\omega}$ are both of class C^2, in order to easily perform some of the computations to follow.

First of all, it is a matter of straightforward calculations, to check that in our case

$$(\mathbf{v}\cdot\nabla)\mathbf{v} = \operatorname{div}(\mathbf{v}\otimes\mathbf{v}).$$

If we now denote by $\mathbf{v}_h = \mathbf{v}_h(x,t)$ an integral average of $\mathbf{v}$ over a ball in space-time of radius h centered at (x,t), it follows from (8.2) that there must exist a regular function p_h, such that

$$\partial_t \mathbf{v}_h - \nu\Delta\mathbf{v}_h = -\operatorname{div}(\mathbf{v}\otimes\mathbf{v})_h + \mathbf{f}_h - \nabla p_h. \tag{17.3}$$

If we take the curl of all the terms in the previous equation, and switch the derivation order, we obtain

$$\partial_t\boldsymbol{\omega}_h - \nu\Delta\boldsymbol{\omega}_h = -\operatorname{curl}\operatorname{div}(\mathbf{v}\otimes\mathbf{v})_h,$$

where we have taken into account that $\operatorname{curl}\mathbf{f}_h = 0$, since $\mathbf{f}$ is conservative. Again, it is a matter of straightforward computations to see that

$$-\operatorname{curl}\operatorname{div}(\mathbf{v}\otimes\mathbf{v})_h = \operatorname{div}((N-1)\operatorname{curl}\mathbf{v}\wedge\mathbf{v})_h = \operatorname{div}((N-1)\,\boldsymbol{\omega}\wedge\mathbf{v})_h = \operatorname{div}\mathbf{g}_h,$$

so that we can write

$$\partial_t\boldsymbol{\omega}_h - \nu\Delta\boldsymbol{\omega}_h = \operatorname{div}\mathbf{g}_h. \tag{17.4}$$

Now, let

$$\mathbf{B}_h \overset{\text{def}}{=} \boldsymbol{\omega}_h - \nabla\Gamma * \mathbf{g}_h;$$

we have

$$\begin{aligned}\partial_t\mathbf{B}_h &= \partial_t\boldsymbol{\omega}_h - \partial_t\left(\nabla\Gamma * \mathbf{g}_h\right) = \partial_t\boldsymbol{\omega}_h - \left(\nabla\partial_t\Gamma\right) * \mathbf{g}_h,\\ \nu\Delta\mathbf{B}_h &= \nu\Delta\boldsymbol{\omega}_h - \nu\Delta\left(\nabla\Gamma * \mathbf{g}_h\right) = \nu\Delta\boldsymbol{\omega}_h - \nu\left(\nabla\Delta\Gamma\right) * \mathbf{g}_h.\end{aligned}$$

Hence, since Γ is the fundamental equation of (16.3), owing to (17.4) we conclude that

$$\begin{aligned}\partial_t\mathbf{B}_h - \nu\Delta\mathbf{B}_h &= \partial_t\boldsymbol{\omega}_h - \nu\Delta\boldsymbol{\omega}_h - \nabla(\partial_t\Gamma - \nu\Delta\Gamma) * \mathbf{g}_h\\ &= \partial_t\boldsymbol{\omega}_h - \nu\Delta\boldsymbol{\omega}_h - (\partial_t\Gamma - \nu\Delta\Gamma) * \operatorname{div}\mathbf{g}_h\\ &= \partial_t\boldsymbol{\omega}_h - \nu\Delta\boldsymbol{\omega}_h - \operatorname{div}\mathbf{g}_h\\ &= 0.\end{aligned}$$

As $\mathbf{B}_h$ is in $L^{1,1}(E_T;\mathbb{R}^N)$ uniformly with respect to h, we can then pass to the limit as $h \to 0$ and conclude. If $\mathbf{v}$ and $\boldsymbol{\omega}$ are not in C^2, the previous computations can be concluded by standard limiting arguments. ■

18 Proof of Theorem 15.1 Concluded

As a consequence of Proposition 17.1, in any $\Omega_{t_1,t_2} \subset\subset E_T$, we can write

$$\boldsymbol{\omega} = \nabla\Gamma * \mathbf{g} + \mathbf{B},$$

where $\mathbf{g} = (N-1)\,\boldsymbol{\omega}\wedge\mathbf{v}$. Hence, $|\mathbf{g}| \le \gamma|\boldsymbol{\omega}||\mathbf{v}|$, where γ depends only on N.

We first prove that in any $\Omega_{t_1,t_2} \subset\subset E_T$ we have $\boldsymbol{\omega} \in L^\infty(\Omega_{t_1,t_2})$. If $\mathbf{v} \in L^{p,q}(E_T;\mathbb{R}^N)$ and $\boldsymbol{\omega} \in L^{r,s}(E_T)$, then, by Hölder's inequality, $\mathbf{g} \in L^{\rho,\sigma}(E_T)$, where ρ, $\sigma \ge 1$ are given by

$$\frac{1}{\rho} = \frac{1}{p} + \frac{1}{r}, \qquad \frac{1}{\sigma} = \frac{1}{q} + \frac{1}{s}.$$

We define the positive constant $\kappa \in (0,1)$ by

$$(N+3)\kappa = 1 - \left(\frac{N}{p} + \frac{2}{q}\right),$$

and also

$$\varrho = \frac{r}{1-\kappa r}, \qquad \varsigma = \frac{s}{1-\kappa s},$$

where $\varrho = \infty$ if $\kappa r \ge 1$, and analogously $\varsigma = \infty$ if $\kappa s \ge 1$. It is straightforward to check that

$$1 \le \rho \le \varrho, \quad 1 \le \sigma \le \varsigma, \quad N\left(\frac{1}{\rho} - \frac{1}{\varrho}\right) + 2\left(\frac{1}{\sigma} - \frac{1}{\varsigma}\right) = 1 - \kappa < 1.$$

By Lemma 16.1 and Proposition 17.1, we conclude that $\boldsymbol{\omega} \in L^{\varrho,\varsigma}(E_T)$, where ϱ, ς are larger than r, s, so that $\boldsymbol{\omega}$ actually enjoys a higher integrability with respect to what was originally assumed. The process can be repeated an arbitrary number of times, beginning with $r = s = 2$. After a finite number of steps, one has $\boldsymbol{\omega} \in L^{\varrho,\varsigma}(E_T)$ with $\varrho = \varsigma \geq \kappa^{-1}$; at the next step $\varrho = \varsigma = \infty$, and we have finished the first part of the proof.

By Theorem 17.1, we now have

$$\mathbf{v}(y,t) = \int_{\Omega} \nabla_x H(y-x) \wedge \boldsymbol{\omega}(x,t)\, dx + \boldsymbol{A}(y,t), \tag{18.1}$$

where $\mathbf{v} \in L^{2,\infty}(\Omega_{t_1,t_2}; \mathbb{R}^N)$, and we have just proven that $\boldsymbol{\omega} \in L^{\infty}(\Omega_{t_1,t_2})$. Hence, the function $\boldsymbol{A} = \boldsymbol{A}(x,t)$ must be bounded on compact subsets of Ω, both as a function of x and of t, and consequently, $\mathbf{v} \in L^{\infty}(\Omega_{t_1,t_2}; \mathbb{R}^N)$. By the usual potential theoretic estimates for heat kernel convolutions (see for example Watson [271]), $\boldsymbol{\omega}$ is Hölder continuous with respect to the space variables in any compact subregion of E_T, with arbitrary exponent $\alpha \in (0,1)$. By the Hölder continuity of $\boldsymbol{\omega}$ and (18.1), we have that $\mathbf{v}$ is also Hölder continuous.

This yields that $\mathbf{g}$ is Hölder continuous, and by the same potential theoretic estimates for the heat kernel convolution we have just relied upon, we have that $\nabla\boldsymbol{\omega}$ is Hölder continuous. From here on, we can bootstrap, and conclude that $\mathbf{v} \in C^{\infty}$ with respect to the space variables.

Up to now, we have not yet used that $\mathbf{v}_t \in L^{2,s}$ with $s \geq 1$. It is rather straightforward to show that (17.2) implies

$$\partial_t \boldsymbol{\omega} - \nu \Delta \boldsymbol{\omega} = \operatorname{div} \mathbf{g}. \tag{18.2}$$

In turn this yields that $\partial_t \boldsymbol{\omega}$ is of class C^{∞} in the space variables, and its derivatives are bounded on compact subsets of E_T. On the other hand, if we differentiate (18.1) with respect to time, we have

$$\mathbf{v}_t(y,t) = \int_{\Omega} \nabla_x H(y-x) \wedge \boldsymbol{\omega}_t(x,t)\, dx + \boldsymbol{A}_t(y,t).$$

Thus, $\mathbf{v}_t$ is of class C^{∞} in the space variables, and each derivative is of class L^s in time. Finally, we recover that equation (15.1) holds almost everywhere in E_T, by letting $h \to 0$ in (17.3). ■

19 Regularity of the Initial-Boundary Value Problem

As we have already discussed in Section 15, solutions of the Navier–Stokes equations behave globally with respect to time, as they are instantaneously determined by the boundary conditions, but they are somehow purely local as far as the space variables are concerned. This suggests that one can hope to gain time regularity from the assigned initial-boundary value problem. We will not go into detail here, and we limit to state a result, whose proof can be found in Galdi [93], § 5.

Theorem 19.1. *Let* $\mathbf{v}$ *be a weak solution in* E_T *of the initial-boundary value problem* (8.1) *with* $\mathbf{f} \equiv 0$ *and* $\mathbf{v}_o \in H$. *Assume that* $\mathbf{v}$ *satisfies at least one of the following two conditions:*

(i) $\mathbf{v} \in L^{p,q}\left(E_T; \mathbb{R}^N\right)$, *for some* p, q *such that* $\frac{N}{p} + \frac{2}{q} = 1, \quad p \in (N, \infty]$;
(ii) $\mathbf{v} \in C^0\left([0,T]; L^N(E)\right)$.

Then, if E *is uniformly of class* C^∞, *we have* $\mathbf{v} \in C^\infty(\bar{E} \times (0,T])$.

Remark 19.1 For $E = \mathbb{R}^3$, Theorem 19.1 was first proved by Leray [165], pp. 224–227, whereas for $E = \mathbb{R}^N$ with $N \geq 2$, and $p < \infty$ it is attributed to Fabes et al. [70]. Sohr proved Theorem 19.1(i) with $p < \infty$, for domains with a bounded boundary [243]. That condition (ii) implies that regularity was first discovered by von Wahl [270], in the case of a bounded domain. This latter result was extended to domains with a bounded boundary by Giga [99].

20 Recovering the Pressure in the Time-Dependent Equations

In Section 12 we have shown how a moderate degree of integrability of ∇p yields a higher regularity on $\mathbf{v}$. We return to this issue and discuss the regularity of p when considering weak solutions of (8.1) for $N = 3$. Instead of dealing with a general domain E_T, just for simplicity we work with $B_1 \times (-1, 0)$. Moreover, we take $\nu = 1$. We prove the following.

Proposition 20.1 (Sohr and von Wahl [245]) *Let* $\mathbf{v}_o \in L^2\left(B_1; \mathbb{R}^3\right)$ *be weakly divergence-free in* B_1 *and* $\mathbf{f} \equiv 0$. *If* $\mathbf{v}$ *is the corresponding weak solution of* (8.1) *in* $B_1 \times (-1, 0)$, *then* $p \in L^{\frac{5}{3}}\left(-1, 0; L^{\frac{5}{3}}(B_1)\right)$.

Proof. If we rely on (12.1) written over B_1 with $p = \frac{15}{14}$ and $pq = 2$, we have

$$\begin{aligned} \|(\mathbf{v}\cdot\nabla)\mathbf{v}\|^{\frac{5}{3}}_{\frac{15}{14};B_1} &\leq \left(\int_{B_1} |\nabla \mathbf{v}|^2 \, dx\right)^{\frac{5}{6}} \left(\int_{B_1} |\mathbf{v}|^{\frac{30}{13}} \, dx\right)^{\frac{13}{18}} \\ &\leq C\left[\|\nabla \mathbf{v}\|^2_{2;B_1} + \|\mathbf{v}\|^{10}_{\frac{30}{13};B_1}\right]. \end{aligned} \tag{20.1}$$

Now we rely on the following interpolation inequality, which can be proven, for example, relying on Proposition 18.1 and Theorem 19.1 of Chapter IX of DiBenedetto [50].

Lemma 20.1 *Let* $r > 0$. *For* $v \in W^{1,2}(B_r)$ *we have*

$$\int_{B_r} |\mathbf{v}|^q \, dx \leq C \left[\int_{B_r} |\nabla \mathbf{v}|^2 \, dx\right]^a \left[\int_{B_r} |\mathbf{v}|^2 \, dx\right]^{\frac{q}{2}-a} + \frac{C}{r^{2a}} \left[\int_{B_r} |\mathbf{v}|^2 \, dx\right]^{\frac{q}{2}}$$

for all $q \in [2, 6]$, $a = \frac{3(q-2)}{4}$.

If we choose $q = \frac{30}{13}$ and $a = \frac{3}{13}$ in Lemma 20.1, we obtain

$$\|\mathbf{v}\|_{\frac{30}{13};B_1} \leq C \|\nabla \mathbf{v}\|_{2;B_1}^{\frac{1}{5}} \|\mathbf{v}\|_{2;B_1}^{\frac{4}{5}} + C \|\mathbf{v}\|_{2;B_1}$$

and also,

$$\|\mathbf{v}\|_{\frac{30}{13};B_1}^{10} \leq C \|\nabla \mathbf{v}\|_{2;B_1}^{2} \|\mathbf{v}\|_{2;B_1}^{8} + C \|\mathbf{v}\|_{2;B_1}^{10}. \tag{20.2}$$

If we take both (20.1) and (20.2) into account, we conclude that

$$\|(\mathbf{v} \cdot \nabla)\mathbf{v}\|_{\frac{15}{14};B_1}^{\frac{5}{3}} \leq C \left[\|\nabla \mathbf{v}\|_{2;B_1}^{2} + \|\nabla \mathbf{v}\|_{2;B_1}^{2} \|\mathbf{v}\|_{2;B_1}^{8} + \|\mathbf{v}\|_{2;B_1}^{10}\right],$$

and integrating with respect to time over $(-1, 0)$ yields

$$\begin{aligned} &\int_{-1}^{0} \left(\int_{B_1} |(\mathbf{v} \cdot \nabla)\mathbf{v}|^{\frac{15}{14}} \, dx\right)^{\frac{14}{15} \cdot \frac{5}{3}} dt \\ &\leq C \int_{-1}^{0} \int_{B_1} |\nabla \mathbf{v}|^2 \, dx dt \\ &\quad + C \int_{-1}^{0} \left(\int_{B_1} |\nabla \mathbf{v}|^2 \, dx\right) \left(\int_{B_1} |\mathbf{v}|^2 \, dx\right)^4 dt \\ &\quad + C \int_{-1}^{0} \left(\int_{B_1} |\mathbf{v}|^2 \, dx\right)^5 dt, \end{aligned}$$

where all the terms on the right-hand side are bounded, since $\mathbf{v} \in W$. Hence, we conclude that

$$(\mathbf{v} \cdot \nabla)\mathbf{v} \in L^{\frac{5}{3}} \left(-1, 0; L^{\frac{15}{14}}(B_1)\right). \tag{20.3}$$

Take $\boldsymbol{\varphi} \in \mathcal{V}$, where in (8.2) we assume $E = B_1$. If we use such a $\boldsymbol{\varphi}$ in the weak formulation of Navier–Stokes equations, we obtain

$$\begin{aligned} \left(\frac{\partial \mathbf{v}}{\partial t}, \boldsymbol{\varphi}\right) &= -(\nabla \mathbf{v}, \nabla \boldsymbol{\varphi}) - ((\mathbf{v} \cdot \nabla)\mathbf{v}, \boldsymbol{\varphi}) \\ \left|\left(\frac{\partial \mathbf{v}}{\partial t}, \boldsymbol{\varphi}\right)\right| &= |-(\nabla \mathbf{v}, \nabla \boldsymbol{\varphi}) - ((\mathbf{v} \cdot \nabla)\mathbf{v}, \boldsymbol{\varphi})| \\ &\leq \|\nabla \mathbf{v}(\cdot, t)\|_{2;B_1} \|\nabla \boldsymbol{\varphi}\|_{2;B_1} + \|\mathbf{v}(\cdot, t)\|_{2;B_1} \|\nabla \mathbf{v}(\cdot, t)\|_{2;B_1} \|\boldsymbol{\varphi}\|_{2;B_1} \\ &\leq \left[\|\nabla \mathbf{v}(\cdot, t)\|_{2;B_1} + \|\mathbf{v}(\cdot, t)\|_{2;B_1} \|\nabla \mathbf{v}(\cdot, t)\|_{2;B_1}\right] \|\boldsymbol{\varphi}\|_{W^{2,2}(B_1)}. \end{aligned}$$

Hence,

$$\frac{\partial \mathbf{v}}{\partial t} - \Delta \mathbf{v} \in L^2(-1, 0; Z)$$

where Z is the dual space of $W_0^{2,2}(B_1)$. We define

$$\mathbf{g} = \frac{\partial \mathbf{v}}{\partial t} - \Delta \mathbf{v},$$

and notice that for almost every $t \in (-1, 0)$,

$$\operatorname{div} \mathbf{g} = \frac{\partial}{\partial t}(\operatorname{div} \mathbf{v}) - \Delta(\operatorname{div} \mathbf{v}) = 0, \quad \operatorname{curl} \mathbf{g} = \operatorname{curl}((\mathbf{v} \cdot \nabla)\mathbf{v}) \quad \text{in } B_1.$$

Then, by the elliptic estimates of Morrey [185], Chapter 7,

$$\|\mathbf{g}\|_{\frac{15}{14}, B_1}^{\frac{5}{3}} \le C \left[\|(\mathbf{v} \cdot \nabla)\mathbf{v}\|_{\frac{15}{14}, B_1}^{\frac{5}{3}} + \|\mathbf{g}\|_{Z}^{\frac{5}{3}} \right].$$

Therefore, integrating we have

$$\frac{\partial \mathbf{v}}{\partial t} - \Delta \mathbf{v} \in L^{\frac{5}{3}}\left(-1, 0; L^{\frac{15}{14}}(B_1)\right); \tag{20.4}$$

(20.3)–(20.4) imply that

$$\nabla p \in L^{\frac{5}{3}}\left(-1, 0; L^{\frac{15}{14}}(B_1)\right)$$

and by the Sobolev embedding theorem, we conclude that

$$p \in L^{\frac{5}{3}}\left(-1, 0; L^{\frac{5}{3}}(B_1)\right).$$

Remark 20.1 The proof of Proposition 20.1 is taken from Lin [173].

Problems and Complements

1c Navier–Stokes Equations in Dimensionless Form

A fluid is viscous if its infinitesimal particles at x at time t, moving with velocity $\mathbf{v}(x, t)$, encounter a nonzero resistance $\mathbf{R} = -f(|\mathbf{v}|)\mathbf{v}$, where f is a smooth, non-negative function whose form is determined from experiments. For sufficiently slow motions $f(|\mathbf{v}|) = \text{const}$ (in the air $|\mathbf{v}| \le 2$ m/sec). In such a case the motion is said to be in *viscous regime*. For an ideal fluid particle assimilated to a ball of sufficiently small radius r

$$f(|\mathbf{v}|) = 6\pi\mu r \quad \text{for} \quad |\mathbf{v}| \ll 1 \quad \text{(viscous regime)}$$

where μ is the *dynamic viscosity*. This form of $f(|\mathbf{v}|)$ implies that μ has dimensions $\rho[V][L]$, where ρ is the density of the fluid. The *dynamic* viscosity is a measure of a resistance offered by a fluid when forced to change its shape. It is a sort of internal friction measured as the resistance elicited by two ideal parallel planes, immersed in the fluid, when forced into a mutual sliding motion. The unit of measure is the *poise*, after J.L.M. Poiseuille. It is measured in $dyne \cdot s/cm^2$ and is the force distributed tangentially on a planar surface of 1 cm^2, needed to cause a variation of velocity of 1 cm/sec between two ideal parallel planes immersed in the fluid and separated by a distance of 1 cm. For water at 20^oC, the dynamic viscosity is 0.01002 poise. The *kinematic* viscosity is the ratio of the dynamic viscosity to the density of the fluid. The *c.g.s.* unit of kinematic viscosity is the *stoke*, after G. G. Stokes.

For greater speeds, $f(|\mathbf{v}|)$ is proportional to $|\mathbf{v}|$ and the motion is said to be in *hydraulic regime* (in the air 2 m/sec< $|\mathbf{v}|$ ≤200 m/sec). For an ideal fluid particle penetrating the fluid and assimilated to a ball of sufficiently small radius r

$$f(|\mathbf{v}|) = 5\pi\mu r^2 |\mathbf{v}| \qquad \text{(hydraulic regime).}$$

4c Nonhomogeneous Boundary Data

Since E is bounded, by the embedding inequalities (2.5)–(2.6), the norm $\|\cdot\|_V$ is equivalent to $\|\nabla\cdot\|_2$. Thus, we regard V as a Hilbert space by the inner product $\langle\cdot,\cdot\rangle = (\nabla\cdot, \nabla\cdot)$. For a fixed pair $(\mathbf{u},\mathbf{w}) \in V$, let $\mathbf{T}(\mathbf{w},\mathbf{u})$ be the linear bounded functional in V defined by

$$\begin{aligned}\langle T(\mathbf{w},\mathbf{u}), \boldsymbol{\varphi}\rangle = \nu \int_E \nabla\mathbf{u} : \nabla\boldsymbol{\varphi}\, dx \\ - \int_E \{\mathbf{w}\cdot(\mathbf{u}\cdot\nabla) + \mathbf{u}\cdot(\mathbf{b}\cdot\nabla) + \mathbf{b}\cdot(\mathbf{u}\cdot\nabla)\}\,\boldsymbol{\varphi}\, dx.\end{aligned}$$

With $\mathbf{g}$ given by (4.4) consider also formally, the functional equation

$$\mathbf{T}(\mathbf{w},\mathbf{u}) = \mathbf{g} \qquad \text{in } V^*. \tag{4.1c}$$

Then a weak solution of (4.1) is an element $\mathbf{u} \in V$ such that $\mathbf{T}(\mathbf{u},\mathbf{u}) = \mathbf{g}$.

Proposition 4.1c *Let the assumptions on* $\mathbf{f}$ *and* $\mathbf{a}$ *be in force so that in particular (4.9) holds. Then for all* $\mathbf{w},\mathbf{u} \in V$ *with* $\|\mathbf{u}\|_V > 0$

$$\|\mathbf{T}(\mathbf{w},\mathbf{u})\| \ge \frac{1}{2}\nu. \tag{4.2c}$$

Moreover, for any fixed $\mathbf{w}\in V$*, any solution* $\mathbf{u}\in V$ *of (4.1c) satisfies*

$$\|\nabla\mathbf{u}\|_2 \le \frac{2\gamma}{\nu}\left[\|\mathbf{f}\|_{\frac{6}{5}} + \nu\|\nabla\mathbf{b}\|_2 + \|\mathbf{b}\|_4^2\right], \tag{4.3c}$$

where γ *is the constant of the embedding of* V *into* $L^6(E;\mathbb{R}^3)$*.*

Remark 4.1c These estimates are independent of $\mathbf{w}$. Thus, in particular, they hold for solutions of $\mathbf{T}(\mathbf{u},\mathbf{u})=\mathbf{g}$.

Proof.

$$\|\mathbf{T}(\mathbf{u},\mathbf{w})\| = \sup_{\|\boldsymbol{\varphi}\|=1}\langle\mathbf{T}(\mathbf{u},\mathbf{w}),\boldsymbol{\varphi}\rangle \ge \frac{\langle\mathbf{T}(\mathbf{u},\mathbf{u}),\mathbf{u}\rangle}{\|\mathbf{u}\|_V}. \qquad \blacksquare$$

4.1c Solving (4.1) by Galerkin Approximations

The space V is a separable Hilbert space by the inner product $(\nabla\cdot,\nabla\cdot)$ and hence it admits a countable base $(\mathbf{e}_1,\dots,\mathbf{e}_n,\dots)$, orthonormal in $(\nabla\cdot,\nabla\cdot)$. Setting $V_n=\operatorname{span}\{\mathbf{e}_1,\dots,\mathbf{e}_n\}$, every $\mathbf{w}\in V$ can be written as

$$\mathbf{w}=\mathbf{w}_n+\sum_{j>n} w_j\mathbf{e}_j \qquad \text{where} \qquad \mathbf{w}_n=\sum_{j=1}^{n} w_j\mathbf{e}_j\in V_n \tag{4.4c}$$

for scalar w_j. If $\mathbf{u}\in V$ is a solution of (4.1) in the sense of (4.4)–(4.5), the latter holds for $\boldsymbol{\varphi}=\mathbf{e}_i$. In the resulting expression write $\mathbf{u}$ in the form (4.4c), and notice that the terms involving $\sum_{j>n}u_j\mathbf{e}_j$ tend to zero as $n\to\infty$. This suggests defining an approximate solution of (4.1) a function $\mathbf{u}_n\in V_n$, satisfying (4.5) for $\boldsymbol{\varphi}=\mathbf{e}_i$, for all $i=1,\dots,n$, i.e.,

$$\begin{aligned}\sum_{j=1}^{n}\Big\{\nu\int_E\nabla\mathbf{e}_j:\nabla\mathbf{e}_i dx+\int_E\mathbf{e}_i\cdot(\mathbf{u}_n\cdot\nabla)\mathbf{e}_j dx\\ +\int_E\mathbf{e}_i\cdot(\mathbf{b}\cdot\nabla)\mathbf{e}_j dx+\int_E\mathbf{e}_i\cdot(\mathbf{e}_j\cdot\nabla)\mathbf{b}dx\Big\}_{ij}u_j=\int_E\mathbf{g}\cdot\mathbf{e}_i dx.\end{aligned}$$

The elements $\{\cdots\}_{ij}$ are the entries of a $n\times n$ matrix $\big(T_{ij}(\mathbf{u}_n)\big)$. The right-hand side defines a vector $(g_1,\dots,g_n)\in\mathbb{R}^n$, identified with $\mathbf{g}_n\in V_n$. More generally, for $\mathbf{w}_n\in V_n$ define $T_{ij}(\mathbf{w}_n)$ as $T_{ij}(\mathbf{u}_n)$, with $\mathbf{w}_n$ replacing $\mathbf{u}_n$, and seek solutions $(u_1,\dots,u_n)\in\mathbb{R}^n$ of

$$T_{ij}(\mathbf{w_n})u_j=g_i \quad \text{for } i=1,\dots,n. \tag{4.5c}$$

The corresponding $\mathbf{u}_n\in V_n$ is a solution of $\mathbf{T}(\mathbf{w}_n,\mathbf{u}_n)=\mathbf{g}_n$. The Galerkin approximations of (4.1) is function $\mathbf{u}_n\in V_n$ satisfying

$$\langle\mathbf{T}(\mathbf{u_n},\mathbf{u_n}),\boldsymbol{\varphi}_n\rangle=\langle\mathbf{g}_n,\boldsymbol{\varphi}_n\rangle \quad \text{for all } \boldsymbol{\varphi}_n\in V_n. \tag{4.6c}$$

Proposition 4.2c

(i). *For all n there exists a Galerkin approximation $\mathbf{u}_n$ to (4.1).*
(ii). *A sequence $\{\mathbf{u}_n\}$ of Galerkin approximations is equibounded in V.*
(iii). *Any $\mathbf{u}$ in the weak closure of $\{\mathbf{u}_n\}$ is a solution of (4.1).*

Prove the proposition by the following steps:

- **Step 1.** Use (4.2c) to prove that $\det\big(T_{ij}(\mathbf{w}_n)\big) \ge \frac{1}{2}\nu$, for all $\mathbf{w}_n \in V_n$. Therefore, for all $\mathbf{g}_n \in V_n$ there exists a unique $\mathbf{u}_n \in V_n$ satisfying (4.5c).
- **Step 2.** Introduce the map $\mathbf{B}(\mathbf{w}_n) = \mathbf{u}_n$ from $\mathbb{R}^n$ into itself. Prove that such a map and its inverse $\mathbf{B}^{-1}$ are well defined and continuous in $\mathbb{R}^n$.
- **Step 3.** Use (4.3c) to prove that map $\mathbf{B}^{-1}$ maps the ball of radius $2\gamma\left[\|\mathbf{f}\|_{\frac{6}{5}} + \nu\|\nabla\mathbf{b}\|_2 + \|\mathbf{b}\|_4^2\right]/\nu$ into itself.
- **Step 4.** Therefore, $\mathbf{B}(\cdot)$ has a fixed point by the Brouwer fixed point theorem [21]. Any such fixed point, identified with an element $\mathbf{u}_n \in V_n$, solves (4.6c).
- **Step 5.** Use (4.3c) to prove that $\|\nabla\mathbf{u}_n\|_2 \le 2\gamma\left[\|\mathbf{f}\|_{\frac{6}{5}} + \nu\|\nabla\mathbf{b}\|_2 + \|\mathbf{b}\|_4^2\right]/\nu$ for all $n \in \mathbb{N}$. Therefore, the embedding $\{\mathbf{u}_n\} \subset L^p(E;\mathbb{R}^3)$ is compact for all $1 < p < 6$.
- **Step 6.** Having fixed $\mathbf{u}$ in the weak closure of $\{\mathbf{u}_n\}$, a subsequence can be selected and relabeled with n, such that $\{\nabla\mathbf{u}_n\} \to \nabla\mathbf{u}$ weakly in $L^2(E;\mathbb{R}^3)$ and $\{\mathbf{u}_n\} \to \mathbf{u}$ strongly in $L^4(E;\mathbb{R}^3)$.
- **Step 7.** Let $n \to \infty$ in (4.6c), justifying the limits of each term, to establish the existence of a solution of (4.1) in the form (4.5).

4.2c Extending Fields $\mathbf{a} \in W^{\frac{1}{2},2}(\partial E;\mathbb{R}^3)$, Satisfying (4.2) into Solenoidal Fields $\mathbf{b} \in W^{1,2}(E;\mathbb{R}^3)$

We prove the following result.

Proposition 4.3c *Let E be a bounded, simply connected, open set in $\mathbb{R}^N$ ($N = 2, 3$) with boundary ∂E of class C^1 having one connected component, and satisfying the segment property. For every vector field $\mathbf{a} \in W^{\frac{1}{2},2}(\partial E;\mathbb{R}^N)$ satisfying*

$$\int_{\partial E} \mathbf{a}\cdot\mathbf{n}\, d\sigma = 0,$$

where $\mathbf{n}$ is the outward unit normal to ∂E, there exists a vector field $\boldsymbol{\psi} \in W^{2,2}(E;\mathbb{R}^N)$ such that $\mathbf{b} = \operatorname{curl}\boldsymbol{\psi}$ is an extension of $\mathbf{a}$ into E. The function $\boldsymbol{\psi}$ can be chosen to be compactly supported about ∂E. Furthermore, for every fixed $\epsilon > 0$ the vector field $\boldsymbol{\psi}$ can be chosen so that for every $\mathbf{u} \in V$

$$\||\mathbf{u}|\,|\operatorname{curl}\boldsymbol{\psi}|\|_2 \le \chi(\epsilon)\|\nabla\mathbf{u}\|_2 \quad \text{in } E, \tag{4.7c}$$

where $\chi(\epsilon) \to 0$ as $\epsilon \to 0$. Finally, if $\mathbf{a} \in C^k(\partial E;\mathbb{R}^N)$, for some $k = 1, \dots,$ and ∂E is of class C^{k+1}, then $\boldsymbol{\psi}$ can be taken of class $C^{k+1}(E;\mathbb{R}^N)$.

We need some preliminary Lemmas. The first and the second ones are taken from Galdi [91], Chapter III, Section 6. In the last one, we follow the approach developed in Ladyzhenskaya [149], Chapter 1, Section 2 and in Finn [73], Lemma 2.1; see also Galdi [92], Chapter VIII, Section 4.

Lemma 4.1c *Let E be a bounded, open set in $\mathbb{R}^N$ and let*

$$\delta(x) = \operatorname{dist}(x, \partial E).$$

For any $\epsilon > 0$ define $\gamma(\epsilon) = \exp\left(-\frac{1}{\epsilon}\right)$. Then, there exists a function $\varphi_\epsilon \in C^\infty(\overline{E})$ such that

- $|\varphi_\epsilon(x)| \le 1$ *for all $x \in E$,*
- $\varphi_\epsilon(x) = 1$ *if $\delta(x) < \gamma^2(\epsilon)/(2\kappa_1)$,*
- $\varphi_\epsilon(x) = 0$ *if $\delta(x) \ge 2\gamma(\epsilon)$,*
- $|\nabla\varphi_\epsilon(x)| \le \kappa_2\epsilon/\delta(x)$ *for all $x \in E$,*

where κ_1, κ_2 depend only on N.

Proof. We first recall the following result, for whose proof we refer to Stein [246], Chapter VI, Theorem 2.

There exists a function $\rho \in C^\infty(E)$ such that for all $x \in E$

1. $\delta(x) \le \rho(x)$;
2. for any partial derivative of order α, $|\alpha| \ge 0$, we have

$$|D^\alpha \rho(x)| \le \kappa_{|\alpha|+1}[\delta(x)]^{1-|\alpha|},$$

where all $\kappa_{|\alpha|+1}$ depend only on α and N.

Now consider the function $\xi_\epsilon : \mathbb{R} \to \mathbb{R}$ defined by

$$\xi_\epsilon(t) = \begin{cases} 1 & \text{if } t < \gamma^2(\epsilon), \\ \epsilon \ln\left(\frac{\gamma(\epsilon)}{t}\right) & \text{if } \gamma^2(\epsilon) < t < \gamma(\epsilon), \\ 0 & \text{if } t > \gamma(\epsilon). \end{cases}$$

Now, choose $\eta = \gamma^2(\epsilon)/2$, a mollifier j_η, and consider the mollified function $\Xi_\epsilon \equiv \xi_\epsilon * j_\eta$. It is not hard to check that

- $\Xi_\epsilon(t) = 1$ for $t < \gamma^2(\epsilon)/2$,
- $\Xi_\epsilon(t) = 0$ for $t > 2\gamma(\epsilon)$,
- $|\Xi_\epsilon(t)| \le 1$ for all $t \in \mathbb{R}$,
- $|\Xi'_\epsilon(t)| \le \epsilon/t$ for all $t \in \mathbb{R}$.

We now let $\varphi_\epsilon(x) = \Xi_\epsilon(\rho(x))$; taking into account 1. and 2. above and the bound on $|\Xi'_\epsilon|$, we conclude that

$$\begin{aligned} \varphi_\epsilon(x) &= 1 \quad \text{if } \delta(x) < \gamma^2(\epsilon)/2\kappa_1, \\ \varphi_\epsilon(x) &= 0 \quad \text{if } \delta(x) > 2\gamma(\epsilon), \\ |\nabla\varphi_\epsilon(x)| \le \kappa_2\epsilon/\rho(x) \le \kappa_2\epsilon/\delta(x) & \quad \text{for all } x \in E, \end{aligned}$$

for proper, positive κ_1 and κ_2, which depend only on N. ■

Lemma 4.2c *Let $E \subset \mathbb{R}^N$ be a bounded, Lipschitz, open set. Then, there exists $c = c(E)$ such that for all $u \in W_o^{1,2}(E)$ we have*

$$\|\frac{u}{\delta}\|_2 \le c\,\|\nabla u\|_2,$$

where $\delta = \delta(x)$ is the function, which has been defined above.

Proof. By density it suffices to assume $u \in C_o^\infty(E)$. By Corollary 3.1 of Chapter 9, for every open set $E' \subset\subset E$, we have

$$\|u\|_{2;E'} \le c_1\,\|\nabla u\|_{2;E},$$

where $c_1 = c_1(N,E)$. In order to conclude, we have to take into account the behavior close to the boundary ∂E. We recall that a bounded domain $E \subset \mathbb{R}^N$ is said to be a Lipschitz domain, if there exists a radius r_o, such that for each $y \in \partial E$, in an appropriate coordinate system,

$$\begin{aligned} E \cap B_{8r_o}(y) &= \{x = (x', x_{\scriptscriptstyle N}) \in \mathbb{R}^N : x_{\scriptscriptstyle N} > \varPhi(x')\} \cap B_{8r_o}(y), \\ \partial E \cap B_{8r_o}(y) &= \{x = (x', x_{\scriptscriptstyle N}) \in \mathbb{R}^N : x_{\scriptscriptstyle N} = \varPhi(x')\} \cap B_{8r_o}(y), \end{aligned}$$

where $\varPhi$ is a Lipschitz function, with $\|\nabla\varPhi\|_{L^\infty} \le L$. The quantities r_o and L are independent of $y \in \partial E$. We say that L is the *Lipschitz constant* of E.

If we set

$$G(y) = E \cap B_{8r_o}(y),$$

it is not hard to check that $\exists c_2 = c_2(E)$, such that

$$\forall x \in G(y) \quad x_{\scriptscriptstyle N} - \varPhi(x') \le c_2\,\delta(x).$$

Therefore, if we let $y = (y', y_{\scriptscriptstyle N})$, and $B'_{r_o}(y')$ denotes the $(N-1)$-dimensional ball centered at y', we have

$$\int_{G(y)} \frac{1}{\delta^2(x)} |u(x)|^2\,dx \le c_2 \int_{B'_{8r_o}(y')} dx' \int_{\varPhi(x')}^{\varPhi(x')+16r_o} \frac{|u(x', x_{\scriptscriptstyle N})|^2}{|x_{\scriptscriptstyle N} - \varPhi(x')|^2}\,dx_{\scriptscriptstyle N},$$

and the wanted estimate follows from the one-dimensional inequality

$$\int_0^\infty \frac{|h(t)|^q}{t^q}\,dt \le \frac{q}{q-1} \int_0^\infty \left|\frac{dh}{dt}\right|^q dt,$$

which holds for any $h \in C_o^\infty(\mathbb{R}_+)$ and for any $q > 1$, and which can be easily proved integrating the identity

$$\frac{|h(t)|^q}{t^q} = \frac{d}{dt}\left[\frac{t^{1-q}}{1-q}|h(t)|^q\right] - \frac{t^{1-q}}{1-q}\frac{d}{dt}|h(t)|^q.$$

■

Lemma 4.3c *Let E be a bounded, simply connected, open set in $\mathbb{R}^N$ ($N = 2, 3$) with boundary ∂E of class C^1, having one connected component, and satisfying the segment property. For every vector field $\mathbf{a} \in W^{\frac{1}{2},2}(\partial E; \mathbb{R}^N)$ satisfying*

$$\int_{\partial E} \mathbf{a} \cdot \mathbf{n} \, d\sigma = 0, \tag{4.8c}$$

where $\mathbf{n}$ is the outward unit normal to ∂E, there exists a vector field $\mathbf{w} \in W^{2,2}(E; \mathbb{R}^N)$ such that $\mathbf{a} = \operatorname{curl} \mathbf{w}$ in the sense of traces on ∂E. Moreover,

$$\|\mathbf{w}\|_{W^{2,2}(E)} \le c \, \|\mathbf{a}\|_{W^{1/2,2}(\partial E)}, \tag{4.9c}$$

where c depends on N and E.

Proof. For the moment we assume ∂E smooth, without further specification, to the extent that all the necessary operations can be performed. At the end we briefly discuss how conditions can be relaxed in a way that all the needed estimates are still justified.

First we consider the case $N = 3$; later on we briefly deal with $N = 2$, which is considerably simpler. Let $\mathbf{n}$ be the outward unit normal to ∂E and rewrite $\mathbf{a}$ as

$$\mathbf{a} = \mathbf{a}_\tau + a_n \mathbf{n},$$

where $a_n = \mathbf{a} \cdot \mathbf{n}$ and $\mathbf{a}_\tau$ is the component of $\mathbf{a}$ tangential to ∂E.

We first look for a *solenoidal* vector field $\mathbf{b}_1 : E \to \mathbb{R}^3$, $\mathbf{b}_1 \in W^{1,2}(E; \mathbb{R}^3)$, such that

$$\begin{cases} \mathbf{b}_1 = \nabla \varphi & \text{in } E, \\ \mathbf{b}_1 \cdot \mathbf{n} = a_n & \text{on } \partial E. \end{cases}$$

Since $\operatorname{div} \mathbf{b}_1 = 0$, this implies that φ is a solution of

$$\begin{cases} \Delta \varphi = 0 & \text{in } E, \\ \dfrac{\partial \varphi}{\partial n} = a_n & \text{on } \partial E. \end{cases}$$

This is a Neumann problem for the Laplacean in E, and condition (4.8c) ensures that a solution $\varphi \in W^{2,2}(E)$ exists, up to an arbitrary constant. Hence, $\mathbf{b}_1 \in W^{1,2}(E; \mathbb{R}^3)$ is well defined. Moreover, since E is simply connected, by well-known results, there exists $\mathbf{w}_1 \in W^{2,2}(E; \mathbb{R}^3)$ such that

$$\mathbf{b}_1 = \operatorname{curl} \mathbf{w}_1.$$

If we now let

$$\mathbf{b} \overset{\text{def}}{=} \mathbf{b}_1 + \mathbf{b}_2,$$

the vector $\mathbf{b}$ is completely determined, if $\mathbf{b}_2$ solves

$$\begin{cases} \operatorname{div} \mathbf{b}_2 = 0 & \text{in } E, \\ \mathbf{b}_2 = \mathbf{a} - \mathbf{b}_1 & \text{on } \partial E, \end{cases} \tag{4.10c}$$

taking into account that, by construction,

$$(\mathbf{a}-\mathbf{b}_1)\cdot\mathbf{n}=0 \quad \text{on} \quad \partial E. \tag{4.11c}$$

In order to explain the main ideas underlying the construction of $\mathbf{b}_2$, we first consider the simple situation of

$$E=\{(x_1,x_2,x_3)\in\mathbb{R}^3:\ x_3>0\},\ \partial E=\{(x_1,x_2,x_3)\in\mathbb{R}^3:\ x_3=0\},$$

before dealing with general E and ∂E. At this step E is not bounded, but it is immaterial for what we are going to do. We have

$$\begin{aligned}\mathbf{a}=&a_1(x_1,x_2)\mathbf{e}_1+a_2(x_1,x_2)\mathbf{e}_2+a_3(x_1,x_2)\mathbf{e}_3,\\ \mathbf{b}_1=&b_{1,1}(x_1,x_2,x_3)\mathbf{e}_1+b_{1,2}(x_1,x_2,x_3)\mathbf{e}_2+b_{1,3}(x_1,x_2,x_3)\mathbf{e}_3,\end{aligned}$$

where $b_{1,3}(x_1,x_2,0)=a_3(x_1,x_2)$. By (4.11c), we have

$$\mathbf{a}-\mathbf{b}_1\big|_{\partial E}=(a_1(x_1,x_2)-b_{1,1}(x_1,x_2,0))\mathbf{e}_1+(a_2(x_1,x_2)-b_{1,2}(x_1,x_2,0))\mathbf{e}_2.$$

If we let

$$h_i(x_1,x_2)=a_i(x_1,x_2)-b_{1,i}(x_1,x_2) \ \text{ for } \ i=1,2, \quad h_3(x_1,x_2)=0,$$

and $\mathbf{b}_2=\operatorname{curl}\mathbf{w}_2$, solving (4.10c) reduces to determining $\mathbf{w}_2: E\to\mathbb{R}^3$, $\mathbf{w}_2\in W^{2,2}(E;\mathbb{R}^3)$, such that $\operatorname{curl}\mathbf{w}_2\big|_{x_3=0}=\mathbf{h}$ in the sense of traces, i.e.,

$$\begin{aligned}\frac{\partial w_{2,3}}{\partial x_2}(x_1,x_2,0)-\frac{\partial w_{2,2}}{\partial x_3}(x_1,x_2,0)&=h_1(x_1,x_2)\\ \frac{\partial w_{2,1}}{\partial x_3}(x_1,x_2,0)-\frac{\partial w_{2,3}}{\partial x_1}(x_1,x_2,0)&=h_2(x_1,x_2)\\ \frac{\partial w_{2,2}}{\partial x_1}(x_1,x_2,0)-\frac{\partial w_{2,1}}{\partial x_2}(x_1,x_2,0)&=0.\end{aligned}$$

Choosing

$$\frac{\partial w_{2,3}}{\partial x_1}(x_1,x_2,0)=\frac{\partial w_{2,3}}{\partial x_2}(x_1,x_2,0)=\frac{\partial w_{2,2}}{\partial x_1}(x_1,x_2,0)=\frac{\partial w_{2,1}}{\partial x_2}(x_1,x_2,0)=0$$

yields

$$\begin{aligned}-\frac{\partial w_{2,2}}{\partial x_3}(x_1,x_2,0)&=h_1(x_1,x_2)\\ \frac{\partial w_{2,1}}{\partial x_3}(x_1,x_2,0)&=h_2(x_1,x_2).\end{aligned}$$

If we assume that $w_{2,1}(x_1,x_2,0)=w_{2,2}(x_1,x_2,0)=w_{2,3}(x_1,x_2,0)=0$, we conclude that a solution is given by

$$\begin{aligned}w_{2,1}(x_1,x_2,x_3)&=x_3h_2(x_1,x_2)\\ w_{2,2}(x_1,x_2,x_3)&=-x_3h_1(x_1,x_2)\\ w_{2,3}(x_1,x_2,x_3)&=0.\end{aligned}$$

Notice that $\mathbf{w}_2(x_1, x_2, 0) = 0$ for any $(x_1, x_2) \in \mathbb{R}^2$. The vector field we were looking for is then

$$\mathbf{b} = \operatorname{curl}\mathbf{w}_1 + \operatorname{curl}\mathbf{w}_2 = \operatorname{curl}(\mathbf{w}_1 + \mathbf{w}_2) = \operatorname{curl}\mathbf{w}.$$

Now, we turn to consider the case of a general simply connected, bounded open set $E \subset \mathbb{R}^3$, with smooth boundary ∂E having one connected component. As before, we can proceed with the construction of the vector field $\mathbf{b}_1 = \nabla\varphi = \operatorname{curl}\mathbf{w}_1$, so that it only remains to determine $\mathbf{w}_2$ in this new context.

Consider a partition of the unity for the set E, namely a collection of C^∞ functions ψ_k with compact support Δ_k, such that

$$\sum_k \psi_k(x) = 1 \quad \forall\, x \in E.$$

Without loss of generality, we can assume each ψ_k to be defined on all $\mathbb{R}^3$. Let ∂E_k be the intersection of ∂E with the domain where $\psi_k \not\equiv 0$, provided such a domain does indeed have a non-empty intersection with ∂E. For each fixed ψ_k, we can now introduce a smooth change of variables $(y_{1,k}, y_{2,k}, y_{3,k})$ such that in the new coordinates, ∂E_k is the graph of $y_3 = 0$ in a compact set $D_k \subset \mathbb{R}^2$, and the coordinate system is orthogonal on ∂E_k.

If we let $(\mathbf{a} - \mathbf{b}_1)_k = \psi_k(\mathbf{a} - \mathbf{b}_1)$, we are going to build a vector field $(\mathbf{b}_2)_k = \operatorname{curl}(\mathbf{w}_2)_k$ such that $(\mathbf{b}_2)_k = (\mathbf{a} - \mathbf{b}_1)_k$ on ∂E, and

$$\mathbf{b} = \operatorname{curl}\mathbf{w}_1 + \sum_k (\mathbf{b}_2)_k = \operatorname{curl}\mathbf{w}_1 + \sum_k \operatorname{curl}(\mathbf{w}_2)_k.$$

Take $(\mathbf{a} - \mathbf{b}_1)_k$, perform the previously mentioned change of variables that flattens the portion ∂E_k of the boundary ∂E, and let $\mathbf{h}_k$ be the restriction on $y_3 = 0$ of the new vector field thus obtained. By construction, $\mathbf{h}_k$ has compact support, and we also have $\mathbf{h}_k = (h_{1,k}(y_1, y_2), h_{2,k}(y_1, y_2), 0)$.

Consider $P = P(x_1^*, x_2^*, x_3^*) \in \partial E$. If $P \in \partial E \backslash \partial E_k$, then $(\mathbf{a} - \mathbf{b}_1)_k(P) = 0$ and we can take $(\mathbf{w}_2)_k(P) = \nabla(\mathbf{w}_2)_k(P) = 0$.

On the other hand, if $P \in \partial E_k$, then the corresponding point

$$Q = Q(y_1^*, y_2^*, 0) \in \operatorname{supp}\mathbf{h}_k,$$

and we can proceed with the construction of $(\mathbf{w}_2)_k$ as we have done before for the set $E = \{y_3 > 0\}$. The vector field $(\mathbf{w}_2)_k = (\mathbf{w}_2)_k(y_1, y_2, y_3)$ vanishes as $(y_1, y_2) \notin D_k$, but there is no condition on y_3. On the other hand, a careful inspection of the construction for $E = \{y_3 > 0\}$ shows that if we consider a function $f \in C_o^\infty(\mathbb{R})$ with $f(0) = 0$, $f'(0) = 1$, $\operatorname{supp} f = [-r, r]$ and $r > 0$ arbitrary, the vector field

$$\begin{aligned} w_{2,1}(x_1, x_2, x_3) &= f(x_3)h_2(x_1, x_2) \\ w_{2,2}(x_1, x_2, x_3) &= -f(x_3)h_1(x_1, x_2) \\ w_{2,3}(x_1, x_2, x_3) &= 0. \end{aligned}$$

is also a solution. Therefore, the support of $(\mathbf{w}_2)_k(y)$ can be contained in a neighborhood of D_k of height r. Once $(\mathbf{w}_2)_k(y)$ has been built, applying the inverse change of variable, we obtain $(\mathbf{b}_2)_k(x) = \operatorname{curl}(\mathbf{w}_2)_k(x)$.

Since we have defined $(\mathbf{w}_2)_k(P)$ in two different ways, namely taking into account whether $P \in \partial E \backslash \partial E_k$ or $P \in \partial E_k$, we still need to check that the values of $(\mathbf{w}_2)_k$ and its derivatives are all compatible. Again, a careful control of the proof, shows that the only requirement is that the tangential derivatives have to vanish, and this is surely satisfied by our construction.

As for the smoothness of $(\mathbf{w}_2)_k$, it is a direct consequence of the smoothness of $\mathbf{a}$, and also of the smoothness of ∂E, which affects the regularity of the change of variables. In particular, if ∂E is of class C^1 and has the segment property, and $\mathbf{a} \in W^{\frac{1}{2},2}(\partial E)$, then it is a matter of straightforward computations to see that the previous construction yields $\mathbf{b} \in W^{1,2}(E)$ and $\mathbf{b} = \operatorname{curl}\mathbf{w}_1 + \operatorname{curl}\mathbf{w}_2$ with $\mathbf{w}_1$, $\mathbf{w}_2 \in W^{2,2}(E)$. As for (4.9c), it is a consequence of standard elliptic estimates.

On the other hand, if we consider regularity in the class of continuous functions, once more it is relatively easy to see, as pointed out in Ladyzhenskaya [149], that if ∂E is a C^2 surface and $\mathbf{a}$ is continuous on ∂E, then $\mathbf{b}$ is continuous on $\overline{E}$.

When $E \subset \mathbb{R}^2$, things are much easier. Recalling that E is simply connected, we look for $\mathbf{b}$ in the form

$$\mathbf{b} = \left(\frac{\partial w}{\partial x_2}, -\frac{\partial w}{\partial x_1}\right).$$

The condition $\mathbf{b}\big|_{\partial E} = \mathbf{a}$ gives the values of $\frac{\partial w}{\partial n}$ and $\frac{\partial w}{\partial \tau}$ on ∂E. From the values of $\frac{\partial w}{\partial \tau}$ on ∂E, we determine w on ∂E up to an arbitrary constant, and w is a single valued continuous function, as

$$\int_{\partial E} \frac{\partial w}{\partial n}\, d\sigma = \int_{\partial E} \mathbf{a} \cdot \mathbf{n}\, d\sigma = 0.$$

Once we know $w\big|_{\partial E}$ and $\dfrac{\partial w}{\partial n}\big|_{\partial E}$, we can finally build w in E. ∎

4.3c Proof of Proposition 4.3c

By Lemma 4.3c, if $N = 3$ there exists $\mathbf{w} \in W^{2,2}(E)$ (if $N = 2$, we have $w \in W^{2,2}(E)$) such that $\operatorname{curl}\mathbf{w}(x) = \mathbf{a}(x)$ for any $x \in \partial E$. For any $\epsilon > 0$, let $\varphi_\epsilon \in C^\infty(\overline{E})$ be the function built in Lemma 4.1c and set

$$\boldsymbol{\psi} \overset{\text{def}}{=} \varphi_\epsilon \mathbf{w}, \quad \mathbf{b} = \operatorname{curl}\boldsymbol{\psi} = \operatorname{curl}(\varphi_\epsilon \mathbf{w}).$$

By construction $\boldsymbol{\psi} \in W^{2,2}(E)$, it is compactly supported about ∂E, and for any $x \in \partial E$ we have

$$\boldsymbol{\psi}(x) = \mathbf{w}(x) \quad \Rightarrow \quad \operatorname{curl}\boldsymbol{\psi}(x) = \operatorname{curl}\mathbf{w}(x) = \mathbf{a}(x).$$

Moreover, owing to its very definition, and to (4.9c)

$$\|\operatorname{curl}\boldsymbol{\psi}\|_{W^{1,2}(E)} \le c\,\|\mathbf{a}\|_{W^{1/2,2}(\partial E)}.$$

It remains to show the validity of (4.7c). Lemma 4.1c yields

$$|\mathbf{b}(x)| \le \frac{\epsilon\kappa_2}{\delta(x)}|\mathbf{w}(x)| + |\nabla\mathbf{w}(x)| \quad \text{ if } \ \delta(x) < 2\gamma(\epsilon),$$

and

$$\mathbf{b}(x) = 0 \quad \text{ if } \delta(x) \ge 2\gamma(\epsilon).$$

By the Sobolev embedding theorem

$$\begin{aligned} |\mathbf{w}(x)| &\le c\,\|\mathbf{w}\|_{2,2}, \\ \|\nabla\mathbf{w}\|_3 &\le c\,\|\mathbf{w}\|_{2,2}, \end{aligned}$$

and by (4.9c) this implies

$$|\mathbf{w}(x)| + \|\nabla\mathbf{w}\|_3 \le c\,\|\mathbf{a}\|_{W^{1/2,2}(\partial E)}. \tag{4.12c}$$

Therefore, for every $\mathbf{u} \in V$, we have

$$\begin{aligned} \||\mathbf{u}|\operatorname{curl}\boldsymbol{\psi}|\|_2 \le & c\,\epsilon\|\mathbf{a}\|_{W^{1/2,2}(\partial E)}\|\frac{1}{\delta}\mathbf{u}\|_2 \\ & + \left(\int_{\delta(x)<2\gamma(\epsilon)} |\mathbf{u}|^2|\nabla\mathbf{w}|^2\,dx\right)^{\frac{1}{2}} \\ \le & c\,\epsilon\|\mathbf{a}\|_{W^{1/2,2}(\partial E)}\|\frac{1}{\delta}\mathbf{u}\|_2 + \|\nabla\mathbf{u}\|_2\|\nabla\mathbf{w}\|_{3;E_\epsilon}, \end{aligned}$$

where

$$E_\epsilon \overset{\text{def}}{=} \{x \in E : \ \delta(x) < 2\gamma(\epsilon)\}.$$

Owing to (4.12c), we have that $\zeta(\epsilon) \overset{\text{def}}{=} \|\nabla\mathbf{w}\|_{3;E_\epsilon}$ vanishes as $\epsilon \to 0$, whereas Lemma 4.2c yields $\|\delta^{-1}\mathbf{u}\|_2 \le c\,\|\nabla\mathbf{u}\|_2$; therefore, we can conclude

$$\||\mathbf{u}|\operatorname{curl}\boldsymbol{\psi}|\|_2 \le c\,\left(\epsilon\|\mathbf{a}\|_{W^{1/2,2}(\partial E)} + \zeta(\epsilon)\right)\|\nabla\mathbf{u}\|_2 = \mathcal{X}(\epsilon)\|\nabla\mathbf{u}\|_2,$$

where $\mathcal{X}(\epsilon) \to 0$ as $\epsilon \to 0$.

4.4c The Case of a General Domain *E*

We briefly discuss what happens when $E \subset \mathbb{R}^N$, $N = 2, 3$, is not simply connected, and/or the boundary ∂E has multiple connected components Γ_i, $i = 1, \ldots, m$ with $m > 1$.

Consequently, the natural compatibility condition on the velocity $\mathbf{v}$ at the boundary ∂E, required by the incompressibility of the fluid, is

$$\int_{\partial E} \mathbf{a}\cdot\mathbf{n}\,d\sigma = \sum_{i=1}^{m}\int_{\Gamma_i} \mathbf{a}\cdot\mathbf{n}\,d\sigma = 0,$$

where $\mathbf{n}$ is the outward unit normal to ∂E, whereas the argument we have presented above (which is Leray's original argument [164]) works if the condition

$$\forall i = 1,\dots,m \qquad \int_{\Gamma_i} \mathbf{a}\cdot\mathbf{n}\,d\sigma = 0$$

holds, which is obviously stronger. Moreover, such a strict requirement does not allow for the presence of extended *sinks* and *sources* into the region of flow, which is physically interesting. The question of whether the problem we have considered here, admits a solution only under the natural restriction is a fundamental question in the mathematical theory of the Navier–Stokes equations.

We refrain from further elaborating on this issue here. The reader interested in the solenoidal extension to a bounded open set that is not simply connected, and/or has a boundary with multiple connected components, can refer, for example, to Finn [73], Section 2 and to Galdi [92], Chapter VIII.

5c Recovering the Pressure

5.1c Proof of Proposition 5.1 for $\mathbf{u} \in \boldsymbol{H}^{\perp} \cap \boldsymbol{C}^{\infty}(\boldsymbol{E};\mathbb{R}^3)$

Pick $\mathbf{w} \in C_o^\infty(E;\mathbb{R}^3)$. Then $\operatorname{curl}\mathbf{w} \in \mathcal{V}$ and hence, by the membership $\mathbf{u} \in H^{\perp}$, and by integration by parts

$$\int_E \mathbf{u}\cdot\operatorname{curl}\mathbf{w}\,dx = -\int_E \operatorname{curl}\mathbf{u}\cdot\mathbf{w}\,dx = 0 \qquad \text{for all } \mathbf{w}\in C_o^\infty(E;\mathbb{R}^3).$$

By density this continues to hold for all $\mathbf{w} \in L^2(E;\mathbb{R}^3)$. Therefore, if $\mathbf{u} \in H^{\perp} \cap C_o^\infty(E;\mathbb{R}^3)$ then $\operatorname{curl}\mathbf{u} = 0$ in E. Since E is assumed to be convex, denoting by $\boldsymbol{\eta}$ the coordinates in $\mathbb{R}^3$, the latter is a necessary and sufficient condition for the differential form $d\mathbf{u} = \mathbf{u}\cdot d\boldsymbol{\eta}$ to be exact in E. Having fixed $x, y \in E$ consider a smooth path from y to x, i.e.,

$$\gamma_{x,y} = \left\{\boldsymbol{\eta} \in C^1\left[(\alpha,\beta);\mathbb{R}^3\right],\ \boldsymbol{\eta}(\alpha) = y,\ \boldsymbol{\eta}(\beta) = x;\ |\boldsymbol{\eta}'| > 0\right\}.$$

The path integral

$$p(x,y) = \int_{\gamma_{x,y}} d\mathbf{u} = \int_\alpha^\beta \mathbf{u}\big(\boldsymbol{\eta}(s)\big)\cdot\boldsymbol{\eta}'(s)ds \tag{5.1c}$$

is independent of $\gamma_{x,y}$, and, for a fixed $y \in E$, uniquely defines a function $p(\cdot,y)$ satisfying $\nabla p(\cdot,y) = \mathbf{u}$. Moreover, for any $y_1, y_2 \in E$, by the stated independence of the path integral, $p(\cdot,y_2) = p(\cdot,y_1) + p(y_1,y_2)$. Since $\mathbf{u} \in C_o^\infty(E;\mathbb{R}^3)$ one has $p(\cdot,y) \in C^\infty(E)$. To establish the proposition in the case $\mathbf{u} \in H^{\perp} \cap C^\infty(E;\mathbb{R}^3)$, fix $y \in E$ and determine the function $E \ni x \to p(x,y)$ up to a constant.

5.2c Proof of Proposition 5.1 for $\mathbf{u} \in \boldsymbol{H}^{\perp}$

Having fixed $\mathbf{u} \in H^{\perp}$, regard it as defined in $\mathbb{R}^3$ by extending it to zero outside E. Pick $\mathbf{w} \in C_o^{\infty}(E)$, and for $\epsilon > 0$ denote by $\mathbf{w}_{\epsilon} = J_{\epsilon} * \mathbf{w}$ the ϵ-mollification of $\mathbf{w}$ by the Friedrich's mollifying kernel $J_{\epsilon}(\cdot)$. We choose ϵ sufficiently small, such that $\mathbf{w}_{\epsilon} \in C_o^{\infty}(E;\mathbb{R}^3)$ and $\operatorname{curl}\mathbf{w}_{\epsilon}$ restricted to E is in H. Since $\mathbf{u} \in H^{\perp}$

$$\begin{aligned} 0 &= \int_E \mathbf{u} \cdot \operatorname{curl}\mathbf{w}_{\epsilon}\, dx = \int_{\mathbb{R}^3} \mathbf{u} \cdot \operatorname{curl}\mathbf{w}_{\epsilon}\, dx \\ &= \int_{\mathbb{R}^3} \mathbf{u}_{\epsilon} \cdot \operatorname{curl}\mathbf{w}\, dx = \int_E \mathbf{u}_{\epsilon} \cdot \operatorname{curl}\mathbf{w}\, dx \\ &= -\int_E \operatorname{curl}\mathbf{u}_{\epsilon} \cdot \mathbf{w}\, dx = 0 \end{aligned}$$

for all $\mathbf{w} \in C_o^{\infty}(E;\mathbb{R}^3)$. By density this continues to hold for all $\mathbf{w} \in L^2(E;\mathbb{R}^3)$. Therefore, $\operatorname{curl}\mathbf{u}_{\epsilon} = 0$ in E, and the path integral

$$p(x,y;\epsilon) = \int_{\gamma_{x,y}} d\mathbf{u}_{\epsilon} = \int_{\alpha}^{\beta} \mathbf{u}_{\epsilon}\big(\boldsymbol{\eta}(s)\big) \cdot \boldsymbol{\eta}'(s) ds$$

is independent of $\gamma_{x,y} \subset E$. For a fixed $y \in E$, such an integral uniquely defines a function $p(\cdot,y;\epsilon)$ satisfying $\nabla p(\cdot,y;\epsilon) = \mathbf{u}_{\epsilon}$. Moreover, for any $y_1, y_2 \in E$, by the stated independence of the path integral, $p(\cdot,y_2;\epsilon) = p(\cdot,y_1;\epsilon) + p(y_1,y_2;\epsilon)$.

Proposition 5.1c *There exists $p(\cdot,\cdot) \in L^2(E \times E)$ and a subnet $\{p(\cdot,\cdot;\epsilon')\} \subset \{p(\cdot,\cdot;\epsilon)\}$, relabeled with ϵ such that as $\epsilon \to 0$*

$$\begin{array}{ll} p(\cdot,\cdot;\epsilon) \to p(\cdot,\cdot) & \textit{in } L^2(E \times E) \textit{ and a.e. in } E \times E \\ p(\cdot,y;\epsilon) \to p(\cdot,y) & \textit{in } L^2(E) \textit{ for a.e. } y \in E \\ p(x,y_2) = p(x,y_1) + p(y_1,y_2) & \textit{a.e. in } E \times E \\ \nabla p(\cdot,y;\epsilon) \to \nabla p(\cdot,y) & \textit{weakly in } L^2(E) \textit{ for a.e. } y \in E \\ \nabla p(\cdot,y) = \mathbf{u} & \textit{for a.e. } y \in E. \end{array} \tag{5.2c}$$

We rely on the following result.

Lemma 5.1c *There holds:*

$$\|p(\cdot,\cdot;\epsilon)\|_{2;E\times E} \le 2\sqrt{2\pi}\,\operatorname{diam}(E)^{\frac{3}{2}}\,\|\mathbf{u}_{\epsilon}\|_{2;E} \le 2\sqrt{2\pi}\,\operatorname{diam}(E)^{\frac{3}{2}}\,\|\mathbf{u}\|_{2;E}$$

uniformly in ϵ.

Proof. Fix $\epsilon > 0$ and in computing $p(x,y;\epsilon)$ from (5.1c) take the segment

$$(0,|x-y|) \ni s \to y + s\nu \qquad \text{where} \qquad \nu = \frac{x-y}{|x-y|}.$$

For such a choice, and Hölder's inequality

$$p^2(x,y;\epsilon) \le \operatorname{diam}(E) \int_0^{|x-y|} |\mathbf{u}_\epsilon|^2(y+s\nu)ds.$$

Integrate both sides in dx over E, and compute the resulting integral on the right-hand side in polar coordinates with pole at y and angular variable ν ranging over the unit sphere of $\mathbb{R}^3$. Denote by $R(y,\nu)$ the polar representation of ∂E with pole at y and also set $z = y+s\nu$ so that the polar radius is $s = |z-y|$ and ranges over $\big(0, R(y,\nu)\big)$. This gives

$$\begin{aligned}
\|p(\cdot,y;\epsilon)\|_{2;E}^2 &\le \operatorname{diam}(E)^2 \int_{|\nu|=1} \Big(\int_0^{R(y,\nu)} |\mathbf{u}_\epsilon|^2(y+s\nu)ds \Big) d\nu \\
&= \operatorname{diam}(E)^2 \int_{\|\nu\|=1} \Big(\int_0^{R(y,\nu)} \frac{|\mathbf{u}_\epsilon(z)|^2}{|z-y|^2} |z-y|^2 d|z-y| \Big) d\nu \\
&= \operatorname{diam}(E)^2 \int_E \frac{|\mathbf{u}_\epsilon(z)|^2}{|z-y|^2} dz.
\end{aligned}$$

Next, integrate both sides in dy over E and estimate the resulting integral on the right-hand side by making use of Fubini's theorem to obtain

$$\begin{aligned}
\|p(\cdot,\cdot;\epsilon)\|_{2;E\times E}^2 &\le \operatorname{diam}(E)^2 \int_E |\mathbf{u}_\epsilon(z)|^2 dz \sup_{z\in E} \int_E \frac{1}{|z-y|^2} dy \\
&\le 8\pi \operatorname{diam}(E)^3 \int_E |\mathbf{u}_\epsilon(z)|^2 dz \\
&\le 8\pi \operatorname{diam}(E)^3 \|\mathbf{u}\|_{2;E}^2.
\end{aligned}$$

The last inequality follows from the properties of the mollifying kernels. ■

Corollary 5.1c *For all positive* ϵ_1, ϵ_2

$$\|p(\cdot,\cdot;\epsilon_1) - p(\cdot,\cdot;\epsilon_2)\|_{2;E\times E} \le 2\sqrt{2\pi} \operatorname{diam}(E)^{\frac{3}{2}} \|\mathbf{u}_{\epsilon_1} - \mathbf{u}_{\epsilon_2}\|_{2;E}.$$

Proof (of Proposition 5.1c). Since $\{\mathbf{u}_\epsilon\}$ is Cauchy in $L^2(E;\mathbb{R}^3)$ the net $\{p(\cdot,\cdot;\epsilon)\}$ is Cauchy in $L^2(E\times E)$ and by the completeness of $L^2(E\times E)$ there exists $p(\cdot,\cdot) \in L^2(E\times E)$ such that

$$\lim_{\epsilon\to 0} \|p(\cdot,\cdot;\epsilon) - p(\cdot,\cdot)\|_{2;E\times E} = 0.$$

Subnets can now be selected satisfying the first three statements in (5.2c). Fix $y \in E$ for which the second of (5.2c) holds and for $\boldsymbol{\zeta} \in C_o^\infty(E;\mathbb{R}^3)$ compute

$$\begin{aligned}
\lim_{\epsilon\to 0} \langle \nabla p(\cdot,y;\epsilon), \boldsymbol{\zeta} \rangle_{L^2(E)} &= \lim_{\epsilon\to 0} -\langle p(\cdot,y;\epsilon), \operatorname{div} \boldsymbol{\zeta} \rangle_{L^2(E)} \\
&= \langle \mathbf{u}, \boldsymbol{\zeta} \rangle_{L^2(E)} = -\langle p(\cdot,y), \operatorname{div} \boldsymbol{\zeta} \rangle_{L^2(E)}.
\end{aligned}$$

■

5.3c More General Versions of Proposition 5.1

5.1. The convexity of E has been used in the previous proof, in order to conclude that $d\mathbf{u}$ is exact. Prove that Proposition 5.1c continues to hold if E is not convex, but any two points $x, y \in E$ can be connected by a smooth curve $\gamma_{x,y} \subset E$ of length not exceeding a fixed constant L. This would include bounded, simply connected sets E with smooth boundary ∂E.

5.2. If E is unbounded let $E_n = E \cap \{|x| < n\}$ and assume that each E_n satisfies the condition in **5.1** with the constant L_n possibly depending on n. State and prove a local version of Proposition 5.1.

5.3. The Helmholtz–Weyl decomposition, sometimes also referred to as the Hodge decomposition, can actually be proven for any open set $E \subset \mathbb{R}^N$, if one works in $L^2(E;\mathbb{R}^N)$, as is the case here (see Galdi [91], § III.1). The situation is more complicated if one works in $L^p(E;\mathbb{R}^N)$ with $p \in (1,\infty)$, $p \neq 2$.

8c Time-Dependent Navier–Stokes Equations in Bounded Domains

In Section 8 we considered the Navier–Stokes equations in E_T with $E \subset \mathbb{R}^3$ an open, bounded set with smooth boundary, and stated Hopf's 1951 result about the existence of weak solutions of the initial-boundary value problem (8.1) [123].

As a matter of fact, the first result about the existence of weak solutions dates back to 1934 and is attributed to Leray [165], who studied the problem in the whole space $\mathbb{R}^3$ with divergence-free initial condition $u_o \in L^2(\mathbb{R}^3)$. Somehow, the more difficult case was solved first.

10c Selecting Subsequences Strongly Convergent in $L^2(E_T)$

Lemma 10.1c (Friedrichs [86]) *Let $Q \subset \mathbb{R}^N$ be a cube of edge L and let $\mathbf{u} \in W^{1,p}(Q)$ for some $1 < p < N$. For every $\varepsilon > 0$ there exist a positive integer k_ε depending only on ε and L, and independent of $\mathbf{u}$, and k_ε linearly independent functions $\{\boldsymbol{\psi}_\ell\}_{\ell=1}^{k_\varepsilon} \subset L^p(Q)$ such that*

$$\|\mathbf{u}\|_{p;Q}^p \le \sum_{\ell=1}^{k_\varepsilon} \left| \int_Q \mathbf{u} \cdot \boldsymbol{\psi}_\ell \, dx \right|^p + \varepsilon \|\nabla \mathbf{u}\|_{p;Q}^p. \tag{10.1c}$$

Remark 10.1c The conclusion continues to hold if $u \in W_o^{1,p}(E)$, where E is a bounded open set in $\mathbb{R}^N$. Indeed, E can be included in a cube Q and, since u has zero trace on ∂E, it can be extended in the whole cube by setting it to be zero outside E.

10.1c Proof of Friedrichs' Lemma

The starting point is the Poincaré inequality, which we state next. Let

$$u_Q = \frac{1}{|Q|}\int_Q u\,dx = \fint_Q u\,dx$$

denote the integral average of u over Q.

Theorem 10.1c (Poincaré Inequality). *Let $u \in W^{1,p}(Q)$. There exists a constant γ depending only on the dimension N and p, such that*

$$\|u - u_Q\|_{p_*;Q} \le \gamma \|\nabla u\|_{p;Q} \qquad \textit{where} \qquad p_* = \frac{Np}{N-p}. \tag{10.2c}$$

Proof. See DiBenedetto [50], Chapter 10, § 10.1. ■

Corollary 10.1c *Let $u \in W^{1,p}(Q)$. There exists a constant γ depending only on the dimension N and p, such that*

$$\|u - u_Q\|_{p;Q} \le \gamma L \|\nabla u\|_{p;Q}. \tag{10.3c}$$

Proof. Apply Hölder's inequality to $\|u - u_Q\|_{p;Q}$ and use (10.2c). ■

Let k be a positive integer to be chosen and subdivide Q in k^N equal subcubes Q_ℓ, with pairwise disjoint interior and edge L/k. Then compute and estimate

$$\begin{aligned}
\int_Q |u|^p dx &= \sum_{\ell=1}^{k^N} \int_{Q_\ell} |u|^p dx = \sum_{\ell=1}^{k^N} \int_{Q_\ell} |(u - u_{Q_\ell}) + u_{Q_\ell}|^p dx \\
&\le 2^{p-1}\Big(\frac{k}{L}\Big)^{N(p-1)} \sum_{\ell=1}^{k^N} \Big| \int_{Q_\ell} u dx \Big|^p + 2^{p-1} \sum_{\ell=1}^{k^N} \int_{Q_\ell} |u - u_{Q_\ell}|^p dx \\
&\le \sum_{\ell=1}^{k^N} \Big| \int_Q u\psi_\ell dx \Big|^p + \gamma 2^{p-1} \frac{L}{k} \sum_{\ell=1}^{k^N} \int_{Q_\ell} |\nabla u|^p dx \\
&= \sum_{\ell=1}^{k^N} \Big| \int_Q u\psi_\ell dx \Big|^p + \varepsilon \int_Q |\nabla u|^p dx
\end{aligned}$$

where we have set

$$\psi_\ell = 2^{\frac{p-1}{p}} \Big(\frac{k}{L}\Big)^{N\frac{p-1}{p}} \chi_{Q_\ell} \qquad \text{and} \qquad \varepsilon = \gamma 2^{p-1} \frac{L}{k}. \qquad ■$$

10.2c Compact Embedding of $W^{1,p}$ into $L^q(Q)$ for $1 \le q < p_*$

- Prove a version of (10.3c) with the left-hand side replaced by $\|u - u_Q\|_{q;Q}$ for $1 \le q < p_*$.
- Prove a version of Friedrichs' lemma with the left-hand side of (10.1c) replaced by $\|u\|_{q;Q}$.

- Use such a version to prove the indicated compact embedding.

Such an embedding was stated in Chapter 9, Section 2.1 and proved in Section 2.2c of the Complements. When $E = Q$ the arguments above provide a different proof. If E is bounded, give conditions on ∂E so that $u \in W^{1,p}(E)$ can be extended into a cube containing E with $u \in W^{1,p}(Q)$.

10.3c Solutions Global in Time

Let $\mathbf{f} \in L^2(E_T; \mathbb{R}^3)$. Prove that a Hopf solution of (8.1) satisfies

$$\left.\begin{aligned} &\|\mathbf{v}(t)\|_{2;E} \le \|\mathbf{v}_o\|_{2;E} + \|\mathbf{f}\|_{2;E_t} \\ &\|\nabla \mathbf{v}\|_{2;E_t} \le \frac{1}{\nu\sqrt{2}}\big(\|\mathbf{v}_o\|_{2;E} + \|\mathbf{f}\|_{2;E_t}\big) \end{aligned}\right. \quad \text{for a.e. } t \in (0,T). \tag{10.4c}$$

If $\mathbf{f} \in L^2(\mathbb{R}^+; L^2(E; \mathbb{R}^3))$, then (8.1) has a weak solution global in time, i.e., in $E \times \mathbb{R}^+$. Moreover, such a solution satisfies the energy estimates (10.4c) for all $t \in \mathbb{R}^+$.

11c The Limiting Process and Proof of Theorem 8.1

In Section 11 we underlined that the strong convergence is needed to pass to the limit in the nonlinear term. We now discuss a counterexample, in order to show that weak convergence in general does not suffice.

In particular, we consider a sequence $\{\mathbf{v}_n(x,t)\} \subset L^2(0,T; W^{1,2}(E)) \cap L^\infty(0,T; L^2(E))$, which satisfies the Navier–Stokes equations (8.1) with $\mathbf{f} = 0$ in the weak sense of (8.2). Moreover, we assume that

a) $x \mapsto \mathbf{v}_n(x,t) \in C^\infty(E)$ for a.e. t, uniformly in n;
b) $\frac{\partial^h \mathbf{v}_n}{\partial x_k^h}(x,t) \in L^\infty(E_T)$, $k = 1, 2, \dots, N$, $h \in \mathbb{N}$, uniformly in n;
c) $\frac{\partial}{\partial t}\Delta \mathbf{v}_n \in L^\infty(E_T)$, uniformly in n.

In spite of the great regularity of a)-b)-c), we show that we *do not have*

$$\int_0^T \int_E (\mathbf{v}_n \cdot \nabla)\,\mathbf{v}_n \cdot \boldsymbol{\varphi}\, dxdt \to \int_0^T \int_E (\mathbf{v} \cdot \nabla)\,\mathbf{v} \cdot \boldsymbol{\varphi}\, dxdt,$$

where $\mathbf{v}$ is the weak limit of $\mathbf{v}_n$ in $L^2(E_T)$, and $\boldsymbol{\varphi} \in C_o^\infty(E_T; \mathbb{R}^N)$.
The counterexample is built in the following way. Let ψ be harmonic in E, i.e., it satisfies $\operatorname{div} \nabla\psi \equiv \Delta\psi = 0$. Set

$$\mathbf{v}_n(x,t) = a_n(t)\nabla\psi,$$

where

$$\{a_n\} \subset L^\infty(0,T) \ \text{ uniformly in } \ n, \ \text{ and } \ \{a_n\} \subset C^1(0,T).$$

Then, $\mathbf{v}_n$ satisfies (**a**)-(**b**)-(**c**) above. Moreover, $\mathbf{v}_n$ satisfies the Navier–Stokes equations, that is,

$$\begin{aligned}\int_0^T \int_E a_n'(t)\nabla\psi \cdot \boldsymbol{\varphi}\, dxdt - \int_0^T a_n(t)\int_E \Delta(\nabla\psi)\cdot\boldsymbol{\varphi}\, dxdt \\ + \int_0^T a_n^2(t)\int_\Omega \frac{\partial\psi}{\partial x_i}\frac{\partial^2\psi}{\partial x_i\partial x_j}\,\boldsymbol{\varphi}\, dxdt = I_1 + I_2 + I_3 = 0,\end{aligned}$$

for every $\boldsymbol{\varphi} \in C^\infty(0,T;\mathcal{V})$. Indeed, we have $I_1 = I_2 = 0$ trivially (we rely on the integration by parts in I_1). For I_3 we have

$$\begin{aligned}\int_E \frac{\partial\psi}{\partial x_i}\frac{\partial^2\psi}{\partial x_i\partial x_j}\boldsymbol{\varphi}\, dx &= \frac{1}{2}\int_E \frac{\partial}{\partial x_j}\left(\sum_{i=1}^N\left(\frac{\partial\psi}{\partial x_i}\right)^2\right)\boldsymbol{\varphi}\, dx \\ &= -\frac{1}{2}\int_E |\nabla\psi|^2 \operatorname{div}\boldsymbol{\varphi}\, dx = 0.\end{aligned}$$

Consider now a sequence $\{a_n\} \subset L^\infty(0,T)$ such that $\|a_n(t)\|_{L^2(0,T)} = 1$ and $a_n(t) \rightharpoonup 0$ weakly in $L^2(0,T)$; for example, we could take $a_n(t) = \sqrt{\frac{2}{T}}\sin\frac{n\pi t}{T}$. Then,

$$\mathbf{v}_n(x,t) = a_n(t)\nabla\psi \rightharpoonup 0$$

weakly in $L^2(E_T)$, but if we consider a general $\boldsymbol{\varphi} \in C_o^\infty(E_T;\mathbb{R}^N)$

$$\begin{aligned}\int_0^T \int_E (\mathbf{v}_n\cdot\nabla)\,\mathbf{v}_n\cdot\boldsymbol{\varphi}\, dxdt &= \frac{1}{2}\int_0^T a_n^2(t)\left[\int_E \nabla(|\nabla\psi|^2)\cdot\boldsymbol{\varphi}\, dx\right] dt \\ &= -\frac{1}{2}\,\|a_n\|_{L^2(0,T)}^2\left(\int_E |\nabla\psi|^2 \operatorname{div}\boldsymbol{\varphi}\, dx\right) \\ &= -\frac{1}{2}\left(\int_E |\nabla\psi|^2 \operatorname{div}\boldsymbol{\varphi}\, dx\right) \neq 0.\end{aligned}$$

12c Higher Integrability and Some Consequences

12.1. Explain why (8.2) holding for all $\boldsymbol{\varphi} \in C^\infty(0,T;\mathcal{V})$ implies (12.2) holding weakly for all $\boldsymbol{\varphi} \in C_o^\infty(E_T;\mathbb{R}^N)$.

13c Energy Identity for the Homogeneous Boundary Value Problem with Higher Integrability

The proof of Proposition 13.1 essentially gives a way of taking $\boldsymbol{\varphi} = \mathbf{v}$ in the weak formulation (8.2).

Proposition 13.1c *Let* $\mathbf{v}$ *be a weak solution of* $(8.2)_o$*. Assume moreover that* $\mathbf{v} \in L^{p,q}(E_T; \mathbb{R}^N)$ *with* $p > N$ *and* $q > 2$ *satisfying (12.3). Then,* $\mathbf{v}$ *satisfies the energy estimates (10.4c).*

By the same token, Proposition 14.1 can be extended several ways. For example, one may permit $\mathbf{f}$ not to be zero, or the boundary data for $\mathbf{v}$ and $\mathbf{u}$ not to be zero, provided $\mathbf{w} = (\mathbf{v} - \mathbf{u})$ has zero trace on ∂E. State and prove this version of such facts by writing the corresponding weak formulation for $\mathbf{w}$ and taking $\boldsymbol{\varphi} = \mathbf{w}$ in the indicated approximate sense. This is possible by the assumed higher integrability on both $\mathbf{v}$ and $\mathbf{u}$ and hence $\mathbf{w}$. For $N = 2$ such a higher integrability assumption is redundant.

15c Local Regularity of Solutions with Higher Integrability

The proofs of Theorem 15.1 in Serrin and Struwe [233, 248] are based on a smart study of the vorticity equation (18.2). This is why the pressure does not appear in the statement. A careful analysis of the proof shows that the transport term is dealt with as if it were an external force.

For a different approach see Seregin [230], and also the references therein. Moreover, in Seregin [230] the author extends his formulation of the regularity estimates up to the boundary under homogeneous Dirichlet conditions on a half cylinder.

16c Proof of Theorem 15.1 – Introductory Results

Proposition 16.1c *Let* $k \in L^{p,p'}(\mathbb{R}^N \times \mathbb{R}^M; \mathbb{R})$ *and* $g \in L^{q,q'}(\mathbb{R}^N \times \mathbb{R}^M; \mathbb{R})$ *with* N, $M \geq 1$*, and*

$$1 \leq q \leq r, \quad 1 \leq q' \leq r', \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1, \quad \frac{1}{p'} + \frac{1}{q'} = \frac{1}{r'} + 1.$$

Then for the double convolution

$$h(x,t) \overset{def}{=} \iint_{\mathbb{R}^N \times \mathbb{R}^M} k(x-\xi, t-\tau) g(\xi, \tau)\, d\xi d\tau$$

we have

$$\|h\|_{r,r'} \leq \|k\|_{p,p'} \|g\|_{q,q'}.$$

Proof. First of all, consider the convolution only in one variable, namely

$$(k * g)(x) = \int_{\mathbb{R}^N} k(x-\xi) g(\xi)\, d\xi,$$

where $k \in L^p(\mathbb{R}^N)$, $g \in L^q(\mathbb{R}^N)$, and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$. We have

$$\begin{aligned}|(k*g)(x)| &= \left|\int_{\mathbb{R}^N} k(x-\xi)g(\xi)\,d\xi\right| \\ &\leq \int_{\mathbb{R}^N} |k(x-y)|\cdot|g(\xi)|\,d\xi \\ &= \int_{\mathbb{R}^N} |k(x-\xi)|^{p/r}|g(\xi)|^{q/r}|k(x-\xi)|^{\frac{r-p}{r}}|g(\xi)|^{\frac{r-q}{r}}\,d\xi.\end{aligned}$$

If we apply Hölder's inequality, we conclude that

$$\begin{aligned}|(k*g)(x)| &\leq \left[\int_{\mathbb{R}^N} |k(x-\xi)|^p|g(\xi)|^q d\xi\right]^{1/r}\cdot \\ &\quad \cdot\left[\int_{\mathbb{R}^N} ||k(x-\xi)|^p\,d\xi\right]^{\frac{r-p}{rp}}\left[\int_{\mathbb{R}^N} |g(\xi)|^q\,d\xi\right]^{\frac{r-q}{rq}} \\ &= \left[\int_{\mathbb{R}^N} |k(x-\xi)|^p|g(\xi)|^q d\xi\right]^{1/r}\|k\|_p^{\frac{r-p}{r}}\|g\|_q^{\frac{r-q}{r}}.\end{aligned}$$

Raising both sides to the power r yields

$$|(k*g)(x)|^r \leq \left[\int_{\mathbb{R}^N} |k(x-\xi)|^p|g(\xi)|^q d\xi\right]\|k\|_p^{r-p}\|g\|_q^{r-q}.$$

If we now integrate with respect to x, we have

$$\begin{aligned}\int_{\mathbb{R}^N} |(k*g)(x)|^r dx \leq \int_{\mathbb{R}^N} |g(\xi)|^q &\left[\int_{\mathbb{R}^N} |k(x-\xi)|^p dx\right]d\xi \\ &\cdot\|k\|_p^{r-p}\|g\|_q^{r-q}\end{aligned}$$

and we conclude that

$$\|k*g\|_r \leq \|k\|_p\|g\|_q. \tag{16.1c}$$

The previous proof holds for any $N \geq 1$; inequality (16.1c) is usually known as Young's inequality for the convolution. Now, we want to consider the double convolution with respect to $(x,t) \in \mathbb{R}^N \times \mathbb{R}^M$, namely

$$h(x,t) = \iint_{\mathbb{R}^N\times\mathbb{R}^M} k(x-\xi,t-\tau)g(\xi,\tau)\,d\xi d\tau.$$

We have

$$\begin{aligned}\|h(t)\|_r &= \left[\int_{\mathbb{R}^N}\left|\iint_{\mathbb{R}^N\times\mathbb{R}^M} k(x-\xi,t-\tau)g(\xi,\tau)\,d\xi d\tau\right|^r dx\right]^{1/r} \\ &\leq \left[\int_{\mathbb{R}^N}\left[\int_{\mathbb{R}^M}\left|\int_{\mathbb{R}^N} k(x-\xi,t-\tau)g(\xi,\tau)\,d\xi\right|d\tau\right]^r dx\right]^{1/r}.\end{aligned}$$

For simplicity, let us set for the moment

$$\int_{\mathbb{R}^N} k(x-\xi, t-\tau) g(\xi, \tau)\, d\xi = f(x, t, \tau).$$

Then, we have

$$\|h(t)\|_r \leq \left(\int_{\mathbb{R}^N} \left[\int_{\mathbb{R}^M} |f(x,t,\tau)|\, d\tau \right]^r dx \right)^{1/r}.$$

We can apply the continuous Minkowski inequality (see, for example, DiBenedetto [50], Chapter 6, Prop. 3.3) to obtain

$$\begin{aligned} \|h(t)\|_r &\leq \int_{\mathbb{R}^M} \|f(x,t,\tau)\|_r\, d\tau \\ &= \int_{\mathbb{R}^M} \left\| \int_{\mathbb{R}^N} k(x-\xi, t-\tau) g(\xi,\tau)\, d\xi \right\|_r d\tau \\ &\leq \int_{\mathbb{R}^M} \|k(t-\tau)\|_p \|g(\tau)\|_q\, d\tau, \end{aligned}$$

where we have taken (16.1c) into account. Let us momentarily set

$$u(t-\tau) = \|k(t-\tau)\|_p, \quad v(\tau) = \|g(\tau)\|_q.$$

We can rewrite

$$\|h(t)\|_r \leq \int_{\mathbb{R}^M} u(t-\tau) v(\tau)\, d\tau = (u * v)(t).$$

Once more, by (16.1c) we conclude

$$\begin{aligned} \|h\|_{r,r'} = \left(\int_{\mathbb{R}^M} \|h(t)\|_r^{r'} dt \right)^{1/r'} &= \|u * v\|_{r'} \leq \|u\|_{p'} \|v\|_{q'} \\ = \left(\int_{\mathbb{R}^M} \|k(t-\tau)\|_p^{p'}\, dt \right)^{1/p'} &\left(\int_{\mathbb{R}^M} \|g(\tau)\|_q^{q'}\, d\tau \right)^{1/q'} = \|k\|_{p,p'} \|g\|_{q,q'}. \end{aligned}$$

■

Proposition 16.2c *Let* $k \in L^{p,p'}(\mathbb{R}^N \times \mathbb{R}; \mathbb{R})$ *and* $g \in L^{q,q'}(\Omega \times (t_1,t_2); \mathbb{R})$ *with* $N \geq 1$*,* Ω *a bounded domain in* $\mathbb{R}^N$*,* $(t_1,t_2) \subset (0,\infty)$*, and*

$$1 \leq q \leq r, \quad 1 \leq q' \leq r', \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1, \quad \frac{1}{p'} + \frac{1}{q'} = \frac{1}{r'} + 1.$$

Then for the double convolution

$$h(x,t) \overset{def}{=} \iint_{\Omega \times (t_1,t_2)} k(x-\xi, t-\tau) g(\xi,\tau)\, d\xi d\tau, \quad (x,t) \in \Omega \times (t_1,t_2),$$

we have

$$\|h\|_{r,r'} \leq \|k\|_{p,p'} \|g\|_{q,q'}.$$

Proof. Same as in Proposition 16.1c

20c Recovering the Pressure in the Time-Dependent Equations

In Section 20 we study the regularity of the pressure p for weak solutions of (8.1) in E_T, where E is a bounded, smooth domain of $\mathbb{R}^3$.

In the whole space $\mathbb{R}^3$ the situation is definitely simpler, and we sketch how the analogous corresponding result can be obtained. We follow an argument given in Caffarelli et al. [26].

If we take the divergence of (8.1), we obtain

$$\Delta p = - \sum_{i,j=1}^{3} \frac{\partial^2}{\partial x_i \partial x_j}(v_i v_j)$$

in the sense of distributions in $\mathbb{R}^3 \times (0,T)$, and therefore, in $\mathbb{R}^3 \times \{t\}$ for a.e. $t \in (0,T)$.

Here, p is the sum of classical singular integral operators applied to $v_i v_j$. By the Calderón–Zygmund theory (see Stein [246]), we have

$$\int_0^T \int_{\mathbb{R}^3} |p|^q \, dxdt \le C(q) \int_0^T \int_{\mathbb{R}^3} |\mathbf{v}|^{2q}, \qquad q \in (1,\infty).$$

By the corresponding result in $\mathbb{R}^3 \times (0,T)$ of Lemma 8.1 we have $\mathbf{v} \in L^{\frac{10}{3}}(\mathbb{R}^3 \times (0,T))$, and therefore we conclude that $p \in L^{\frac{5}{3}}(\mathbb{R}^3 \times (0,T))$.

15

QUASI-LINEAR FIRST-ORDER SYSTEMS

1 Hyperbolic Systems

Consider a first-order quasi-linear symmetric hyperbolic system of the type

$$u_{\ell,t} + A_{i,\ell k}(x,t,u)u_{k,x_i} = B_\ell(x,t,u) \qquad \ell = 1,2\dots,m \tag{1.1}$$

where, in summation notation, $\mathbf{A}_i = (A_{i,\ell k})$ for $i = 1,2,\dots,N$ are given $m \times m$, symmetric, nontrivial matrices with real–valued entries defined in $\mathbb{R}^{N+1+m}$, and the functions $(x,t,u) \to B_\ell(x,t,u)$ for $\ell = 1,2,\dots,m$ are given and defined in $\mathbb{R}^{N+1+m}$. Setting $\mathbf{B} = (B_1,\dots,B_m)$ and $\mathbf{u} = (u_1,\dots,u_m)$, the system can be rewritten more concisely as

$$L(\mathbf{u}) \equiv \mathbf{u}_t + \mathbf{A}_i\mathbf{u}_{x_i} = \mathbf{B}.$$

The system is *hyperbolic* if for all $\xi \in \mathbb{R}^N$, the matrix $\mathbf{A}_i\xi_i$ has m real eigenvalues. It is *strictly hyperbolic* if, in addition, the eigenvalues are all distinct. In order to highlight the difference, in the case of multiple real eigenvalues, we talk about *weakly hyperbolic* systems. To justify such a distinction between strictly and weakly hyperbolic, let Γ be a smooth surface in $\mathbb{R}^{N+1}$, and up to a rotation of the coordinate axes, let $t = \varphi(x)$, for a smooth function φ, be its local representation about one of its points. Fix a smooth function $\mathbf{f} : \Gamma \to \mathbb{R}^m$ and consider the Cauchy problem

$$\begin{cases} L(\mathbf{u}) = \mathbf{B} & \text{in a neighborhood of } \Gamma \\ \mathbf{u} = \mathbf{f} & \text{on } \Gamma. \end{cases} \tag{1.2}$$

If such a problem has a unique classical solution $\mathbf{u}$ in a neighborhood of Γ, then $\mathbf{u}$, $\mathbf{u}_{x_i}$, $\mathbf{u}_t$ are uniquely determined by $\mathbf{f}$. In particular, since

$$\mathbf{u}(x,\varphi(x))_{x_i} = \mathbf{f}_{x_i} + \mathbf{u}_t\,\varphi_{x_i} \qquad \text{on} \quad \Gamma,$$

one can compute $\mathbf{u}_t$ on Γ from

© Springer Nature Switzerland AG 2023

E. DiBenedetto, U. Gianazza, *Partial Differential Equations*,
Cornerstones, https://doi.org/10.1007/978-3-031-46618-2_16

$$(\mathbf{I} + \mathbf{A}_i \varphi_{x_i})\, \mathbf{u}_t = \mathbf{B} - \mathbf{A}_i \mathbf{f}_{x_i}.$$

Therefore, a necessary condition for $\mathbf{u}$ to be a classical solution of the Cauchy problem (1.2) is that $\det(\mathbb{I} + \mathbf{A}_i \varphi_{x_i}) \neq 0$ on Γ. A smooth surface $\Gamma \subset \mathbb{R}^{N+1}$ with a local representation $t = \varphi(x)$ is *characteristic* of (1.1) if

$$\det(\mathbb{I} + \mathbf{A}_i \varphi_{x_i}) = 0 \quad \text{on } \ \Gamma. \tag{1.3}$$

For such surfaces, the Cauchy problem (1.4) has no classical solutions. This condition provides an operative way of finding characteristics, by finding those functions φ for which (1.3) holds on some of the level sets $\varphi = t$. These characteristic surfaces depend upon the solution itself; however, for linear systems when $\mathbf{A}_i = \mathbf{A}_i(x,t)$ they can be determined a priori. To find some characteristics, attempt to solve (1.1) for functions depending only on one variable, i.e., of the type

$$\varphi(x_1, \dots, x_N) = \varphi(x_k) \quad \text{for some fixed } \ k \in \{1, 2, \dots, N\}.$$

On the surface $t = \varphi(x_k)$

$$dt = \varphi' dx_k \quad \text{or} \quad \frac{dx_k}{dt} = \frac{1}{\varphi'} \quad \text{provided } \ \varphi' \neq 0.$$

This in (1.3) gives

$$\det\left(\frac{dx_k}{dt}\mathbf{I} + \mathbf{A}_k\right) = 0. \tag{1.4}$$

Assume that all the matrices $\mathbf{A}_k$ have each m distinct real eigenvalues $\lambda_{k,j}$ with corresponding eigenvectors $\mathbf{v}_{k,j}$, and let $\mathbf{V}_k = (\mathbf{v}_{k,1} \dots \mathbf{v}_{k,m})$ be the matrix of the eigenvectors. Then, from (1.4),

$$\det\left[\mathbf{V}_k^{-1}\left(\frac{dx_k}{dt}\mathbb{I} + \mathbf{A}_k\right)\mathbf{V}_k\right] = 0$$

from which

$$\frac{dx_k}{dt} = \lambda_{k,j} \quad \text{for all} \quad j = 1, 2, \dots, m. \tag{1.5}$$

These determine m distinct characteristic curves in the plane $\{x_k, t\}$. Therefore, the strict hyperbolicity condition implies that for every direction η in $\mathbb{R}^N$, the hyperplane $\{\eta, t\}$ contains m distinct characteristic curves. Thus, there are at least m distinct characteristic surfaces associated with (1.1).

2 Some Examples

Since the matrices $\mathbf{A}_i$ are nontrivial, the hyperplane $t = 0$ is not characteristic and the corresponding Cauchy problem

$$\begin{cases} L(\mathbf{u}) = \mathbf{B} & \text{in a neighborhood of } t = 0 \\ \mathbf{u}(\cdot, 0) = \mathbf{u}_o & \text{for } t = 0 \end{cases} \tag{2.1}$$

is called the initial value problem for the initial datum $\mathbf{u}_o$. By the previous discussion this problem is in principle permitted to have a smooth solution, at least for smooth initial data $\mathbf{u}_o$.

Next we give examples of systems of the type of (1.1) taken from physics.

2.1 Incompressible Euler Equations

Consider a fluid of density $(x,t) \to \rho(x,t)$ so that the infinitesimal mass $d\mu$ about the point x at time t is ρdx where dx is the infinitesimal volume about x. If $d\mu$ moves along a Lagrangian trajectory $x \to x(t)$ with velocity $\mathbf{u}$, then $\dot{x} = \mathbf{u}$. The fluid moves because of internal $\mathbf{F}^{(\mathbf{i})}$ and external $\mathbf{F}^{(\mathbf{e})}$ body force–densities, acting on $d\mu$. By the conservation of momentum

$$\rho dx \frac{d}{dt} u(x(t),t) = \big(F^{(i)} + F^{(e)}\big) dx. \tag{2.2}$$

The internal force–densities are the Kelvin body forces due to pressures and those due to friction. Denoting by ∇ the gradient with respect to the space variables, the Kelvin forces are given by $-\nabla p$, where p is the pressure, and the internal friction is proportional to the space–variation of the stress tensor. A fluid is said to be *Newtonian* if the stress tensor is *linearly* proportional to the matrix $(\nabla \mathbf{u})$. Therefore, for a Newtonian fluid the friction is given by $\nu \operatorname{div}(D_x \mathbf{u})$ where ν is the kinematic viscosity. The external force–densities $\mathbf{F}^{(e)}$ are due to gravity and/or buoyancy and are cumulatively denoted by $\mathbf{f}(x,t,\mathbf{u})$. With these stipulations (2.2) can be rewritten as

$$\rho(u_{\ell,t} + u_i u_{\ell,x_i}) = -p_{x_l} + \nu \Delta u_\ell + f_\ell(x,t,\mathbf{u}), \quad \ell = 1,2,3.$$

These are the Navier–Stokes equations introduced in Section 7 of the Preliminaries, and studied in Chapter 14.

The fluid is *inviscid* if there is no internal friction, i.e., if $\nu = 0$. In such a case the system is rewritten as

$$\rho \mathbf{u}_t + \rho\,(\mathbf{u} \cdot \nabla)\,\mathbf{u} = -\nabla p + \mathbf{f}, \tag{2.3}$$

and we obtain the so-called *Euler equations* (see also Section 8 of the Preliminaries). As we have seen in Chapter 14, they can be supplemented with the further equation

$$\operatorname{div} \mathbf{u} = 0, \tag{2.4}$$

so that (2.3)–(2.4) are usually referred to as the incompressible Euler equations. Taking the force $\mathbf{f} = 0$ for simplicity, and assuming the pressure p and the density ρ as known functions of (x,t), (2.3) is a system of three equations in the unkowns u_i, $i = 1, 2, 3$. If $\rho(x,t) > 0$ for all (x,t) in its domain of definition, the corresponding matrices $\mathbf{A}_i$ are $\rho u_i \mathbb{I}$; thus, the system is hyperbolic.[1] Notice that, strictly speaking, the system (2.3)–(2.4) is not hyperbolic.

2.2 Reacting Gas Flow in 1–Space Dimension

Consider the motion of a gas due to some internal fast chemical reaction, say, for example, combustion or detonation. Assume that the motion is laminar unidimensional and along the x-axis so that all the physical quantities depend only upon x and t. It is assumed that the reacting mixture has only two species, i.e., burnt and unburnt gas and that unburnt gas is converted into burnt gas by an irreversible process. Let u, ρ, p, and T denote velocity, density, pressure, and temperature and denote by

e	absolute internal chemical energy per unit mass
z	the mass fraction of unburnt gas.

These quantities are linked by the nonlinear system [179]

$$
\begin{aligned}
\rho_t + (\rho u)_x &= 0 && (2.5)\\
(\rho u)_t + (\rho u^2 + p)_x &= \nu u_{xx} && (2.6)\\
(\tfrac{1}{2}\rho u^2 + \rho e)_t + \big(u(\tfrac{1}{2}\rho u^2 + \rho e)\big)_x + (pu)_x &= (\nu u u_x)_x + (\lambda T_x)_x && (2.7)\\
(\rho z)_t + (u\rho z)_x &= k\varphi(T)\rho z + d(\rho z_x)_x, && (2.8)
\end{aligned}
$$

where k is a positive parameter measuring roughly the *liberated energy* at the burning front, λ is heat conductivity, and d is the species diffusivity.

In the system the first equation represents the conservation of mass and the second is the Navier–Stokes equation (2.3) with $\mathbf{f} \equiv 0$, and where ρ_t has been substituted from (2.5). Equation (2.7) is the balance of energy. In (2.8) the function $\varphi(\cdot)$ is given by

$$
\varphi(T) = \begin{cases} 0 & \text{if } T \le T_i \\ 1 & \text{if } T > T_i. \end{cases} \tag{2.9}
$$

The number T_i is the *ignition temperature* and $\varphi(\cdot)$ acts as a switch, turning the combustion on and off according to critical values of the temperature. Equation (2.8) is the balance of mass of the unburnt portion of the gas.

Remark 2.1 *Experiments suggest that $\varphi(\cdot)$ in fact increases sharply but smoothly from 0 to 1, i.e., there exists a (small) positive constant γ such that*

$$
\varphi(T) = \begin{cases} 0 & \textit{for } T \le T_i \\ \begin{array}{l}\textit{increases smoothly}\\ \textit{and continuously}\end{array} & \textit{for } T_i < T < T_i + \gamma \\ 1 & \textit{for } T \ge T_i + \gamma. \end{cases} \tag*{(2.9)$'$}
$$

---

[1] In general p and ρ are not known and (2.2) is complemented by an equation of state linking them, and the continuity equation (2.4) of the Preliminaries.

Introduce the new variables

$$\xi = \int_{x_o}^{x} \rho(s,t)ds \qquad \tau = t$$

and continue to denote by u, e, T, and z the transformed functions and by x and t the new variables ξ and τ. Let also $v = \rho^{-1}$ denote the the specific volume of the gas. Then, with this notation, the system (2.5)–(2.8) takes the form,

$$\begin{aligned} v_t - u_x &= 0 && (2.5)' \\ u_t + p_x &= \nu(v^{-1}u_x)_x && (2.6)' \\ (\tfrac{1}{2}u^2 + e)_t + (pu)_x &= \nu(v^{-1}uu_x)_x + (\lambda v^{-1}T_x)_x && (2.7)' \\ z_t &= -k\varphi(T)z + (dz_x)_x. && (2.8)' \end{aligned}$$

This is a system of four equations in the five unknowns u, v, e, T, and z. Various combustion models are derived from $(2.5)'$–$(2.8)'$ by “guessing” the form of the energy function e and a fifth equation linking the variables.

2.3 A Weakly Hyperbolic System Arising in Magnetohydrodynamics

Magnetohydrodynamics is the study of the properties and behavior of electrically conducting fluids. A typical example, although not the only one, of such fluids is given by plasma. The equations of magnetohydrodynamics are given by the coupling of the Navier–Stokes (see Chapter 14) or the Euler equations of gas dynamics (see Section 2.1) and the induction equation for the magnetic field. The physical aspects of these systems are described, for example, in Davidson and in Landau and Lifshitz [43, 152].

Here, for simplicity, we consider one-dimensional inviscid, incompressible processes, and we follow the presentation of magnetohydrodynamics for an ideal gas as given in Torrilhon [258]. The variables used in the description of the fluid are the density ρ, the velocity $\mathbf{v}$, the magnetic field $\mathbf{B}$, and total energy e. These variables constitute the component of the vector $\mathbf{u}$.

Owing to our simplified situation the vector fields $\mathbf{v}$ and $\mathbf{B}$ are decomposed into their scalar normal components v_n and B_n in the direction of the space variable, which we denote by x, and the two-dimensional transversal parts $\mathbf{v}_T$ and $\mathbf{B}_T$. Hence, we have

$$\mathbf{B} = (B_n, \mathbf{B}_T) \quad \text{and} \quad \mathbf{v} = (v_n, \mathbf{v}_T).$$

Owing to the divergence-free condition that has to be imposed on the magnetic field $\mathbf{B}$, its normal component B_n has to be constant in space, as we are dealing with one-dimensional processes; for simplicity, we assume B_n to be non-negative.

The remaining components of the variable $\mathbf{u}$, namely $(\rho, v_n, \mathbf{v}_T, \mathbf{B}_T, e)$, are the set of unknowns for the one-dimensional magnetohydrodynamics. Since we deal only with ideal gases, the total energy e is related to the pressure p by

$$e = \frac{1}{\gamma - 1} p + \frac{1}{2}\rho v_n^2 + \frac{1}{2}\rho \mathbf{v}_T^2 + \frac{1}{2}\mathbf{B}_T^2,$$

where γ is usually assumed to be $5/3$, and for simplicity $\mathbf{B}_T^2$ and $\mathbf{v}_T^2$ stand for $|\mathbf{B}_T|^2$ and $|\mathbf{v}_T|^2$. In the previous relation, the contribution of the normal field B_n is cancelled.

Finally, the magnetohydrodynamics equations in the one-dimensional setting are

$$\begin{aligned}
\rho_t + (\rho v_n)_x &= 0 \\
(\rho v_n)_t + (\rho v_n^2 + p + \frac{1}{2}\mathbf{B}_T^2)_x &= 0 \\
(\rho \mathbf{v}_T)_t + (\rho v_n \mathbf{v}_T - B_n \mathbf{B}_T)_x &= 0 \\
(\mathbf{B}_T)_t + (v_n \mathbf{B}_T - B_n \mathbf{v}_T)_x &= 0 \\
e_t + ((e + p + \frac{1}{2}\mathbf{B}_T^2) v_n - B_n \mathbf{B}_T \cdot \mathbf{v}_T)_x &= 0.
\end{aligned} \tag{2.10}$$

The equation for the normal component of the magnetic field reduces to the statement that B_n is also constant in time; in such a way, B_n can be simply considered as if it were a parameter. The system (2.10) is hyperbolic and its eigenvalues, namely

$$\begin{aligned}
&\lambda_1 = v - c_f,\ \lambda_2 = v - c_A,\ \lambda_3 = v - c_s, \\
&\lambda_4 = v, \\
&\lambda_5 = v + c_s,\ \lambda_6 = v + c_A,\ \lambda_7 = v + c_f,
\end{aligned}$$

have the physical meaning of velocities; the so-called *fast and slow magnetoacoustic velocities* c_f and c_s, and the *Alfven velocity* c_A, are given by

$$\begin{aligned}
c_f &= \sqrt{\frac{1}{2}\left(\frac{B_n^2 + \mathbf{B}_T^2}{\rho} + a^2\right) + \sqrt{\frac{1}{4}\left(\frac{B_n^2 + \mathbf{B}_T^2}{\rho} + a^2\right)^2 - a^2 \frac{B_n^2}{\rho}}}, \\
c_s &= \sqrt{\frac{1}{2}\left(\frac{B_n^2 + \mathbf{B}_T^2}{\rho} + a^2\right) - \sqrt{\frac{1}{4}\left(\frac{B_n^2 + \mathbf{B}_T^2}{\rho} + a^2\right)^2 - a^2 \frac{B_n^2}{\rho}}}, \\
c_A &= \sqrt{\frac{B_n^2}{\rho}},
\end{aligned}$$

where $a = \sqrt{\gamma \frac{p}{\rho}}$. Since

$$c_s \leq c_A \leq c_f,$$

the eigenvalues may coincide for special values of the flow, and therefore, the system (2.10) is in general only weakly hyperbolic.

We give another example of a (possibly) weakly hyperbolic system in Section 2c of the Complements.

3 Uniqueness of Smooth Solutions

Consider the initial value problem (2.1). In physically relevant problems the matrices $\mathbf{A}_i$ are not regular and the initial datum $\mathbf{u}_o$ is only in $L^1_{\text{loc}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. The main difficulty in establishing an existence and uniqueness theorem resides precisely in this lack of smoothness. Moreover, even if the data are regular, the existence of smooth solutions holds, in general, only for small times.[2]

Fix some $0 < T < \infty$, denote by S_T the $(N+1)$-dimensional strip $\mathbb{R}^N \times (0,T)$, and consider solutions of the Cauchy problem (2.1) in the class[3]

$$\begin{cases} \mathbf{u} \in L^\infty(S_T) \cap L^2(S_T) & \\ \mathbf{u}(x,\cdot) \in C^1(0,T) & \text{for all } x \in \mathbb{R}^N \\ \mathbf{u}_t \in L^\infty(S_T) & \\ \mathbf{u}(\cdot,t) \in C^1(\mathbb{R}^N) & \text{for all } t \in (0,T) \\ D\mathbf{u}(\cdot,t) \in L^\infty(\mathbb{R}^N) & \text{uniformly in } t \in (0,T), \end{cases} \tag{3.1}$$

where D denotes the gradient with respect to the space variables only.

On the matrices $\mathbf{A}_i$ and the vector $\mathbf{B}$ assume that

$$\begin{cases} \mathbf{A}_i,\ \mathbf{B} \in C(S_T \times \mathbb{R}) & \\ \mathbf{A}_i(x,t,\cdot),\ \mathbf{B}(x,t,\cdot) \in C^1(\mathbb{R}) & \text{for all } (x,t) \in S_T \\ \mathbf{A}_{i,z},\ \mathbf{B}_z \in L^\infty(\mathbb{R}^N) & \text{uniformly in } t \in (0,T). \end{cases} \tag{3.2}$$

These are regularity assumptions that imply that the system (2.1) holds pointwise in S_T. A structural assumption placed on the system is that the matrices $\mathbf{A}_i$ are symmetric, i.e.,

$$A_{i,\ell k} = A_{i,k\ell} \quad \forall\, i = 1,\dots,N; \quad \forall\, \ell, k = 1,\dots,m. \tag{3.3}$$

Theorem 3.1. *Let (3.2)–(3.3) hold. There exists at most one solution in the class (3.1) of the Cauchy problem (2.1) in S_T.*

---

[2] See, for example, the discussion on solutions to the initial value problem for the Burgers equation in Section 7 of Chapter 7.

[3] The symbolism is meant to indicate that each scalar component of the indicated classes belongs to the proper space of scalar functions. For example, the last one means that $u_{\ell,x_i}(\cdot,t)$ is essentially bounded in $\mathbb{R}^N$, uniformly in t, for all $\ell = 1,\dots,m$ and all $i = 1,\dots,N$.

Proof. Let $\mathbf{u}$ and $\mathbf{v}$ belong both to the class (3.1), and let both be solutions of (2.1) in S_T, originating from the same initial datum $\mathbf{u}_o$. By the assumptions of the class (3.1),

$$\mathbf{w}(t) \overset{\text{def}}{=} (\mathbf{u}-\mathbf{v})(t) \to 0 \quad \text{in } L^2_{\text{loc}}(\mathbb{R}^N), \text{ as } t \to 0.$$

Differencing (2.1) written for $\mathbf{u}$ and $\mathbf{v}$, we obtain

$$\begin{aligned} \mathbf{w}_t + \mathbf{A}_i(x,t,\mathbf{u})\mathbf{w}_{x_i} = &\{\mathbf{A}_i(x,t,\mathbf{v}) - \mathbf{A}_i(x,t,\mathbf{u})\}\,\mathbf{v}_{x_i} \\ &+ \mathbf{B}(x,t,\mathbf{u}) - \mathbf{B}(x,t,\mathbf{v}). \end{aligned} \tag{3.4}$$

For $R > 0$ denote by B_R the ball $\{|x| < R\}$ centered at the origin of $\mathbb{R}^N$ and radius R, and let $x \to \zeta_R(x)$ denote a non-negative piecewise smooth cutoff function in B_{2R}, vanishing for $|x| > 2R$, and such that

$$\zeta_R(x) = 1 \quad \text{for } |x| < R\,; \quad |D\zeta_R| \le \frac{1}{R}.$$

Multiply the ℓ^{th} equation in (3.4) by $w_\ell\zeta_R$, add over $\ell = 1, 2, \dots, m$ and integrate over the cylindrical domain

$$Q_{2R}(t) \equiv B_{2R} \times (0,t), \qquad \text{for } 0 < t \le T.$$

Standard calculations, using (3.2), give

$$\begin{aligned} \frac{1}{2}\iint_{Q_{2R}(t)} \frac{\partial}{\partial t}|\mathbf{w}|^2\zeta\,dxd\tau \le &-\iint_{Q_{2R}(t)} A_{i,\ell k}(x,t,\mathbf{u})w_{k,x_i}w_\ell\zeta_R\,dxd\tau \\ &+ \gamma\left(1 + \|D\mathbf{v}\|_{\infty;S_T}\right)\iint_{Q_{2R}(t)} |\mathbf{w}|^2\zeta_R\,dxd\tau. \end{aligned} \tag{3.5}$$

Here, γ is a positive constant that is determined a priori by the upper bounds arising from (3.2). Transform the first term on the right-hand side of (3.5) by performing an integration by parts as follows.

$$\begin{aligned} \iint_{Q_{2R}(t)} A_{i,\ell k}w_{k,x_i}w_\ell\zeta_R\,dxd\tau = &-\iint_{Q_{2R}(t)} A_{i,\ell k}w_{\ell,x_i}w_k\zeta_R\,dxd\tau \\ -\iint_{Q_{2R}(t)} \frac{\partial}{\partial x_i}A_{i,\ell k}w_kw_\ell\zeta_R\,dxd\tau &- \iint_{Q_{2R}(t)} A_{i,\ell k}(x,t,\mathbf{u})w_kw_\ell\frac{\partial}{\partial x_i}\zeta_R\,dxd\tau. \end{aligned}$$

Therefore, since $\mathbf{A}_i$ are symmetric

$$\begin{aligned} &2\iint_{Q_{2R}(t)} A_{i,\ell k}w_{k,x_i}w_\ell\zeta_R\,dxd\tau \\ &\qquad\qquad \le \sum_{i,k,\ell} \|DA_{i,\ell k}\|_{\infty;S_T\times\mathbb{R}^m}\iint_{Q_{2R}(t)} |\mathbf{w}|^2dxd\tau \end{aligned}$$

$$+\frac{1}{R}\sum_{i,k,\ell}\|A_{i,\ell k}\|_{\infty;S_T\times\mathbb{R}^m}\iint_{Q_{2R}(t)}|\mathbf{w}|^2dxd\tau.$$

Combining these calculations in (3.5) and letting $R\to\infty$ gives

$$\operatorname*{ess\,sup}_{0\le\tau\le t}\int_{\mathbb{R}^N}|\mathbf{w}|^2(\tau)\,dx\ \le\ C\,t\operatorname*{ess\,sup}_{0\le\tau\le t}\int_{\mathbb{R}^N}|\mathbf{w}|^2(\tau)\,dx$$

for a constant C that can be determined a priori from the upper bounds in (3.2). Thus, $|\mathbf{w}|\equiv 0$ in S_{t_1} where $t_1=1/2C$. The argument can be repeated a finite number of times up to covering the whole interval $(0,T)$. ∎

4 Existence of Solutions: The Linear Theory

We assume first that the matrices $\mathbf{A}_i$ and the vector $\mathbf{B}$ are independent of u, and consider the linear Cauchy problem

$$\begin{cases}\mathbf{u}_t+\mathbf{A}_i\mathbf{u}_{x_i}=\mathbf{B}(x,t) & \text{in } S_T\equiv\mathbb{R}^N\times(0,T)\\ \mathbf{u}(\cdot,0)=\mathbf{u}_o.\end{cases}\tag{4.1}$$

Let α be a multi-index of size $|\alpha|\le n_o\equiv\left\lfloor\frac{N+1}{2}\right\rfloor+1$, where $\forall y\in\mathbb{R}$, $\lfloor y\rfloor$ denotes the largest integer that does not exceed y. On the matrices $\mathbf{A}_i$, the vector $\mathbf{B}$ and the initial datum $\mathbf{u}_o$ we assume

$$\begin{cases}\|\mathbf{u}_o\|_{n_o}\equiv\sum_{|\alpha|\le n_o}\|D^\alpha\mathbf{u}_o\|_{2;\mathbb{R}^N}<\infty,\\ \|\mathbf{A}\|_{n_o}\equiv\operatorname{ess\,sup}_{t\ge0}\sum_{|\alpha|\le n_o}\sum_{i,k,l}\|D^\alpha A_{i,lk}\|_{2;\mathbb{R}^N}(t)<\infty,\\ \|\mathbf{B}\|_{n_o}\equiv\operatorname{ess\,sup}_{t\ge0}\sum_{|\alpha|\le n_o}\sum_{l}\|D^\alpha B_l\|_{2;\mathbb{R}^N}(t)<\infty,\end{cases}\tag{4.2}$$

where D^α denotes the differential operator of index α with respect to the space variables only.

Theorem 4.1. *Let (4.2) hold. Then there exists* $T=T\left(\|\mathbf{A}\|_{n_o},\|\mathbf{B}\|_{n_o}\right)$ *such that there exists a unique solution* $\mathbf{u}$ *of (4.1) in* S_T*, satisfying*

$$\operatorname*{ess\,sup}_{0\le t\le T}\|\mathbf{u}(\cdot,t)\|_{n_o}<\infty.$$

Remark 4.1 Assumption (4.2) on $\mathbf{u}_o$ ensures that $\mathbf{u}_o\in C^1(\mathbb{R}^N)\cap W^{1,\infty}(\mathbb{R}^N)$. If $f\in C_o^s(\mathbb{R}^N)$, from the inverse Fourier transform formula

$$f(x)=(2\pi)^{\frac{N}{2}}\int_{\mathbb{R}^N}e^{ix\cdot\xi}\hat{f}(\xi)\,d\xi,$$

we get

$$\begin{aligned} |f(x)|^2 &= (2\pi)^N \left| \int_{\mathbb{R}^N} e^{ix\cdot\xi}(1+|\xi|)^s (1+|\xi|)^{-s} \hat{f}(\xi)\, d\xi \right|^2 \\ &\le \Gamma_N \int_{\mathbb{R}^N} (1+|\xi|)^{2s} |\hat{f}(\xi)|^2\, d\xi, \end{aligned} \tag{4.3}$$

where

$$\Gamma_N = (2\pi)^N \int_{\mathbb{R}^N} (1+|\xi|)^{-2s} d\xi\,, \qquad s > \frac{N}{2}.$$

Next, from the formulae

$$i\xi^\alpha \hat{f} = \widehat{D^\alpha f}\,, \qquad (1+|\xi|)^s = \sum_{k=0}^{s} \binom{s}{k} |\xi|^k,$$

it follows that

$$\begin{aligned} (1+|\xi|)^s |\hat{f}(\xi)| &= \sum_{k=0}^{s} \binom{s}{k} |\xi|^k |\hat{f}(\xi)| \\ &\le 2^s \sum_{k=0}^{s} s^{\frac{k}{2}} |\xi|^k |\hat{f}(\xi)| \le 2^s s^{\frac{s}{2}} \sum_{|\alpha|\le s} \left| \xi^\alpha \hat{f}(\xi) \right|. \end{aligned}$$

Substituting this in (4.3) yields

$$|f(x)|^2 \le C(N,s) \sum_{|\alpha|\le s} \int_{\mathbb{R}^N} |D^\alpha f|^2\, dx. \tag{4.4}$$

Applying (4.4) with f replaced by $f_{x_h}\,, h = 1, 2, \dots, N$ we have

$$|Df(x)|^2 \le C \int_{\mathbb{R}^N} |D^\alpha Df|^2\, dx \tag*{(4.4)$'$}$$

provided $s = |\alpha| - 1 > \frac{N}{2}$.

4.1 A Family of Approximating Problems

Let $0 < T < \infty$ be arbitrary but fixed, and for $\varepsilon > 0$ let

$$Q_\varepsilon(T) \overset{\text{def}}{=} \left\{ |x| < 2n_o \varepsilon^{-1} \right\} \times (0,T);$$

consider the boundary value problem

$$\begin{cases} \frac{\partial}{\partial t} u_{\varepsilon l} - \varepsilon \Delta u_{\varepsilon l} + A_{i,lk}(x,t) u_{\varepsilon k, x_i} = B_{\varepsilon l}(x,t) \quad \text{in} \quad Q_\varepsilon(T), \\ u_{\varepsilon l}(\cdot, t) = 0 \quad \text{for} \quad |x| = 2n_o\varepsilon^{-1}; \quad 0 \le t \le T, \\ u_{\varepsilon l}(\cdot, 0) = u_{o,\varepsilon l}(\cdot), \end{cases} \tag{4.5}$$

where

$$\begin{cases} \mathbf{B}_\varepsilon = \cos^2 \frac{\pi}{2}(\varepsilon|x| - (2n_o - 1))_+ \, \mathbf{B}, \\ \mathbf{u}_{o,\varepsilon} = \cos^2 \frac{\pi}{2}(\varepsilon|x| - (2n_o - 1))_+ \, \mathbf{u}_o. \end{cases} \tag{4.6}$$

By Theorem 7.1 of Ladyzhenskaya et al. [151], Chapter VII, § 7, the system (4.5)–(4.6) has a unique solution $\mathbf{u} \in C^{2+\alpha,1+\alpha/2}(Q_\varepsilon(T))$.

Remark 4.2 The definition of $\mathbf{B}_\varepsilon$ and $\mathbf{u}_{o,\varepsilon}$ in (4.6) permit the compatibility conditions near $|x| = 2n_o\varepsilon^{-1}$, $t = 0$, in Theorem 7.1 of Ladyzhenskaya et al. [151] to be veriied.

Next we derive a priori estimates on u_ε. Consider the balls

$$B_o \equiv \{|x| < \varepsilon^{-1}\}, \qquad B_j \equiv \{|x| < (j+1)\varepsilon^{-1}\}, \qquad j = 1, 2, \dots, n_o,$$

and let $x \to \zeta_j(x)$ denote a smooth non-negative cutoff function in B_j that equals one on B_{j-1} and such that $|D^\alpha \zeta_j| \le C\varepsilon^{|\alpha|}$ for some constant C and for all $j = 1, 2, \dots, n_o$. To simplify the estimates to follow, we multiply (4.5) by ζ_j , set $\mathbf{v} = \zeta_j \, \mathbf{u}_\varepsilon$, $\tilde{\mathbf{B}} = \zeta_j \, \mathbf{B}$ and obtain

$$\begin{cases} \frac{\partial}{\partial t} v_l - \varepsilon \varDelta v_l + A_{i,lk}(x,t) v_{k,x_i} = \tilde{B}_l + \varPhi_l \quad \text{in} \quad Q_j(T), \\ v_l(\cdot, 0) = v_{o,l}(\cdot) \equiv u_{o,\varepsilon l} \zeta_j, \end{cases} \tag{4.7}$$

where

$$Q_j(T) \overset{\text{def}}{=} \left\{|x| < (j+1)\varepsilon^{-1}\right\} \times (0,T),$$

$$\varPhi_l \overset{\text{def}}{=} A_{i,lk} u_{\varepsilon k} \zeta_{j,x_i} - \varepsilon u_{\varepsilon l} \varDelta \zeta_j - 2\varepsilon D u_{\varepsilon l} \cdot D\zeta_j .$$

In the estimates to follow, we let C denote a generic positive constant, depending only upon N, and independent of ε, j. We also introduce the norms

$$\begin{cases} \|f(t)\|_{n,j} \equiv \sum_{|\alpha| \le n} \|D^\alpha f(\cdot, t)\|_{2;B_j}, \quad n, j \in \mathbb{N}, \\ \|f\|_{n,j} \equiv \operatorname{ess\,sup}_{0 \le t \le T} \sum_{|\alpha| \le n} \|D^\alpha f(\cdot, t)\|_{2;B_j}. \end{cases}$$

Let α be a multi-index of size $|\alpha| = n \le n_o$ and apply the operator D^α to (4.7) to obtain

$$\begin{cases} \frac{\partial}{\partial t} D^\alpha v_l - \varepsilon \varDelta D^\alpha v_l + D^\alpha \left[A_{i,lk}(x,t) v_{k,x_i}\right] = D^\alpha \tilde{B}_l + D^\alpha \varPhi_l \\ D^\alpha v_l(\cdot, 0) = D^\alpha v_{o,l}(\cdot). \end{cases}$$

We multiply by $D^\alpha v_l$, add over $l = 1, 2, \dots, m$ and over all multi-indices $|\alpha| \le n$, and integrate over $Q_j(T)$ to obtain by standard calculations

$$\begin{aligned}
&\|\mathbf{v}\|^2_{n,j} + \varepsilon \int_0^T \|\mathbf{v}(t)\|^2_{n+1,j} dt - CT\|\mathbf{B}\|_{n,j}\|\mathbf{v}\|_{n,j} - \|\mathbf{v}_o\|^2_{n,j} \\
&\leq CT \operatorname*{ess\,sup}_{0\leq t\leq T} \left| \sum_{|\alpha|\leq n} \int_{B_j} D^\alpha \left[A_{i,lk}(x,t) v_{k,x_i}\right] D^\alpha v_l \, dx \right| \\
&\quad + C \left| \sum_{|\alpha|\leq n} \iint_{Q_j} D^\alpha \left[A_{i,lk}(x,\tau) u_\varepsilon \zeta_{j,x_i}\right] D^\alpha v_l \, dx d\tau \right| \qquad (4.8) \\
&\quad + C \left| \sum_{|\alpha|\leq n} \iint_{Q_j} D^\alpha \left[-\varepsilon u_{\varepsilon l} \Delta\zeta_j - 2\varepsilon D u_{\varepsilon l} \cdot D\zeta_j\right] D^\alpha v_l \, dx d\tau \right| \\
&\equiv H_1 + H_2 + H_3.
\end{aligned}$$

4.2 Estimate of H_i, $i = 1, 2, 3$

As for H_3, after one integration by parts we easily have

$$H_3 \leq \varepsilon^2 \left(\int_0^T \|\mathbf{v}(t)\|^2_{n+1,j} \, dt \right)^{\frac{1}{2}} \left(\int_0^T \|\mathbf{u}_\varepsilon(t)\|^2_{n,j} \, dt \right)^{\frac{1}{2}}.$$

To estimate H_1 we transform the integrals as follows

$$\begin{aligned}
&\int_{B_j} D^\alpha \left[A_{i,lk} v_{k,x_i}\right] D^\alpha v_l \, dx = \int_{B_j} D^\alpha A_{i,lk} v_{k,x_i} D^\alpha v_l \, dx \\
&+ \int_{B_j} A_{i,lk} D^\alpha v_{k,x_i} D^\alpha v_l \, dx + \sum_{\substack{|\beta|+|\gamma|=|\alpha| \\ |\beta|,|\gamma|\neq 0}} c_{\beta,\gamma} \int_{B_j} D^\beta A_{i,lk} D^\gamma v_{k,x_i} D^\alpha v_l \, dx \\
&\equiv H_1^{(1)} + H_1^{(2)} + H_1^{(3)}.
\end{aligned}$$

Here, β, γ are multi-indices such that $\beta + \gamma = \alpha$.

To estimate $H_1^{(2)}$ we use the symmetry of the matrices $\mathbf{A}_i$ and integrate by parts using the differential factor $D^\alpha v_{l,x_i}$ as follows:

$$\begin{aligned}
H_1^{(2)} &= -\frac{1}{2} \int_{B_j} \frac{\partial}{\partial x_i} A_{i,lk} D^\alpha v_k D^\alpha v_l dx \\
&\leq C\|D\mathbf{A}\|_{\infty;S_T} \|\mathbf{v}\|^2_{n,j} \leq C\|\mathbf{A}\|_{n_o} \|\mathbf{v}\|^2_{n,j},
\end{aligned}$$

where $(4.4)'$ has been used.

Proceeding to estimate $H_1^{(3)}$ we have

$$H_1^{(3)} \leq \|\mathbf{v}\|_{n,j} \sum_{\substack{|\beta|+|\gamma|=|\alpha| \\ |\beta|,|\gamma|\neq 0}} c_{\beta,\gamma} \left\{ \int_{B_j} |D^\beta \mathbf{A}|^2 |D^\gamma D\mathbf{v}|^2 dx \right\}^{\frac{1}{2}}. \qquad (4.9)$$

If $|\beta| = 1$, then $|\gamma| = |\alpha| - 1$ and

$$\int_{B_j} |D^\beta \mathbf{A}|^2 |D^\gamma D\mathbf{v}|^2 dx \le \|D\mathbf{A}\|_{\infty;S_T} \|\mathbf{v}\|_{n,j} \le C\|\mathbf{A}\|_{n_o} \|\mathbf{v}\|_{n,j}.$$

To deal with the cases when $|\beta| \ge 2$, $|\gamma| = |\alpha| - 2$, we recall the following particular case of the Sobolev embedding Theorem (see Adams [5], Theorem 4.12, Case C, Part I and III). Sobolev spaces of higher order have been defined in Section 1c of the Complements of Chapter 9.

Theorem 4.2. *Let m denote a positive integer and let η be a multi-index of size $|\eta| \le m$. There exists a constant γ depending only upon m and N, such that for every $\varphi \in W^{m,2}(\mathbb{R}^N)$*

$$\left(\int_{\mathbb{R}^N} |\varphi|^q dx \right)^{\frac{1}{q}} \le \gamma \sum_{|\eta| \le m} \left(\int_{\mathbb{R}^N} |D^\eta \varphi|^2 dx \right)^{\frac{1}{2}}$$

where, as long as $2m < N$,

$$q = \frac{2N}{N - 2m}.$$

Next, for $|\beta| \ge 2$ we estimate the term in braces on the right-hand side of (4.9) as follows

$$\begin{aligned} &\left[\int_{B_j} |D^\beta \mathbf{A}|^2 |D^\gamma D\mathbf{v}|^2 \, dx \right]^{\frac{1}{2}} \\ &\qquad \le \left[\int_{B_j} |D^\gamma D\mathbf{v}|^{\frac{2N}{N-2(n-|\gamma|-1)}} \, dx \right]^{\frac{N-2(n-|\gamma|-1)}{2N}} \\ &\qquad\quad \cdot \left[\int_{B_j} |D^\beta \mathbf{A}|^{\frac{N}{2(n-|\gamma|-1)}} \, dx \right]^{\frac{2(n-|\gamma|-1)}{2N}}. \end{aligned} \tag{4.10}$$

We estimate the first integral on the right-hand side of (4.10) by making use of Theorem 4.2. This is possible since

$$n - |\gamma| - 1 < n_o - |\gamma| - 1 = \left\lfloor \frac{N+1}{2} \right\rfloor - |\gamma| \le \frac{N}{2}, \qquad \text{as } |\gamma| \ge 1.$$

We obtain

$$\begin{aligned} &\left[\int_{B_j} |D^\gamma D\mathbf{v}|^{\frac{2n}{N-2(n-|\gamma|-1)}} \, dx \right]^{\frac{N-2(n-|\gamma|-1)}{2N}} \\ &\qquad \le C \sum_{|\eta| \le n-|\gamma|-1} \left[\int_{B_j} |D^\eta D^\gamma D\mathbf{v}|^2 \, dx \right]^{\frac{1}{2}} \le C\|\mathbf{v}\|_{n,j}. \end{aligned}$$

To estimate the second integral on the right-hand side of (4.10), rewrite the exponent as

$$\frac{2N}{N-2(\frac{N}{2}-n+|\gamma|+1)}=\frac{2N}{N-2(\frac{N}{2}-|\beta|+1)}\,,\qquad \text{since } \ |\beta|+|\gamma|=|\alpha|=n.$$

Assume first that the number $m=\frac{N}{2}-|\beta|+1$ is an integer. Since $|\beta|\ge 2$, $m<\frac{N}{2}$ and by Theorem 4.2 we have

$$\left[\int_{B_j}|D^\beta\mathbf{A}|^{\frac{N}{2(n-|\gamma|-1)}}\,dx\right]^{\frac{2(n-|\gamma|-1)}{2N}}\le\sum_{|\eta|\le n_o}\left[\int_{\mathbb{R}^N}|D^\eta\mathbf{A}|^2\,dx\right]^{\frac{1}{2}}\equiv\|\mathbf{A}\|_{n_o}.$$

Proceeding in this fashion we find that all the integrals making up $H_1^{(3)}$, are majorized by $C\|\mathbf{A}\|_{n_o}\|\mathbf{v}\|_{n,j}$. We conclude that

$$H_1^{(3)}\le C\|\mathbf{A}\|_{n_o}\|\mathbf{v}\|_{n,j}^2.$$

We proceed to estimate $H_1^{(1)}$. We have

$$H_1^{(1)}\le C\|\mathbf{v}\|_{n,j}\left(\int_{B_j}|D^\alpha\mathbf{A}|^2|D\mathbf{v}|^2\,dx\right)^{\frac{1}{2}}.$$

If $|\alpha|=n_o$, by (4.4)$'$

$$\begin{aligned}&\left(\int_{B_j}|D^\alpha\mathbf{A}|^2|D\mathbf{v}|^2dx\right)^{\frac{1}{2}}\\&\quad\le\|D\mathbf{v}\|_{\infty;B_j}\left(\int_{B_j}|D^\alpha\mathbf{A}|^2dx\right)^{\frac{1}{2}}\le C\|\mathbf{A}\|_{n_o}\|\mathbf{v}\|_{n_o,j}.\end{aligned}$$

If $|\alpha|=n\le n_o-1$

$$\begin{aligned}&\left(\int_{B_j}|D^\alpha\mathbf{A}|^2|D\mathbf{v}|^2dx\right)^{\frac{1}{2}}\\&\quad\le\left(\int_{B_j}|D\mathbf{v}|^{\frac{2N}{N-2(n-1)}}\,dx\right)^{\frac{N-2(n-1)}{2N}}\left(\int_{\mathbb{R}^N}|D^\alpha\mathbf{A}|^{\frac{N}{n-1}}\,dx\right)^{\frac{n-1}{2N}}\\&\quad\le\|\mathbf{v}\|_{n,j}\left(\int_{\mathbb{R}^N}|D^\alpha\mathbf{A}|^{\frac{2N}{N-2(\frac{N}{2}-n+1)}}\,dx\right)^{\frac{n-1}{2N}}\le\|\mathbf{A}\|_{n_o}\|\mathbf{v}\|_{n,j}.\end{aligned}$$

Therefore, in either case

$$H_1^{(1)}\le C\|\mathbf{A}\|_{n_o}\|\mathbf{v}\|_{n,j}^2.$$

We conclude that

$$H_1 \le CT\|\mathbf{A}\|_{n_o}\|\mathbf{v}\|_{n,j}^2.$$

The estimate of H_2 is carried out exactly as in the estimation of $H_1^{(3)}$, except that now all the terms contain at least one derivative of ζ_j, and therefore, they contain at least a factor of ε. The net result is

$$H_1 \le C\varepsilon\|\mathbf{A}\|_{n_o}\|\mathbf{v}\|_{n,j}\left(\int_0^T \|\mathbf{u}_\varepsilon(t)\|_{n,j}\,dt\right)^{\frac{1}{2}}.$$

4.3 Proof of Theorem 4.1

Carrying these estimates in (4.8) we deduce

$$\begin{aligned}\|\mathbf{v}\|_{n,j}^2+\varepsilon\int_0^T \|\mathbf{v}(t)\|_{n+1,j}^2\,dt &\le \|\mathbf{v}_o\|_{n,j}^2 + CT\|\mathbf{B}\|_{n_o}\\ &+ CT\left\{\|\mathbf{A}\|_{n_o}+\|\mathbf{B}\|_{n_o}\right\}\|\mathbf{v}\|_{n,j}^2\\ &+ C\varepsilon\left\{\|\mathbf{A}\|_{n_o}+\|\mathbf{B}\|_{n_o}\right\}\left(\int_0^T \|\mathbf{u}_\varepsilon(t)\|_{n,j}\,dt\right).\end{aligned}\tag{4.11}$$

We first choose T to be so small that

$$CT\left\{\|\mathbf{A}\|_{n_o}+\|\mathbf{B}\|_{n_o}\right\} \le \frac{1}{2}.\tag{4.12}$$

Then we recall the definition of the cutoff function ζ_j to conclude that

$$C_o\|\mathbf{u}_\varepsilon\|_{n,j-1} \le \|\mathbf{v}\|_{n,j} \le C_1\|\mathbf{u}_\varepsilon\|_{n,j}$$

for two constants C_i, $i=0,1$, depending only upon N. Substituting this in (4.11) gives

$$\begin{aligned}\|\mathbf{u}_\varepsilon\|_{n,j-1}^2+\varepsilon\int_0^T \|\mathbf{u}_\varepsilon(t)\|_{n+1,j-1}^2\,dt\\ \le C\left(1+\|\mathbf{u}_o\|_{n_o}\right)+\varepsilon\int_0^T \|\mathbf{u}_\varepsilon(t)\|_{n,j}^2\,dt.\end{aligned}$$

We iterate these estimates for n_o, $n_o-1,\dots,1$, 0 and $j-1=n_o$, $j-2,\dots,$ and obtain

$$\|\mathbf{u}_\varepsilon\|_{n_o,n_o}^2 \le C\left(1+\|\mathbf{u}_o\|_{n_o}\right)+\iint_{Q_\varepsilon(T)} |\mathbf{u}_\varepsilon|^2\,dxd\tau.$$

To estimate the last integral, multiply the first of (4.5) by $u_{\varepsilon l}$, add over $l=1,2,\dots,m$ and integrate by parts by using the homogeneous boundary conditions of $\mathbf{u}$. The term involving the matrices $\mathbf{A}_i$ is treated by means of the symmetry device repeatedly used earlier. We get

$$\begin{aligned}&\operatorname*{ess\,sup}_{0\le t\le T}\int_{\{|x|<\frac{2n_o}{\varepsilon}\}}|\mathbf{u}_\varepsilon|^2dx\\ &\qquad\le C\left(1+\|\mathbf{u}_o\|_{n_o}\right)+CT\|\mathbf{A}\|_{n_o}\operatorname*{ess\,sup}_{0\le t\le T}\int_{\{|x|<\frac{2n_o}{\varepsilon}\}}|\mathbf{u}_\varepsilon|^2dx.\end{aligned}$$

Thus, if T is chosen as in (4.12), this last estimate implies that

$$\|\mathbf{u}_\varepsilon\|_{n_o,n_o}\le C\left(1+\|\mathbf{u}_o\|_{n_o}\right)$$

for $C=C(N)$.

The proof of the theorem is concluded by means of a standard limiting process. In particular, we find a unique solution $\mathbf{u}:S_T\to\mathbb{R}^m$ satisfying

$$\begin{cases}\sup_{0<t<T}\|\mathbf{u}(t)\|_{n_o}\le C\left(1+\|\mathbf{u}_o\|_{n_o}\right),\\ \|\mathbf{u}_t\|_{\infty;S_T}\le C\left(1+\|\mathbf{u}_o\|_{n_o}\right),\\ TC\left(\|\mathbf{A}\|_{n_o}+\|\mathbf{B}\|_{n_o}\right)\le\frac{1}{2},\end{cases}\tag{4.13}$$

where the estimate on $\mathbf{u}_t$ follows from the system. ∎

5 Existence of Solutions: The Nonlinear Theory

Consider the Cauchy problem

$$\begin{cases}u_{l,t}+A_{i,kl}(x,t,u)u_{k,x_i}=B_l(x,t,u)\,, & \text{in } S_T\\ u_l(\cdot,0)=u_{o,l}\quad l=1,2,\dots,m.\end{cases}\tag{5.1}$$

On the coefficients $A_{i,kl}$, the forcing term B_l, and the initial datum $\mathbf{u}_o$ we assume

$$\begin{aligned}&\|\mathbf{u}_o\|_{n_o}<\infty,\\ &\|\mathbf{A}\|_{n_o}+\|\mathbf{B}\|_{n_o}\le\gamma\left(1+\|\mathbf{v}\|_{n_o}^s\right),\end{aligned}\tag{5.2}$$

where $\mathbf{v}:S_T\to\mathbb{R}^m$ is a smooth vector satisfying $\|\mathbf{v}\|_{n_o}<\infty$, and for given positive constants $\gamma\,,s$. Here the norms $\|\cdot\|_{n_o}$, with $n_o\equiv\left\lfloor\frac{N+1}{2}\right\rfloor+1$ have been defined in (4.2).

Theorem 5.1. *There exists a positive time T_o depending upon $\|\mathbf{u}_o\|_{n_o}$ such that the Cauchy problem (5.1) has a unique solution in the strip S_{T_o}.*

Proof. For $T>0$, let X_T denote the Banach space

$$X_T\equiv L^\infty\left(0,T;W^{n_o,2}(\mathbb{R}^N)\right).$$

Fix $\mathbf{v}\in X_T$ and let $\mathcal{F}(\mathbf{v})\in X_{T_o}$ be the unique solution of the Cauchy problem

$$\begin{cases} \frac{\partial}{\partial t}\mathcal{F}_l(\mathbf{v}) + A_{i,kl}(x,t,\mathbf{v})\mathcal{F}_{k,x_i}(\mathbf{v}) = B_l(x,t,\mathbf{v})\,, & \text{in } S_{T_o} \\ \mathcal{F}_l(\mathbf{v})(\cdot,0) = \mathbf{u}_{o,l}, \quad l = 1,2,\dots,m, \end{cases} \tag{5.3}$$

where, according to (4.12) and (5.2)

$$T_o\left(1 + \|\mathbf{v}\|_{n_o}^s\right) \le \frac{1}{2C\gamma} \equiv \gamma_1.$$

Moreover, by (4.13), $\mathcal{F}(\mathbf{v})$ satisfies

$$\begin{cases} \|\mathcal{F}(\mathbf{v})\|_{n_o} \le C\left(1 + \|\mathbf{u}_o\|_{n_o}\right), & \text{in } S_{T_o} \\ \|\frac{\partial}{\partial t}\mathcal{F}(\mathbf{v})\|_{\infty,S_{T_o}} \le C\left(1 + \|\mathbf{u}_o\|_{n_o}\right), \end{cases} \tag{5.4}$$

where C depends only upon N and the constant γ in (5.2). Thus, if we pick $\mathbf{v}$ out of the ball

$$B_o \equiv \{\mathbf{v} \in X_{T_o} : \|\mathbf{v}\|_{n_o} \le C\left(1 + \|\mathbf{u}_o\|_{n_o}\right) \equiv K_o\}$$

and choose

$$T_o = \frac{\gamma_1}{1 + K_o^s},$$

then $\mathcal{F}$ maps the ball B_o in X_{T_o} into itself. Next, we observe that by possibly choosing T_o even smaller, $\mathcal{F}(\cdot)$ is a contraction in $L^\infty\left(0,T_o;L^2(\mathbb{R}^N)\right)$. This follows exactly as in the proof of the uniqueness Theorem 3.1 of Section 3. Therefore, $\mathcal{F}(\cdot)$ has a fixed point $\mathbf{u}$ limit, in $L^\infty\left(0,T_o;L^2(\mathbb{R}^N)\right)$, of a sequence $\{\mathbf{u}_j\}$, which is constructed by iteration. By (5.4)

$$\begin{cases} \sup_{0\le t\le T_o}\|\mathbf{u}_j(t)\|_{n_o} \le C\left(1 + \|\mathbf{u}_o\|_{n_o}\right) \\ \|\frac{\partial}{\partial t}\mathbf{u}_j\|_{\infty;S_{T_o}} \le C\left(1 + \|\mathbf{u}_o\|_{n_o}\right); \qquad j = 1,2,\dots, \end{cases}$$

where C depends only upon N and γ. It follows that a subsequence converges to $\mathbf{u}$ in $C^{1-\varepsilon}\left(0,T_o;C^1(\mathcal{K})\right)$ for all $\varepsilon \in (0,1)$ and all compact subsets $\mathcal{K} \subset \mathbb{R}^N$. However, since the whole sequence $\{\mathbf{u}_j\} \longrightarrow \mathbf{u}$ in $L^2(S_{T_o})$, we have that the entire sequence converges to $\mathbf{u}$ in $C^{1-\varepsilon}\left(0,T_o;C^1(\mathcal{K})\right)$ for all $\varepsilon \in (0,1)$ and all compact subsets $\mathcal{K} \subset \mathbb{R}^N$.

Let $\varphi \in C^1(0,T_o)$, write (5.3) for $\mathbf{v} = \mathbf{u}_j$ and $\mathcal{F}(\mathbf{u}_j) = \mathbf{u}_{j+1}$, multiply by φ, and integrate over $(0,t)\,, \forall\, 0 < t < T_o$. We obtain

$$\begin{aligned} \mathbf{u}_{j+1l}(x,t)\varphi(t) - \int_0^t \mathbf{u}_{j+1l}(x,\tau)\varphi_\tau(\tau)\,d\tau + \int_0^t A_{i,kl}(x,\tau,\mathbf{u}_j)\mathbf{u}^{j+1k,x_i}\varphi(\tau)\,d\tau \\ = \int_0^t B_l(x,\tau,\mathbf{u}_j)\varphi(\tau)\,d\tau + \mathbf{u}_{o,l}(x)\varphi(0). \end{aligned}$$

Letting $j \to \infty$ proves the theorem. ■

Remark 5.1 The previous argument is briefly sketched in § 2 of Lax [161]. For a more thorough discussion Lax himself refers to Courant and Hilbert [38], vol. II, Chapter VI, § 10, Par. 2, where Schauder is credited with the original idea [224, 225]. A somewhat similar argument is given in Courant and Lax [39].

6 An Interlude: Counterexamples to Uniqueness in the Linear Case

In the previous sections, the symmetry of the matrices $\mathbf{A}_i$ has played a central role in ensuring the existence and uniqueness of solutions of the Cauchy problem (2.1), at least for a short time interval $[0, T]$.

It is well-known that real, symmetric matrices are diagonalizable, even when they have multiple eigenvalues. A fortiori, this holds for matrices, whose eigenvalues are all real and distinct, since in such a case the symmetry plays no role.

Even remaining within the framework of linear systems, the situation becomes more challenging, as far as uniqueness is concerned, if we have multiple real eigenvalues, and therefore, the corresponding linear system is only weakly hyperbolic. Besides their intrinsic mathematical interest, as we have seen in Section 2.3, weakly hyperbolic systems arise in physical applications, such as magnetohydrodynamics. In view of Holmgren's theorem (see Chapter 1), the lack of uniqueness cannot occur when the coefficients are analytic functions.

The first example of the non-uniqueness of solutions to a linear system with smooth but non-analytic coefficients, was given by Andrzej Pliś in 1954 [204]. A few months later, in a completely independent way, Ennio DeGiorgi published a similar result for a linear evolution equation, which is weakly hyperbolic [45]. Without pretending to fully develop the theory of weakly hyperbolic systems and equations, in the following we present the counterexample by Pliś.

We consider a 2×2 system, we work in a neighborhood of the origin $O(0, 0)$, and for simplicity, we disregard lower order terms. Hence, we have

$$\frac{\partial u_i}{\partial t} = \sum_{j=1}^{2} a_{ij}(x,t) \frac{\partial u_j}{\partial x}, \qquad i = 1, 2.$$

We are going to build the coefficients a_{ij} and the corresponding solutions u_q with $i, j, q = 1, 2$, such that they all belong to $C^\infty(\mathbb{R}^2)$ and vanish together with all their derivatives for $t = 0$, without being identically zero in a neighborhood of the origin.

Define

$$A(s) = \frac{\displaystyle\int_0^s \exp\left(\frac{1}{t(t-\frac{1}{4})}\right) dt}{\displaystyle\int_0^{\frac{1}{4}} \exp\left(\frac{1}{t(t-\frac{1}{4})}\right) dt}, \quad s \in [0, \frac{1}{4}],$$

$$A_{n,k}(t) = A\left(n(n+1)t - n - \frac{k}{4}\right), \quad n = 1, 2, \ldots, \ k = 0, 1, 2, 3,$$

$$F_n(x,t) = \frac{\left(\dfrac{1}{3}\right)^{n+1} e^{-2^n} A'_{n,1}(t)}{1 + \dfrac{1}{3} e^{-2^n} A_{n,1}(t)(\sin 3^{n+1}x \sin 3^n x - \cos 3^{n+1}x \cos 3^n x)},$$

$$G_n(x,t) = \frac{\left(\dfrac{1}{3}\right)^{n} e^{-7\cdot 2^n} A'_{n,3}(t)}{1 + 3e^{-7\cdot 2^n}(1 - A_{n,3}(t))(\sin 3^{n+1}x \sin 3^n x - \cos 3^{n+1}x \cos 3^n x)}.$$

For $t \le 0$, we let

$$\begin{cases} a_{ij}(x,t) = 0 & i,j = 1,2, \\ u_q(x,t) = 0 & q = 1,2; \end{cases}$$

for $t \in \left(\frac{1}{n+1}, \frac{1}{n+1} + \frac{1}{4n(n+1)}\right]$, $n = 1,2,\ldots$, we let

$$\begin{aligned} a_{11}(x,t) &= a_{12}(x,t) = a_{22}(x,t) = 0, \\ a_{21}(x,t) &= \frac{2A'_{n,0}(t)}{3^{n+1}}, \\ u_1(x,t) &= e^{-2^{n+1}} \cos 3^{n+1}x, \\ u_2(x,t) &= (1 - 2A_{n,0}(t)) e^{-2^{n+1}} \sin 3^{n+1}x; \end{aligned}$$

for $t \in \left(\frac{1}{n+1} + \frac{1}{4n(n+1)}, \frac{1}{n+1} + \frac{2}{4n(n+1)}\right]$, $n = 1,2,\ldots$, we let

$$\begin{aligned} a_{11}(x,t) &= -F_n(x,t) \sin 3^{n+1}x \cos 3^n x, \\ a_{12}(x,t) &= -F_n(x,t) \cos 3^{n+1}x \cos 3^n x, \\ a_{21}(x,t) &= -F_n(x,t) \sin 3^{n+1}x \sin 3^n x, \\ a_{22}(x,t) &= -F_n(x,t) \cos 3^{n+1}x \sin 3^n x, \\ u_1(x,t) &= e^{-2^{n+1}} \cos 3^{n+1}x + A_{n,1}(t) e^{-3\cdot 2^n} \cos 3^n x, \\ u_2(x,t) &= -e^{-2^{n+1}} \sin 3^{n+1}x + A_{n,1}(t) e^{-3\cdot 2^n} \sin 3^n x; \end{aligned}$$

for $t \in \left(\frac{1}{n+1} + \frac{2}{4n(n+1)}, \frac{1}{n+1} + \frac{3}{4n(n+1)}\right]$, $n = 1,2,\ldots$, we let

$$\begin{aligned} a_{11}(x,t) &= a_{22}(x,t) = 0, \\ a_{12}(x,t) &= \frac{2\cdot 2^n A'_{n,2}(t)}{3^n}, \\ a_{21}(x,t) &= -\frac{2\cdot 2^n A'_{n,2}(t)}{3^n}, \\ u_1(x,t) &= e^{-2^n(2+6A_{n,2}(t))} \cos 3^{n+1}x + e^{-2^n(3-2A_{n,2}(t))} \cos 3^n x, \\ u_2(x,t) &= -e^{-2^n(2+6A_{n,2}(t))} \sin 3^{n+1}x + e^{-2^n(3-2A_{n,2}(t))} \sin 3^n x; \end{aligned}$$

for $t \in \left(\frac{1}{n+1} + \frac{3}{4n(n+1)}, \frac{1}{n}\right]$, $n = 1,2,\ldots$, we let

$$\begin{aligned}
a_{11}(x,t) &= G_n(x,t)\cos 3^{n+1}x \sin 3^n x,\\
a_{12}(x,t) &= -G_n(x,t)\cos 3^{n+1}x \cos 3^n x,\\
a_{21}(x,t) &= -G_n(x,t)\sin 3^{n+1}x \sin 3^n x,\\
a_{22}(x,t) &= G_n(x,t)\sin 3^{n+1}x \cos 3^n x,\\
u_1(x,t) &= e^{-8\cdot 2^n}(1-A_{n,3}(t))\cos 3^{n+1}x + e^{-2^n}\cos 3^n x,\\
u_2(x,t) &= -e^{-8\cdot 2^n}(1-A_{n,3}(t))\sin 3^{n+1}x + e^{-2^n}\sin 3^n x;
\end{aligned}$$

finally, for $t \geq 1$, we let

$$\begin{aligned}
a_{11}(x,t) = a_{12}(x,t) &= a_{21}(x,t) = a_{22}(x,t) = 0,\\
u_1(x,t) &= e^{-2}\cos 3x,\\
u_2(x,t) &= e^{-2}\sin 3x.
\end{aligned}$$

It is rather easy to check that $A = A(s)$ is of class C^∞ in $[0, \frac{1}{4}]$, $A(0) = 0$, $A(\frac{1}{4}) = 1$, $A(s) \in [0,1]$ for $s \in [0, \frac{1}{4}]$ and $\dfrac{d^k A(s)}{ds^k} = 0$ in $s = 0$ and $s = \frac{1}{4}$ for any $k = 1, 2, \ldots$.

As a consequence, for $i, j, q = 1, 2$, the coefficients a_{ij} and the functions u_q belong to C^∞ for $t \neq 0$. Moreover, it is not hard to show that, again for any $i, j, q = 1$, 2,

$$\begin{aligned}
\left|\frac{\partial^{k+p} u_q}{\partial t^k \partial x^p}\right| &\leq L_{kp} e^{-2^n}[n(n+1)]^k 3^{(n+1)p} 2^{kn},\\
\left|\frac{\partial^{k+p} a_{ij}}{\partial t^k \partial x^p}\right| &\leq M_{kp} \max\left\{e^{-2^n}[n(n+1)]^{k+1} 3^{np}, \left(\frac{2}{3}\right)^n [n(n+1)]^{k+1}\right\},
\end{aligned}$$

for $t \in \left(\frac{1}{n+1}, \frac{1}{n}\right]$ and for certain constants L_{kp}, M_{kp}, independent of n.

From these inequalities, it follows that

$$\begin{aligned}
&\lim_{t\to 0} \frac{\partial^{k+p} u_q}{\partial t^k \partial x^p} = 0 \qquad k, p = 0, 1, 2, \ldots,\ q = 1, 2,\\
&\lim_{t\to 0} \frac{\partial^{k+p} a_{ij}}{\partial t^k \partial x^p} = 0 \qquad k, p = 0, 1, 2, \ldots,\ i, j = 1, 2,
\end{aligned}$$

which is precisely what we wanted. It can be easily verified that the coefficients a_{ij} and the functions u_q satisfy the system, and that both u_1 and u_2 do not vanish in any neighborhood of the origin. Since $u_1 = u_2 \equiv 0$ is also a solution to the system in $\mathbb{R}^2$, we have shown the non-uniqueness of solutions.

DeGiorgi's counterexample is built along similar lines. The only difference is that he deals with a linear equation in $\mathbb{R}^2$ of eighth order, namely

$$\frac{\partial^8 u}{\partial t^8} = a(x,t)\frac{\partial^4 u}{\partial x^4} + b(x,t)\frac{\partial^2 u}{\partial x^2} + c(x,t)u,$$

instead of a system. Also in this case, the coefficients a, b, c, and the solution u belong to C^∞ and

$$\lim_{t\to 0}\frac{\partial^{k+p}u}{\partial t^k\partial x^p}=0 \qquad k,p=0,1,2,\dots,$$
$$\lim_{t\to 0}\frac{\partial^{k+p}a}{\partial t^k\partial x^p}=\lim_{t\to 0}\frac{\partial^{k+p}b}{\partial t^k\partial x^p}=\lim_{t\to 0}\frac{\partial^{k+p}c}{\partial t^k\partial x^p}=0 \qquad k,p=0,1,2,\dots.$$

These counterexamples were systematically studied by Paul Cohen in 1960 [35].[4] Although Cohen's work on non-uniqueness remained in the preprint stage, and therefore, is not so easily accessible, it was nevertheless promoted a few years later by Lars Hörmander [127], Theorem 8.9.2.

It is interesting to present Cohen's results in their original formulation. In order to do that we need some preliminary notation.

We consider $P_o(t,x_1,\dots,x_k)$ a homogeneous polynomial of degree n such that the coefficient of t^n is 1. We let

$$P(D;x,t)=\sum_{|\alpha|=n}a_\alpha(x,t)D_\alpha,\quad D_\alpha=\frac{\partial^{\alpha_o}}{\partial t^{\alpha_o}}\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}}\cdots\frac{\partial^{\alpha_k}}{\partial x_k^{\alpha_k}}$$

be a differential operator of order n, such that

$$P(D;0,0)=P_o(\frac{\partial}{\partial t},\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_k}).$$

If $Q(D)$ is a differential operator with variable coefficients of order less than or equal to $n-1$, uniqueness is said to hold for the equation

$$P(D)u+Q(D)u=0, \tag{6.1}$$

if any solution u that vanishes for $t\le 0$ is identically zero, or at least vanishes in some neighborhood of the origin, depending only upon the equation, and not on u.

We have the following.

Theorem 6.1. *Let $P(t,x_1,\dots,x_k)$ be a homogeneous polynomial of degree n with the coefficient of t^n equal to* 1*. If there exist real numbers $\xi_1,\dots,\xi_k$ not all zero such that the equation*

$$P(t,\xi_1,\dots,\xi_k)=0$$

has a real root $t=\tau$ of multiplicity $r>1$, then there exists a differential operator Q of order less than or equal to $n-r+1$ with C^∞ coefficients such that (6.1) *has a C^∞ solution u, and both u and Q vanish for $t\le 0$.*

In Cohen's construction, both u and Q in Theorem 6.1 are complex-valued, as it is clear from the proof, which is given in the Complements. However, it is possible to give a corresponding statement for a real equation.

---

[4] A few years later Cohen shifted his scientific interests toward Mathematical Logic, and ended up winning the Fields Medal for his groundbreaking contributions to the Continuum Hypothesis.

Theorem 6.2. *Let P be a differential operator with real coefficients of degree n. If P has a real characteristic of multiplicity $r > 1$, then a real function u and a real differential operator Q, both vanishing for $t \leq 0$, can be found such that $P(D)u + Q(D)u = 0$ and the degree of Q is less than or equal to $n - r + 2$. If $r \geq 3$, Q is thus an operator of degree $\leq n - 1$, whereas if $r = 2$, Q involves a term of order n of the form $a(x,t)\dfrac{\partial^n}{\partial x^n}$.*

In both Pliś [204] and DeGiorgi [45] there is no *quantitative* consideration whatsoever about the regularity of the coefficients, beyond their belonging to C^∞, as the authors simply content themselves with showing that the counterexamples actually exist. In DeGiorgi [46], published in the same year, although not in an explicit, direct way, the author gives a quantitative statement about conditions for non-uniqueness. Indeed, given the set $R = [a_1, a_2] \times [t_1, t_2]$ he studies the Cauchy problem

$$\begin{cases} \dfrac{\partial^m u}{\partial t^m} = \displaystyle\sum_{h=0}^{m-1} \sum_{k=0}^{\lfloor \alpha(m-h) \rfloor} c_{hk}(x,t) \frac{\partial^{h+k} u}{\partial t^h \partial x^k} \quad \text{in} \quad R, \\ \dfrac{\partial^h u}{\partial t^h}(x, t_1) = 0 \quad x \in [a_1, a_2], \ h = 0, 1, \ldots, m-1, \end{cases}$$

where $\alpha \in (0,1)$. As in the previous sections $\lfloor \gamma \rfloor$ denotes the largest integer that does not exceed γ, the coefficients c_{hk} are continuous in R, and of class C^∞ with respect to the variable x. The main result of the paper is a uniqueness theorem. Referring to DeGiorgi [45], the author points out that, if for any $\epsilon > 0$, there exists a positive ρ such that

$$\lim_{n \to \infty} \frac{\rho^n}{n!} \left| \frac{\partial^n c_{hk}}{\partial x^n} \right|^{\alpha(1-\epsilon)} = 0,$$

then there is no uniqueness of solutions to the Cauchy problem.

A more thorough characterization of the conditions under which a lack of uniqueness can be expected, is given by Jean Leray [167], in the context of the study that, together with his collaborators Yujiro Ohya and Lucien Waelbroeck, he carried out about weakly hyperbolic systems (see, for example, Leray and Ohya, and Ohya [168, 195]).

For functions $b_j = b_j(x,t)$, $j = 0, \ldots, q$ with $b_j \in C^\infty(\mathbb{R}^2)$, Leray defines the differential operator

$$b(x,t,D_x) \stackrel{\text{def}}{=} \sum_{j=0}^{q} b_j(x,t) \frac{\partial^j}{\partial x^j}$$

and considers the Cauchy problem in $\mathbb{R} \times (0,T]$

$$\begin{cases} \dfrac{\partial^m u}{\partial t^m} = b(x,t,D_x)u, \\ \dfrac{\partial^h u}{\partial t^h}(x,0) = 0 \quad x \in \mathbb{R}, \ h = 0, 1, \ldots, m-1, \end{cases}$$

where $m \geq 1$ and $q \geq 1$. Leray quantifies in a precise way the previous qualitative statement by DeGiorgi. In order to do so, he relies on Gevrey classes, which we now introduce. We follow the same notation used by Leray [167].

For a sequence $\{\Phi_s\}_{s\in\mathbb{N}} \subset \mathbb{R}$, we define the *formal* series

$$\Phi(\rho) = \sum_{s=0}^{\infty} \frac{\rho^s}{s!} \Phi_s, \tag{6.2}$$

and we say that $\Phi(\rho) >> 0$, whenever $\Phi_s \geq 0$ for any $s \geq 0$. Given a strip $\mathbb{R} \times [0,T]$ and a real-valued function u defined in the strip, we let $S_t = \mathbb{R} \times \{t\}$ with $t \in [0,T]$, and introduce the quasi-norm

$$|u, S_t| \overset{\text{def}}{=} \sup_{x\in\mathbb{R}} |u(x,t)|$$

and the *formal* quasi-norm

$$|D^{h,\infty} u, S_t, \rho| \overset{\text{def}}{=} \sum_{s=0}^{\infty} \frac{\rho^s}{s!} \sup_j \left| \frac{\partial^{j+s} u}{\partial t^j \partial x^s}, S_t \right|, \quad \text{where} \quad 0 \leq j \leq h.$$

This is a formal series, of the kind just defined in (6.2). We say that $\Phi = \Phi(\rho)$ belongs to the *formal* Gevrey class $\Gamma^{(\alpha)}$ if there exists a positive constant c, which depends on Φ, such that

$$\Phi_s \leq c^s (s!)^\alpha,$$

and that u belongs to the Gevrey class $\gamma^{h,(\alpha)}$, if there exists a formal series $\Phi = \Phi(\rho)$, independent of t, such that

$$|D^{h,\infty} u, S_t, \rho| \leq \Phi(\rho) \in \Gamma^{(\alpha)}.$$

Leray points out that in DeGiorgi's counterexample one has $m = 8$, $q = 4$ and $\forall h \in \mathbb{N}$

$$u \in \gamma^{h,(\alpha)}, \qquad b_j \in \gamma^{h,(\alpha)},\ j = 0, 2, 4,$$

where $\alpha > \frac{m}{q} = 2$. It is precisely this value of α above the critical threshold $\frac{m}{q}$ that allows for more than one solution to exist.

7 Back to Quasi-Linear First-Order Strictly Hyperbolic Systems

Assume for simplicity that $\mathbf{B} \equiv 0$, and that the matrices $\mathbf{A}_i$ depend only on $\mathbf{u}$, but not on (x,t). With a small extra effort, the results of Section 3 and of Section 5 (see Majda [180], Chapter 2 for the details), can be recast in the following way.

Theorem 7.1. *Assume that* $\mathbf{A}_i$ *are of class* C^1, $\mathbf{u}_o \in W^{n_o,2}(\mathbb{R}^N;\mathbb{R}^m)$ *with* $n_o > \frac{N}{2}+1$ *and* $u_o \in G_1$, $\bar{G}_1 \subset\subset G \subset \mathbb{R}^m$. *Then there exists a time interval* $[0,T]$ *with* $T>0$ *such that the initial value problem* (2.1) *has a unique classical solution* $\mathbf{u} \in C^1(\mathbb{R}^N \times [0,T])$, *with* $\mathbf{u}(x,t) \in G_2$, $\bar{G}_2 \subset\subset G$ *for* $(x,t) \in \mathbb{R}^N \times [0,T]$. *Furthermore,*

$$\mathbf{u} \in C\left([0,T];W^{n_o,2}(\mathbb{R}^N;\mathbb{R}^m)\right) \cap C^1\left([0,T];W^{n_o-1,2}(\mathbb{R}^N;\mathbb{R}^m)\right),$$

and T *depends on* $\|\mathbf{u}_o\|_{n_o}$ *and* G_1.

As we have already seen in Chapter 7 dealing with quasi-linear hyperbolic equations, in general, T is small and therefore, it is of little use in the applications. We need to develop a theory of weak solutions for systems, similar to what we did in Chapter 7 for single equations.

Here, we limit ourselves to the case $N=1$, and for some $T>0$ consider the Cauchy problem for the system of conservation laws

$$\begin{aligned} \partial_t \mathbf{u} + \partial_x \mathbf{f}(\mathbf{u}) &= 0 \qquad \forall\,(x,t) \in S_T \equiv \mathbb{R} \times (0,T), \\ \mathbf{u}(\cdot,0) &= \mathbf{u}_o \qquad \text{for } x \in \mathbb{R}. \end{aligned} \tag{7.1}$$

Here, $\mathbf{f}:\mathbb{R}^m \to \mathbb{R}^m$ with $m>1$ is sufficiently smooth. Moreover, throughout the next sections, we always assume the system to be strictly hyperbolic, that is, the matrix $\mathbf{A} = \nabla_{\mathbf{u}}\mathbf{f}$ has real and distinct eigenvalues for all values of its argument $\mathbf{u}$. We have the following.

Definition 7.2. *The function* $\mathbf{u}: S_T \to \mathbb{R}^m$ *is a weak solution of* (7.1) *if*

$$\int_0^{+\infty}\int_{\mathbb{R}} [\mathbf{u}\cdot\partial_t\boldsymbol{\varphi} + \mathbf{f}\cdot\partial_x\boldsymbol{\varphi}]\,dxdt + \int_{\mathbb{R}} \mathbf{u}_o(x)\cdot\boldsymbol{\varphi}(x,0)\,dx = 0$$

for all $\boldsymbol{\varphi} \in C^1(\mathbb{R}\times[0,T]);\mathbb{R}^m)$, *which vanish for* $|x|+t$ *large enough.*

It is a matter of straightforward computations (see Section 7c of the Complements) to prove the following.

Proposition 7.1 *Let* Γ *be a regular curve within* S_T, *parametrized by* $x = x(t)$. *Let* u *be of class* C^1 *away from* Γ, *having continuous left and right limits* $\mathbf{u}_l$, $\mathbf{u}_r$ *on* Γ. *Then,* $\mathbf{u}$ *is a weak solution to the Cauchy problem* (7.1) *if and only if*

- $\mathbf{u}$ *is a classical solution away from* Γ;
- *Along* Γ, $\mathbf{u}$ *satisfies the Rankine–Hugoniot jump condition*

$$s[u_i] = [f_i(\mathbf{u})], \qquad i = 1,\dots,m, \tag{7.2}$$

where $s = \dfrac{dx}{dt}$.

Notice that if we eliminate s from (7.2), we have $m-1$ relations.

When $N=m=1$, the assumption $a'(u)>0$ plays a fundamental role for the solvability of the initial value problem (6.1) of Chapter 7. We want to see what the equivalent condition is for systems. Recall that we have m distinct real eigenvalues, which we order

$$\lambda_1(\mathbf{u}) < \lambda_2(\mathbf{u}) < \cdots < \lambda_m(\mathbf{u}), \tag{7.3}$$

with the corresponding right and left eigenvectors $\mathbf{r}_k$ and $\mathbf{l}_k$, which respectively solve

$$(\lambda_k \mathbf{I} + \mathbf{A})\mathbf{r}_k = 0, \qquad \mathbf{l}_k^T(\lambda_k \mathbf{I} + \mathbf{A}) = 0.$$

Eigenvalues and eigenvectors are smooth functions of $\mathbf{u}$. The points of the set $G \subset \mathbb{R}^m$ where $\mathbf{u}$ takes value, are referred to as *states* in the following.

Definition 7.3. *The k-th characteristic family is said to be* genuinely nonlinear *in a region $D \subset \mathbb{R}^m$ if $\mathbf{r}_k \cdot \nabla_{\mathbf{u}}\lambda_k \neq 0$. We can then normalize $\mathbf{r}_k$ such that $\mathbf{r}_k \cdot \nabla_{\mathbf{u}}\lambda_k = 1$.*
On the other hand, if $\mathbf{r}_k \cdot \nabla_{\mathbf{u}}\lambda_k = 0$, we say that the k-th characteristic field is linearly degenerate*.*

By the continuity of first-order derivatives, and by the normalization we have assumed on $\mathbf{r}_k$, since $\mathbf{r}_k \cdot \nabla_{\mathbf{u}}\lambda_k$ is the directional derivative of λ_k along the integral curve of the vector $\mathbf{r}_k$, in the genuinely nonlinear case the value of λ_k is strictly increasing along such a curve. On the other hand, for the same reasons, in the linearly degenerate case, λ_k is constant along every integral curve of $\mathbf{r}_k$.

If $m=1$, we have $u_t + [f(u)]_x = 0$, so that

$$\lambda = a = f'(u), \quad r = 1, \quad r \cdot \nabla_{\mathbf{u}}\lambda = f''(u) > 0,$$

and in this case the genuine nonlinearity reduces to convexity for f.

7.1 A First Example

Consider the so-called p-system (see Smoller [239], Chapter 17), which is a model for isentropic gas dynamics in Lagrangian coordinates (here, v stands for the specific volume and w for the velocity):

$$\begin{bmatrix} v \\ w \end{bmatrix}_t + \begin{bmatrix} 0 & -1 \\ p'(v) & 0 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix}_x = 0, \quad t>0,\, x \in \mathbb{R},$$

where $p' < 0$, and $p'' > 0$. In applications one usually takes

$$p(v) = \kappa v^{-\gamma}, \quad \gamma > 1, \quad \kappa = \frac{(\gamma-1)^2}{4\gamma}.$$

From the physical point of view, the first equation represents the conservation of mass, and the second one the conservation of momentum. Since the temperature is kept constant and energy must be added to the system, the equation corresponding to the conservation of energy is missing. The eigenvalues are

$$\lambda_1 = -\sqrt{-p'(v)}, \qquad \lambda_2 = \sqrt{-p'(v)},$$

and the corresponding eigenvectors, which we have not normalized, are

$$\mathbf{r}_1 = \begin{bmatrix} \frac{1}{\sqrt{-p'(v)}} \\ 1 \end{bmatrix}, \qquad \mathbf{r}_2 = \begin{bmatrix} -\frac{1}{\sqrt{-p'(v)}} \\ 1 \end{bmatrix}.$$

Hence,

$$\mathbf{r}_2 \cdot \nabla\lambda_2 = \begin{bmatrix} -\dfrac{1}{\sqrt{-p'(v)}} & 1 \end{bmatrix} \begin{bmatrix} -\frac{p''(v)}{2\sqrt{-p'(v)}} \\ 0 \end{bmatrix} = -\frac{p''(v)}{2p'(v)} > 0,$$

and an analogous result holds for $\mathbf{r}_1 \cdot \nabla\lambda_1$. It is apparent that both characteristic fields are genuinely nonlinear.

7.2 A Second Example

The complete gas dynamics in Eulerian coordinates is described by the system

$$\begin{bmatrix} \rho \\ u \\ \epsilon \end{bmatrix}_t + \begin{bmatrix} u & \rho & 0 \\ \frac{1}{\rho}\frac{\partial p}{\partial \rho} & u & 0 \\ 0 & \frac{p}{\rho} & u \end{bmatrix} \begin{bmatrix} \rho \\ u \\ \epsilon \end{bmatrix}_x = 0, \quad t > 0,\, x \in \mathbb{R}, \tag{7.4}$$

where ρ is the density, u is the flow velocity, p is the pressure, and ϵ is the specific internal energy. The constitutive relation is $p = p(\rho, \epsilon)$, and for thermodynamical reasons $\frac{\partial p}{\partial \rho} > 0$, $\frac{\partial^2 p}{\partial \rho^2} > 0$.

The eigenvalues are

$$\lambda_1 = u - \sqrt{\frac{\partial p}{\partial \rho}}, \quad \lambda_2 = u, \quad \lambda_3 = u + \sqrt{\frac{\partial p}{\partial \rho}};$$

for λ_2, we have $\mathbf{r}_2 = [0\ 0\ 1]^T$, and we easily obtain $\mathbf{r}_2 \cdot \nabla\lambda_2 = 0$. Hence, the second characteristic field is linearly degenerate.

8 Lax Shock Conditions

If deriving the equivalent statements of the Rankine–Hugoniot jump conditions was rather straightforward, more care is required, in order to formulate the right shock conditions for systems.

Suppose we have a line of discontinuity $\varGamma$ and consider a point P on $\varGamma$. Let $\mathbf{u}_l$ and $\mathbf{u}_r$ be the values at P of the solution on the left and on the right of the discontinuity $\varGamma$ respectively.

Issuing from P in the positive t direction, draw those characteristics with respect to $\mathbf{u}_l$ that stay to the left of $\varGamma$, and those with respect to $\mathbf{u}_r$ which stay to the right of $\varGamma$. As we saw in (1.5), the characteristics are given by

$$\frac{dx}{dt} = \lambda_k \qquad k = 1, \ldots, m \quad \text{or equivalently,} \;\; \frac{dt}{dx} = \frac{1}{\lambda_k},$$

and the eigenvalues are ordered, as in (7.3). We assume that the discontinuity moves with speed s.

If in P

$$\lambda_1(\mathbf{u}_l) < \cdots < \lambda_j(\mathbf{u}_l) < s < \lambda_{j+1}(\mathbf{u}_l) < \cdots < \lambda_m(\mathbf{u}_l),$$

this means that we have j characteristics that stay to the left for $t > t_P$ and therefore, cannot be traced back to the initial time. Hence, we need k additional conditions to fully determine the problem.

On the other hand, if in P

$$\lambda_1(\mathbf{u}_r) < \cdots < \lambda_k(\mathbf{u}_r) < s < \lambda_{k+1}(\mathbf{u}_r) < \cdots < \lambda_m(\mathbf{u}_r),$$

this means that we have $m-k$ characteristics, which stay to the right for $t > t_P$ and therefore, cannot be traced back to the initial time. Hence, we need $m-k$ additional conditions. Altogether, we need $j+m-k$ extra conditions. As we have remarked in the previous section, the Rankine–Hugoniot jump conditions amount to $m-1$ equations.

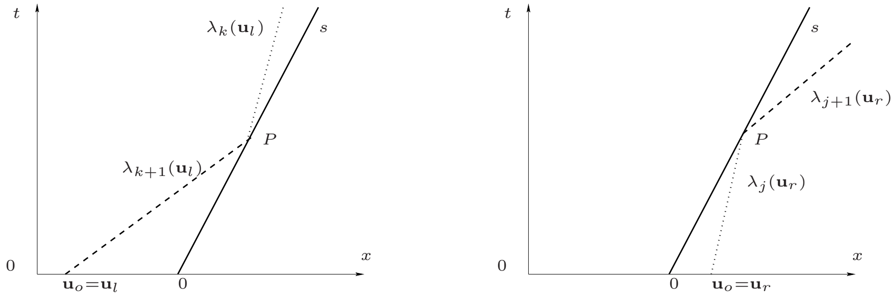
t
$\lambda_k(\mathbf{u}_l)$
s
P
$\lambda_{k+1}(\mathbf{u}_l)$
0
x
$\mathbf{u}_o = \mathbf{u}_l$
0
t
s
$\lambda_{j+1}(\mathbf{u}_r)$
P
$\lambda_j(\mathbf{u}_r)$
0
x
0
$\mathbf{u}_o = \mathbf{u}_r$

Fig. 8.1: The characteristics running into the shock wave, and appearing from it; the solid line represents the shock wave, the other lines are the characteristics from and to the left, and from and to the right, respectively.

The Lax shock conditions for a system tell us that the number of additional conditions required by the characteristics should be equal to the number of equations provided by (7.2), that is

$$j + m - k = m - 1 \quad \Rightarrow \quad k = j + 1.$$

We can then conclude that for some index k with $1 \le k \le m$, we must have

$$\begin{cases} \lambda_{k-1}(\mathbf{u}_l) < s < \lambda_k(\mathbf{u}_l), \\ \quad \lambda_k(\mathbf{u}_r) < s < \lambda_{k+1}(\mathbf{u}_r). \end{cases} \tag{8.1}$$

We can go one step further, and rewrite (8.1) as

$$\begin{cases} \quad \lambda_k(\mathbf{u}_r) < s < \lambda_k(\mathbf{u}_l), \\ \lambda_{k-1}(\mathbf{u}_l) < s < \lambda_{k+1}(\mathbf{u}_r). \end{cases} \tag{8.2}$$

Hence, for one and only one index k is the discontinuity speed s intermediate to the characteristic speeds λ_k on both sides of Γ. A discontinuity across which both (7.2) and (8.2) are satisfied is called a k-shock, where k is its index.

In the previous argument we have assumed that the characteristic speeds with respect to the state on the right and to the state on the left are either smaller or larger than the propagation speed of the discontinuity. Nevertheless, it could happen that the discontinuity Γ is a characteristic curve with respect to the state on one side. Such a discontinuity is called a *contact discontinuity*, and we will get back to them later on.

Coming back to example (7.4), if we let for simplicity $c \overset{\text{def}}{=} \sqrt{\frac{\partial p}{\partial \rho}}$ (c can be interpreted as the sound speed) and in (8.1) we take $k = 3$, we have

$$\begin{aligned} u_l < s < u_l + c_l, \\ u_r + c_r < s. \end{aligned}$$

This means that the the shock speed s is larger than the flow speed both on the right and on the left side of the shock. Consequently, the particles of the gas cross the shock from the right to the left. More specifically, the Lax shock conditions in this case tell us that the shock is supersonic with respect to the right and subsonic with respect to the left. Also notice that this can only be satisfied if $u_r + c_r < u_l + c_l$. If this cannot be satisfied, the shock corresponding to $k = 3$ is not allowed, as it would violate the Lax conditions. As discussed in Courant and Friedrichs [37], Chapter 65, and in Weyl [275], this property of shocks is completely equivalent to the second law of thermodynamics, which requires the entropy of particles to increase as they cross the shock.

From here on, we say that a weak solution is *admissible* if both (7.2) and either (8.1) or (8.2) hold.

9 Shocks

Now we want to investigate the sets of all states $\mathbf{u}_r$, which can be connected to a given state $\mathbf{u}_l$ through a k-shock on the right.

Theorem 9.1. [159] *Any given state* $\mathbf{u}_l$ *can be connected to a one-parameter family of states* $\mathbf{u}_r = \mathbf{u}(\epsilon)$, $\epsilon \in (-\epsilon_o, 0]$, $\mathbf{u}(0) = \mathbf{u}_l$, *on the right through a* k*-shock, provided that the* k*-family of characteristics is genuinely nonlinear. Moreover, the shock speed* s *is a smooth function of* ϵ *too.*

Proof. Since $\mathbf{u}_l$ is fixed, the actual unknowns are the states $\mathbf{u}_r$, which must satisfy the Rankine–Hugoniot conditions (7.2) and the Lax shock conditions (8.1). In particular, as we have already remarked, (7.2) represent $m-1$ relations between $\mathbf{u}_l$ and $\mathbf{u}_r$, which in a general way can be written as

$$\mathbf{g}(\mathbf{u}_l, \mathbf{u}_r) = 0 \quad \text{with} \quad \mathbf{g} : \mathbb{R}^{2m} \to \mathbb{R}^{m-1}.$$

If $\mathbf{u}_l$ is fixed, we end up with $m-1$ relations among m variables; the genuine nonlinearity allows us to apply the implicit function theorem, and conclude that there exists a smooth, one-parameter family of states $\mathbf{u}_r = \mathbf{u}(\epsilon)$, $\mathbf{u}(0) = \mathbf{u}_l$. The shock speed s is also a smooth function of ϵ, namely $s = s(\epsilon)$.

A priori, the parameter ϵ takes values in a symmetric interval $[-\epsilon_o, \epsilon_o]$ about the origin. We now show that ϵ can only be nonpositive. In the following, for simplicity, instead of writing $\mathbf{u}_r$, we simply write $\mathbf{u}$. If we differentiate condition (7.2), we obtain

$$\frac{ds}{d\epsilon}[\mathbf{u} - \mathbf{u}_l] + s\frac{d\mathbf{u}}{d\epsilon} = \nabla \mathbf{f}\,\frac{d\mathbf{u}}{d\epsilon} = \mathbf{A}\frac{d\mathbf{u}}{d\epsilon}. \tag{9.1}$$

If we compute (9.1) at $\epsilon = 0$, we have

$$s(0)\frac{d\mathbf{u}}{d\epsilon}(0) = \mathbf{A}(\mathbf{u}_l)\frac{d\mathbf{u}}{d\epsilon}(0)$$

and this can be satisfied by $\frac{d\mathbf{u}}{d\epsilon}(0) \neq 0$, only if $s(0)$ is an eigenvalue of $\mathbf{A}(\mathbf{u}_l)$, and $\frac{d\mathbf{u}}{d\epsilon}(0)$ is the corresponding eigenvector, that is

$$s(0) = \lambda_k(\mathbf{u}_l), \qquad \frac{d\mathbf{u}}{d\epsilon}(0) = \alpha\,\mathbf{r}_k(\mathbf{u}_l) \tag{9.2}$$

for some $\alpha \in \mathbb{R}$, $\alpha \neq 0$. By a proper parametrization, without loss of generality, we may assume $\alpha = 1$. If we differentiate (9.1) with respect to ϵ once more, we have

$$\frac{d^2 s}{d\epsilon^2}[\mathbf{u} - \mathbf{u}_l] + 2\frac{ds}{d\epsilon}\frac{d\mathbf{u}}{d\epsilon} + s\frac{d^2\mathbf{u}}{d\epsilon^2} = \frac{d\mathbf{A}}{d\epsilon}\frac{d\mathbf{u}}{d\epsilon} + \mathbf{A}\frac{d^2\mathbf{u}}{d\epsilon^2}.$$

If we evaluate it at $\epsilon = 0$, we have

$$2\frac{ds}{d\epsilon}(0)\frac{d\mathbf{u}}{d\epsilon}(0) + s(0)\frac{d^2\mathbf{u}}{d\epsilon^2}(0) = \frac{d\mathbf{A}}{d\epsilon}(0)\frac{d\mathbf{u}}{d\epsilon}(0) + \mathbf{A}(\mathbf{u}_l)\frac{d^2\mathbf{u}}{d\epsilon^2}(0).$$

Taking account of (9.2) yields

$$2\frac{ds}{d\epsilon}(0)\mathbf{r}_k(\mathbf{u}_l) + \lambda_k(\mathbf{u}_l)\frac{d^2\mathbf{u}}{d\epsilon^2}(0) = \frac{d\mathbf{A}}{d\epsilon}(0)\mathbf{r}_k(\mathbf{u}_l) + \mathbf{A}(\mathbf{u}_l)\frac{d^2\mathbf{u}}{d\epsilon^2}(0). \tag{9.3}$$

In order to evaluate the right-hand side, we rely on $\mathbf{r}_k$ being an eigenvalue, that is

$$\mathbf{A}(\mathbf{u})\mathbf{r}_k(\mathbf{u}) = \lambda_k(\mathbf{u})\mathbf{r}_k(\mathbf{u}).$$

If we differentiate with respect to ϵ,

$$\frac{d\lambda_k}{d\epsilon}\mathbf{r}_k + \lambda_k\frac{d\mathbf{r}_k}{d\epsilon} = \mathbf{A}\frac{d\mathbf{r}_k}{d\epsilon} + \frac{d\mathbf{A}}{d\epsilon}\mathbf{r}_k, \tag{9.4}$$

compute (9.4) at $\epsilon = 0$ and subtract it from (9.3), we have

$$\begin{aligned}\lambda_k(\mathbf{u}_l)\left[\frac{d^2\mathbf{u}}{d\epsilon^2}(0) - \frac{d\mathbf{r}_k}{d\epsilon}(0)\right] + \left[2\frac{ds}{d\epsilon}(0) - \frac{d\lambda_k}{d\epsilon}(0)\right]\mathbf{r}_k(\mathbf{u}_l)\\ =\mathbf{A}(\mathbf{u}_l)\left[\frac{d^2\mathbf{u}}{d\epsilon^2}(0) - \frac{d\mathbf{r}_k}{d\epsilon}(0)\right].\end{aligned} \tag{9.5}$$

Taking the scalar product with the left eigenvalue $\mathbf{l}_k(\mathbf{u}_l)$, we end up with

$$2\frac{ds}{d\epsilon}(0) - \frac{d\lambda_k}{d\epsilon}(0) = 0.$$

By the genuine nonlinearity and its normalization, we have

$$\frac{\lambda_k}{d\epsilon}(0) = \nabla\lambda_k(\mathbf{u}_l)\frac{d\mathbf{u}}{d\epsilon}(0) = \nabla\lambda_k(\mathbf{u}_l)\mathbf{r}_k(\mathbf{u}_l) = 1.$$

Hence,

$$2\frac{ds}{d\epsilon}(0) = \frac{d\lambda_k}{d\epsilon}(0) = 1, \tag{9.6}$$

and (9.5) reduces to

$$\mathbf{A}(\mathbf{u}_l)\left[\frac{d^2\mathbf{u}}{d\epsilon^2}(0) - \frac{d\mathbf{r}_k}{d\epsilon}(0)\right] = \lambda_k(\mathbf{u}_l)\left[\frac{d^2\mathbf{u}}{d\epsilon^2}(0) - \frac{d\mathbf{r}_k}{d\epsilon}(0)\right],$$

which implies that

$$\frac{d^2\mathbf{u}}{d\epsilon^2}(0) - \frac{d\mathbf{r}_k}{d\epsilon}(0) = \beta\mathbf{r}_k(\mathbf{u}_l)$$

with $\beta \in \mathbb{R}$. By a further change of variables, without loss of generality, we may assume that $\beta = 0$, so that

$$\frac{d^2\mathbf{u}}{d\epsilon^2}(0) = \frac{d\mathbf{r}_k}{d\epsilon}(0) = \nabla\mathbf{r}_k(0)\cdot\frac{d\mathbf{u}}{d\epsilon}(0) = \nabla\mathbf{r}_k(0)\cdot\mathbf{r}_k(\mathbf{u}_l).$$

From (9.6) we have

$$\begin{aligned}\lambda_k(\mathbf{u}_r) = \lambda_k(\epsilon) = \lambda_k(0) + \epsilon + O(\epsilon^2) = \lambda_k(\mathbf{u}_l) + \epsilon + O(\epsilon^2),\\ s(\epsilon) = s(0) + \frac{\epsilon}{2} + O(\epsilon^2) = \lambda_k(\mathbf{u}_l) + \frac{\epsilon}{2} + O(\epsilon^2),\end{aligned}$$

and (8.2) yields

$$\lambda_k(\mathbf{u}_l) = \lambda_k(0) > s(\epsilon) > \lambda_k(\epsilon) = \lambda_k(\mathbf{u}_r);$$

combining the previous two statements necessarily implies that $\epsilon < 0$.

Collecting all the changes of variables introduced so far, we see that the parametrization is normalized by

$$\frac{d\mathbf{u}}{d\epsilon}(0) = \mathbf{r}_k(\mathbf{u}_l), \qquad \frac{d^2\mathbf{u}}{d\epsilon^2}(0) = \mathbf{r}_k(\mathbf{u}_l) \cdot \nabla \mathbf{r}_k(\mathbf{u}_l),$$

and obviously $\mathbf{r}_k \cdot \nabla\lambda_k = 1$. ■

Remark 9.1 It is apparent that

$$s = \frac{\lambda_k(\mathbf{u}_r) + \lambda_k(\mathbf{u}_l)}{2} + O(\epsilon^2),$$

so that the speed of the shock, up to terms of order $O(\epsilon^2)$, is the arithmetic mean of the speeds in the front and in the back. In the following, to denote the Rankine–Hugoniot speed of the shock, instead of writing s, we use the notation $\lambda_k(\mathbf{u}_l, \mathbf{u}_r)$.

Remark 9.2 Since $\frac{d\mathbf{u}}{d\epsilon}(0) = \mathbf{r}_k(\mathbf{u}_l)$, we can write $\mathbf{u}_r = \mathbf{u}_l + \epsilon\mathbf{r}_k(\mathbf{u}_l) + O(\epsilon^2)$.

9.1 An Example

Consider the so-called chromatography system (see Rhee et al. [212] for its physical justification)

$$\begin{cases} \partial_t v + \partial_x \left(\dfrac{v}{1+v+w} \right) = 0, \\ \partial_t w + \partial_x \left(\dfrac{w}{1+v+w} \right) = 0. \end{cases} \tag{9.7}$$

Its eigenvalues are

$$\lambda_1 = \frac{1}{(1+v+w)^2}, \qquad \lambda_2 = \frac{1}{1+v+w},$$

and the corresponding eigenvectors are

$$\mathbf{r}_1 = -\frac{1}{\sqrt{v^2+w^2}} \begin{pmatrix} v \\ w \end{pmatrix}, \qquad \mathbf{r}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Notice that we cannot have $v = w = 0$. By simple computations, we have

$$\mathbf{r}_1 \cdot \nabla\lambda_1 = \frac{2v+2w}{\sqrt{v^2+w^2}(1+v+w)^3} \neq 0.$$

Hence, the first characteristic field is genuinely nonlinear. If we let $\mathbf{u}_l = (u_1, u_2)$, in order to determine the one-parameter family of states $\mathbf{u}_r$ claimed by Theorem 9.1, we can rely on the Rankine–Hugoniot conditions (7.2). Since $\mathbf{f} = \begin{bmatrix} \frac{v}{1+v+w} \\ \frac{w}{1+v+w} \end{bmatrix}$, we have

$$\begin{cases} s[v - u_1] = \left[\dfrac{v}{1+v+w} - \dfrac{u_1}{1+u_1+u_2}\right], \\ s[w - u_2] = \left[\dfrac{w}{1+v+w} - \dfrac{u_2}{1+u_1+u_2}\right], \end{cases}$$

whence we easily deduce $w - u_2 = \dfrac{u_2}{u_1}(v - u_1)$; this can be rewritten as

$$\begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \epsilon \begin{bmatrix} \frac{u_1}{\sqrt{u_1^2+u_2^2}} \\ \frac{u_2}{\sqrt{u_1^2+u_2^2}} \end{bmatrix},$$

and corresponds to the parametrization given in Remark 9.2. Notice that in this case we do not have the remainder $O(\epsilon^2)$, since the Taylor expansion reduces to the first two terms.

On the other hand, we have

$$\mathbf{r}_2 \cdot \nabla \lambda_2 = 0;$$

hence, we conclude that the second characteristic field is linearly degenerate, and we cannot apply the previous construction. We will get back to this case in Section 11.

10 Centered Rarefaction Waves

We look for an important class of *continuous solutions*, the so-called *centered rarefaction waves*. Namely, we look for functions that depend only on the ratio $\dfrac{x - x_o}{t - t_o}$, (x_o, t_o) being the center of the wave. Once more, we assume that the characteristic k-field we work with is genuinely nonlinear.

Assume for simplicity that $(x_o, t_o) = (0, 0)$, and let $\mathbf{u}(x, t) = \mathbf{h}(\frac{x}{t})$. We want to determine the expression of $\mathbf{h}$. Since

$$\partial_t u = -\frac{x}{t^2}\mathbf{h}'\left(\frac{x}{t}\right), \qquad \partial_x u = \frac{1}{t}\mathbf{h}'\left(\frac{x}{t}\right),$$

the system $\partial_t \mathbf{u} + \mathbf{A}(\mathbf{u})\partial_x \mathbf{u} = 0$ becomes $-\dfrac{x}{t^2}\mathbf{h}' + \dfrac{1}{t}\mathbf{A}(\mathbf{h})\mathbf{h}' = 0$, whence

$$[\mathbf{A}(\mathbf{h}) - \xi \mathbf{I}]\mathbf{h}' = 0, \qquad \text{with} \qquad \xi = \frac{x}{t},$$

which is satisfied assuming

$$\xi = \lambda_k(\mathbf{h}(\xi)), \qquad \mathbf{h}' = \alpha\, \mathbf{r}_k(\mathbf{h}).$$

If we differentiate with respect to ξ, taking into account that the field is genuinely nonlinear, we have

$$1 = \frac{d}{d\xi}\lambda_k(\mathbf{h}(\xi)) = \nabla\lambda_k \cdot \mathbf{h}' = \alpha\, \mathbf{r}_k(\mathbf{h}) \cdot \nabla\lambda_k(\mathbf{h}) = \alpha.$$

Hence, $\alpha = 1$, and we have $\mathbf{h}' = \mathbf{r}_k(\mathbf{h})$. If we let for simplicity $\lambda_k = \lambda_k(\mathbf{u}_l)$, then the Cauchy problem

$$\begin{cases} \quad \mathbf{h}' = \mathbf{r}_k(\mathbf{h}), \\ \mathbf{h}(\lambda_k) = \mathbf{u}_l, \end{cases}$$

has a unique solution $\mathbf{h} = \mathbf{h}(\xi)$. A priori, the proper neighborhood of λ_k, where $\mathbf{h}$ is defined, is centered. The following argument shows that $\xi \in (\lambda_k, \lambda_k + \epsilon_o)$.

Take $\epsilon > 0$ so small that $\mathbf{h}$ is defined at $\lambda_k + \epsilon$, let $\mathbf{u}_r \overset{\text{def}}{=} \mathbf{h}(\lambda_k + \epsilon)$, and consider the following piecewise smooth function defined in $\mathbb{R} \times [0, +\infty)$:

$$\mathbf{u}(x,t) = \begin{cases} \mathbf{u}_l & \text{for } \ x < \lambda_k t, \\ \mathbf{h}\left(\dfrac{x}{t}\right) & \text{for } \ \lambda_t \le x \le (\lambda_k + \epsilon)t, \\ \mathbf{u}_r & \text{for } \ x > (\lambda_k + \epsilon)t. \end{cases}$$

The function $\mathbf{u}$ satisfies the system of conservation laws in each of the three different regions, and by construction is continuous across the lines separating the regions. We say that the states $\mathbf{u}_l$ and $\mathbf{u}_r$ are connected through a centered k-rarefaction wave. Notice that it is essential here that $\epsilon > 0$. Therefore, we have the following result.

Theorem 10.1. [159] *Assume that the system $\partial_t\mathbf{u} + \partial_x\mathbf{f}(\mathbf{u}) = 0$ is strictly hyperbolic and that the characteristic k-field is genuinely nonlinear. Then, given a state $\mathbf{u}_l$, there exists a one-parameter family of states $\mathbf{u}_r = \mathbf{u}_r(\epsilon)$ with $\epsilon \in [0, \epsilon_o]$ for a proper ϵ_o, which can be connected to $\mathbf{u}_l$ through a centered k-rarefaction wave.*

Remark 10.1 In other terms, a centered k-rarefaction wave through $\mathbf{u}_l$ is given by the unique solution of the Cauchy Problem

$$\begin{cases} \dfrac{d\mathbf{h}}{d\epsilon} = \mathbf{r}_k(\mathbf{h}), \\ \mathbf{h}(0) = \mathbf{u}_l. \end{cases}$$

Notice that if $|\mathbf{r}_k(\mathbf{u})| = 1$, then ϵ plays the role of arc length.

Theorems 9.1–10.1 can be connected in a unified statement.

Theorem 10.2. *Assume that the system* $\partial_t \mathbf{u} + \partial_x \mathbf{f}(\mathbf{u}) = 0$ *is strictly hyperbolic and that the characteristic* k*-field is genuinely nonlinear. Then, a given state* $\mathbf{u}_l$ *can be connected to a one-parameter family of states* $\mathbf{u}_r = \mathbf{u}_r(\epsilon)$ *with* $\epsilon \in [-\epsilon_o, \epsilon_o]$ *for a proper* ϵ_o*, on the right of* $\mathbf{u}_l$ *through either a* k*-shock or a centered* k*-rarefaction wave.*

We also have an important statement concerning the regularity.

Proposition 10.1 *The one-parameter family* $\mathbf{u} = \mathbf{u}(\epsilon)$ *is twice continuously differentiable with respect to* ϵ*.*

Proof. It suffices to prove the continuity of $\frac{d\mathbf{u}}{\epsilon}$ and $\frac{d^2\mathbf{u}}{d\epsilon^2}$ at $\epsilon = 0$. For both k-shocks and k-rarefaction waves, since we have always chosen $\alpha = 1$, we have

$$\frac{d\mathbf{u}}{d\epsilon}(0^+) = \frac{d\mathbf{u}}{d\epsilon}(0^-) = \mathbf{r}_k(\mathbf{u}_l).$$

Coming to the second-order derivative, from $\mathbf{h}' = \mathbf{r}(\mathbf{h})$, we obtain

$$\frac{d^2\mathbf{u}}{d\epsilon^2}(0^+) = \mathbf{h}''(\lambda_k) = \nabla \mathbf{r}_k(0) \cdot \mathbf{r}_k(\mathbf{u}_l)$$

and from the regularity of $\mathbf{r}_k$ we conclude. ■

10.1 An Example

When considering (9.7), we have seen that the characteristic 1-field is genuinely nonlinear. Hence, if we let $\mathbf{h} = \begin{pmatrix} v \\ w \end{pmatrix}$, we obtain the system of ordinary differential equations

$$\begin{cases} v' = -\dfrac{v}{\sqrt{v^2 + w^2}}, \\ w' = -\dfrac{w}{\sqrt{v^2 + w^2}}, \\ v(0) = u_1, \\ w(0) = u_2, \end{cases}$$

where the independent variable $\epsilon \in [0, \epsilon_o)$. It is apparent that

$$\frac{v'}{v} = \frac{w'}{w} \quad \Rightarrow \quad |v| = K|w|,$$

and from the initial conditions we obtain $K = \dfrac{|u_1|}{|u_2|}$. Substituting back in the first equation of the system yields

$$v' = -\frac{v}{|v|} \frac{|u_1|}{\sqrt{u_1^2 + u_2^2}}.$$

By continuity, in a small right neighborhood of λ_1 v has the same sign as u_1, so that

$$v' = -\frac{u_1}{\sqrt{u_1^2+u_2^2}} \quad \Rightarrow \quad v = u_1 - \epsilon \frac{u_1}{\sqrt{u_1^2+u_2^2}} + O(\epsilon^2).$$

Moreover, the solution v can be extended on the whole interval $[0, +\infty)$. The same holds for w, so that we conclude

$$\mathbf{h}(\epsilon) = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \epsilon \begin{bmatrix} \frac{u_1}{\sqrt{u_1^2+u_2^2}} \\ \frac{u_2}{\sqrt{u_1^2+u_2^2}} \end{bmatrix},$$

exactly as in Section 9.1; hence, in this case the k-shock and the centered k-rarefaction wave coincide. The full solution is given by the following piecewise smooth function $\mathbf{u}$:

$$\mathbf{u}(x,t) = \begin{cases} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} & \text{for } x < \lambda_1 t, \\ \mathbf{h}\left(\dfrac{x}{t}\right) & \text{for } \lambda_1 t \le x \le (\lambda_1 + \epsilon_o)t, \\ \begin{bmatrix} u_1 - \epsilon_o \frac{u_1}{\sqrt{u_1^2+u_2^2}} \\ u_2 - \epsilon \frac{u_2}{\sqrt{u_1^2+u_2^2}} \end{bmatrix} & \text{for } x > (\lambda_1 + \epsilon_o)t, \end{cases}$$

for some $\epsilon_o > 0$.

11 Contact Discontinuities

We now deal with the case of a characteristic k-field, which is linearly degenerate, that is, $\mathbf{r}_k \cdot \nabla \lambda_k = 0$. In such a case we talk of *contact discontinuity.*

Given a state $\mathbf{u}_l$, consider the system that yields the k-rarefaction wave through $\mathbf{u}_l$, namely

$$\begin{cases} \mathbf{h}'(s) = \mathbf{r}_k(\mathbf{h}(s)) & s \in (-\epsilon_o, \epsilon_o), \\ \mathbf{h}(0) = \mathbf{u}_l, \end{cases}$$

and assume that for some ϵ in the interval $(-\epsilon_o, \epsilon_o)$ we have $\mathbf{u}_r = \mathbf{h}(\epsilon)$. By the linear degeneracy condition, the directional derivative of λ_k along $\mathbf{u}_r$ vanishes, that is, λ_k is constant along $\mathbf{u}_r$. If we choose $\lambda = \lambda_k(\mathbf{u}_l)$, the piecewise constant function given by

$$\mathbf{u}(x,t) = \begin{cases} \mathbf{u}_l & \text{if } x < \lambda t, \\ \mathbf{u}_r & \text{if } x > \lambda t, \end{cases}$$

gives a solution. Indeed, it trivially satisfies the system in the two regions.

Coming to the Rankine–Hugoniot condition, we have

$$\begin{aligned}\mathbf{f}(\mathbf{u}_r) - \mathbf{f}(\mathbf{u}_l) &= \int_0^\epsilon \nabla\mathbf{f}(\mathbf{h}(s))\mathbf{h}'(s)\,ds \\ &= \int_0^\epsilon \nabla\mathbf{f}(\mathbf{h}(s))\mathbf{r}_k(\mathbf{h}(s))\,ds \\ &= \int_0^\epsilon \lambda_k(\mathbf{u}_l)\mathbf{r}_k(\mathbf{h}(s))\,ds \\ &= \lambda_k(\mathbf{u}_l)[\mathbf{h}(\epsilon) - \mathbf{h}(0)] = s[\mathbf{u}_r - \mathbf{u}_l]\end{aligned}$$

and therefore, (7.2) is satisfied. Finally, as far as the Lax shock conditions are concerned, we simply have

$$\lambda_k(\mathbf{u}_r) = s = \lambda_k(\mathbf{u}_l).$$

Notice that in this case, the sign of ϵ is immaterial. Finally, by their own construction, the k-shock and the k-rarefaction wave coincide, when we deal with a contact discontinuity.

There is a strong parallelism with what one usually does when dealing with discontinuities in the linear case (this is one of the motivation for denoting this case as *linearly degenerate*). Indeed, consider the Cauchy problem

$$\begin{cases}\partial_t\mathbf{u} + \mathbf{A}\partial_x\mathbf{u} = 0, & x \in \mathbb{R}, t > 0, \\ \qquad \mathbf{u}(x,0) = \mathbf{u}_o, & \end{cases} \tag{11.1}$$

where $\mathbf{A}$ is a strictly hyperbolic, $m \times m$ constant coefficient matrix, that is, the eigenvalues λ_k are all real and distinct. As usual, we denote the corresponding right and left eigenvectors with $\mathbf{r}_k$, $\mathbf{l}_k$.

The explicit solution of (11.1) can be written as a sum with respect to the basis given by the eigenvectors, namely

$$u(x,t) = \sum_{j=1}^m u_j(x,t)\mathbf{r}_j,$$

where the coefficients $u_j(x,t)$ are the solution of

$$\partial_t u_j + \lambda_j \partial_x u_j = 0;$$

if we take into account the initial condition $\mathbf{u}_o$, we have

$$u_j(x,t) = (\mathbf{l}_j, \mathbf{u}_o(x - \lambda_j t)) \quad \Rightarrow \quad \mathbf{u}(x,t) = \sum_{j=1}^m (\mathbf{l}_j, \mathbf{u}_o(x - \lambda_j t))\mathbf{r}_j.$$

This shows that the eigenvalues λ_j are the propagation velocities of the modes corresponding to $\mathbf{r}_j$.

If we let

$$\mathbf{u}_o = \begin{cases} \mathbf{u}_l & \text{for } x < 0, \\ \mathbf{u}_r & \text{for } x > 0, \end{cases}$$

the corresponding solution is

$$\mathbf{u}(x,t) = \begin{cases} \mathbf{u}_l & \text{for } x < \lambda_1 t, \\ \mathbf{a}_1 & \text{for } \lambda_1 t < x < \lambda_2 t, \\ \vdots \\ \mathbf{a}_{m-1} & \text{for } \lambda_{m-1} t < x < \lambda_m, \\ \mathbf{u}_r & \text{for } x > \lambda_m t, \end{cases}$$

where the vectors $\mathbf{a}_j$ are such that $\mathbf{a}_{j+1} - \mathbf{a}_j$ is proportional to $\mathbf{r}_j$, $j = 1, \ldots, m-1$. Hence, the initial discontinuity splits into m discontinuities, each one propagating at one of the characteristic speeds; we see the same kind of behavior when dealing with the solution of the Riemann problem for linearly degenerate characteristic fields in the next section.

11.1 An Example

When considering (9.7), we have seen that the characteristic 2-field is linearly degenerate. Hence, if we let $\mathbf{h} = \begin{pmatrix} v \\ w \end{pmatrix}$, we obtain the system of ordinary differential equations

$$\begin{cases} v' = \dfrac{1}{\sqrt{2}}, \\ w' = -\dfrac{1}{\sqrt{2}}, \\ v(0) = u_1, \\ w(0) = u_2, \end{cases}$$

where the independent variable $\epsilon \in \mathbb{R}$. Hence,

$$\mathbf{h} = \begin{bmatrix} u_1 + \frac{\epsilon}{\sqrt{2}} \\ u_2 - \frac{\epsilon}{\sqrt{2}} \end{bmatrix}, \qquad \mathbf{u}(x,t) = \begin{cases} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} & \text{if } x < \dfrac{1}{1+u_1+u_2} t, \\ \begin{bmatrix} u_1 + \frac{\epsilon}{\sqrt{2}} \\ u_2 - \frac{\epsilon}{\sqrt{2}} \end{bmatrix} & \text{if } x > \dfrac{1}{1+u_1+u_2} t, \end{cases}$$

and it is trivial to check that $\lambda_2(\mathbf{u}_l) = \lambda_2(\mathbf{u}_r)$.

12 The Riemann Problem

Consider a cylindrical tube, with a diameter small compared with its length. We assume that the tube is filled with a gas, separated by a thin membrane.

The gas is at rest on both sides of the membrane, but pressure and densities on the two sides differ.

At time $t = 0$ the separating membrane is instantaneously and completely removed, and the issue is to determine the motion of the gas for $t > 0$. If we let

$$[u_l \ \ \rho_l \ \ p_l]^T, \qquad [u_r \ \ \rho_r \ \ p_r]^T$$

denote speed, density, and pressure on the two sides of the membrane, and we define the specific energy $E \overset{\text{def}}{=} e + \frac{u^2}{2}$, where the internal energy e is given by a proper equation of state $e = e(p, \rho)$, we have to solve the system

$$\begin{cases} \partial_t \left(\dfrac{1}{\rho}\right) - \partial_x u = 0, \\ \partial_t u + \partial_x p = 0, \\ \partial_t E + \partial_x (pu) = 0, \end{cases} \qquad x \in \mathbb{R},\ t > 0, \tag{12.1}$$

with the initial condition

$$\begin{bmatrix} u \\ \rho \\ p \end{bmatrix} (x, 0) = \begin{cases} \begin{bmatrix} u_l \\ \rho_l \\ p_l \end{bmatrix} & \text{if } x < 0, \\ \begin{bmatrix} u_r \\ \rho_r \\ p_r \end{bmatrix} & \text{if } x > 0. \end{cases}$$

This problem was originally studied by Bernhard Riemann [218] and has been known by his name since then (Figure 12.2).

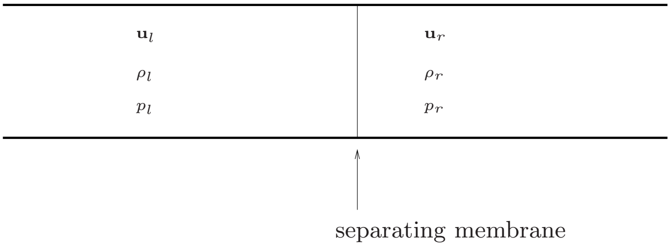

Fig. 12.2

We have already considered simplified examples in Section 6.2 and in Section 6.3c of Chapter 7, but we now deal with it to a greater extent. We get back to (12.1) in Section 13.

What we have developed in Sections 9–11 help us to build a solution to the Riemann problem for a general system of conservation laws $\partial_t \mathbf{u} + \partial_x \mathbf{f}(\mathbf{u}) = 0$, assuming that the initial condition is given by

$$\mathbf{u}_o(x) = \begin{cases} \mathbf{u}^- & \text{if } x < 0, \\ \mathbf{u}^+ & \text{if } x > 0. \end{cases}$$

Once a general solution for the Riemann problem is available, it represents a building block in the fundamental existence result for general initial data by Glimm [103], which we discuss in Section 14.

The solution of the Riemann problem must be a function of the ratio $\frac{x}{t}$. Indeed, if $\mathbf{u} = \mathbf{u}(x,t)$ is a weak solution with the given initial datum, for any positive constant α, the function $\mathbf{u}_\alpha = \mathbf{u}(\alpha x, \alpha t)$ must also be a weak solution of the Riemann problem with the same initial condition; in fact, the discontinuities are only shocks, and it takes the same datum as $t \to 0$. Since we reasonably expect to have a unique solution, we must have $\mathbf{u}(x,t) = \mathbf{u}(\alpha x, \alpha t)$, and this is allowed only if $\mathbf{u}(x,t) = \mathbf{u}(\frac{x}{t})$.

Solving the Riemann problem amounts to finding a set of intermediate states

$$\begin{aligned} \boldsymbol{\omega}_o &= \mathbf{u}^- \\ \boldsymbol{\omega}_1 &= \boldsymbol{\psi}_1(\epsilon_1, \boldsymbol{\omega}_o) \\ &\vdots \\ \boldsymbol{\omega}_m &= \mathbf{u}^+ = \boldsymbol{\psi}_m(\epsilon_m, \boldsymbol{\omega}_{m-1}) \end{aligned}$$

such that a pair of adjacent states $\boldsymbol{\omega}_{k-1}$, $\boldsymbol{\omega}_k$ can be connected by an elementary k-wave, that is, a k-shock, a k-rarefaction wave, or a k-contact singularity.

This construction can actually be realized if $\mathbf{u}^-$ and $\mathbf{u}^+$ are sufficiently close; indeed, in such a case we can apply the implicit function theorem, which ensures the existence of

$$\epsilon_1,\ \epsilon_2,\ \ldots,\ \epsilon_m$$

such that

$$\mathbf{u}^+ = \boldsymbol{\psi}_m(\epsilon_m, \boldsymbol{\omega}_{m-1}) \circ \cdots \circ \boldsymbol{\psi}_1(\epsilon_1, \mathbf{u}^-).$$

The complete solution is then obtained by patching together the solutions of the Riemann problems given by

$$\begin{cases} \partial_t \mathbf{u} + \partial_x \mathbf{f}(\mathbf{u}) = 0, \\ \qquad \mathbf{u}(x,0) = \boldsymbol{\omega}_{k-1} \quad \text{if } x < 0, \\ \qquad \mathbf{u}(x,0) = \boldsymbol{\omega}_k \quad \text{if } x > 0. \end{cases} \tag{12.2}$$

We proceed to constructing solutions to (12.2), which satisfy (7.2) and (8.1)–(8.2), and consist of a simple wave of the three different types discussed before.

We need to distinguish two alternatives:

1. The characteristic k-field is genuinely nonlinear and $\epsilon_k > 0$. Then, the solution of (12.2) consists of a centered k-rarefaction wave $\mathbf{h}_k$ and the characteristic speeds take value in the interval $[\lambda_k^-, \lambda_k^+]$ with $\lambda_k^- = \lambda_k(\boldsymbol{\omega}_{k-1})$, $\lambda_k^+ = \lambda_k(\boldsymbol{\omega}_k)$.

2. The characteristic k-field is either genuinely nonlinear and $\epsilon_k < 0$, or linearly degenerate with ϵ_k of an arbitrary sign. In such a case the solution of (12.2) is given either by an admissible shock or by a contact discontinuity, traveling with speed $\lambda_k^- \overset{\text{def}}{=} \lambda_k^+ \overset{\text{def}}{=} \lambda_k(\boldsymbol{\omega}_{k-1}, \boldsymbol{\omega}_k)$.

As briefly discussed above, the solution of the general problem can now be built patching together all the solutions of (12.2). In fact, if ϵ_1, ϵ_2, ..., ϵ_m are all sufficiently small, the speeds λ_k^+, λ_k^- remain close to the corresponding eigenvalues $\lambda_k(\boldsymbol{\omega}_{k-1})$ of the matrix $\mathbf{A}(\boldsymbol{\omega}_{k-1})$. Hence, by the strict hyperbolicity and continuity, we can suppose that

$$\lambda_1^- \le \lambda_1^+ \le \cdots \le \lambda_m^- \le \lambda_m^+,$$

and a piecewise smooth solution $\mathbf{u} : \mathbb{R} \times [0, +\infty) \to \mathbb{R}^m$ is defined by

$$\mathbf{u}(x,t) = \begin{cases} \mathbf{u}^- & \text{if } \dfrac{x}{t} \in (-\infty, \lambda_1^-), \\ \vdots & \\ \mathbf{h}_k\left(\dfrac{x}{t}\right) & \text{if } \dfrac{x}{t} \in [\lambda_k^-, \lambda_k^+), \\ \boldsymbol{\omega}_k & \text{if } \dfrac{x}{t} \in [\lambda_k^+, \lambda_{k+1}^-), \\ \vdots & \\ \mathbf{u}^+ & \text{if } \dfrac{x}{t} \in [\lambda_m, +\infty). \end{cases}$$

In the Complements, we give a detailed discussion of the solution to the Riemann problem for the model of isentropic gas dynamics in Lagrangian coordinates.

13 Convex Entropies

We have already seen that the system

$$\partial_t \mathbf{u} + \partial_x \mathbf{f}(\mathbf{u}) = 0, \qquad (x,t) \in \mathbb{R} \times (0,T) \tag{13.1}$$

with smooth $\mathbf{f}(\mathbf{u})$, can be equivalently rewritten as

$$\partial_t \mathbf{u} + \mathbf{A} \partial_x \mathbf{u} = 0, \qquad (x,t) \in \mathbb{R} \times (0,T), \tag{13.2}$$

where $\mathbf{A} = \nabla_{\mathbf{u}} f$. If we assume that $\mathbf{u}$ is a solution to (13.1) that vanishes for $|x| >> 1$, and integrate, then

$$\int_{\mathbb{R}} \partial_t \mathbf{u}\, dx = 0 \quad \Rightarrow \quad \partial_t \int_{\mathbb{R}} \mathbf{u}\, dx = 0 \quad \Rightarrow \quad \int_{\mathbb{R}} u_i\, dx = \text{ const}, \ i = 1, \ldots, m.$$

Now we ask ourselves the following question: given a function $U : \mathbb{R}^m \to \mathbb{R}$, $U = U(\mathbf{u})$, where $\mathbf{u}$ is a solution to (13.1), what does it take for U to satisfy

$$\partial_t U + \partial_x F = 0 \tag{13.3}$$

for a proper $F : \mathbb{R}^m \to \mathbb{R}$, $F = F(\mathbf{u})$? We can rewrite (13.3) as

$$\sum_{l=1}^{m} \frac{\partial U}{\partial u_l}\frac{\partial u_l}{\partial t} + \frac{\partial F}{\partial u_l}\frac{\partial u_l}{\partial x} = 0 \quad \Leftrightarrow \quad \nabla U \cdot \partial_t \mathbf{u} + \nabla F \cdot \partial_x \mathbf{u} = 0.$$

By (13.2), we conclude that we must have

$$\nabla U \cdot \mathbf{A} = \nabla F. \tag{13.4}$$

This is a system of m partial differential equations for U and F. If $m \geq 2$, it is overdetermined, and in general it does not have solutions. However, there are special cases where a solution does exist. We consider some of these instances in the following.

We now assume that such a function U exists, so that (13.1) admits a further conservation law, and at the same time we also suppose that U is convex. The pair (U, F) is usually referred to as *entropy, entropy production.*

A general class of systems for which U with the above-given properties exist is given by symmetric systems, namely when

$$\frac{\partial f_j}{\partial u_l} = \frac{\partial f_l}{\partial u_j}. \tag{13.5}$$

By well-known classical results, (13.5) implies that there exists a function $g : \mathbb{R}^m \to \mathbb{R}$, $g = g(\mathbf{u})$, such that $\dfrac{\partial g}{\partial u_j} = f_j$. Then, by direct substitution, it is easy to verify that

$$U(\mathbf{u}) = \frac{1}{2}\sum_{j=1}^{m} u_j^2, \qquad F(\mathbf{u}) = \sum_{j=1}^{m} u_j f_j - g.$$

Let us go back to the general framework. If a smooth, convex function U with the required properties exists, then its Hessian matrix $H \overset{\text{def}}{=} \nabla^2 U$ must be positive definite. We can rewrite (13.4) as

$$\sum_{j=1}^{m} \frac{\partial U}{\partial u_j}\frac{\partial f_j}{\partial u_l} = \frac{\partial F}{\partial u_l}.$$

If we differentiate it with respect to u_h, we get

$$\sum_{j=1}^{m} \frac{\partial^2 U}{\partial u_j \partial u_h}\frac{\partial f_j}{\partial u_l} + \frac{\partial U}{\partial u_j}\frac{\partial^2 f_j}{\partial u_l \partial u_h} = \frac{\partial^2 F}{\partial u_l \partial u_h}.$$

Since the second term on the left-hand side and the one on the right-hand side are symmetric with respect to h and l, the first term on the left-hand side must also necessarily be symmetric with respect to h and l. Hence,

$$\sum_{j=1}^{m} \frac{\partial^2 U}{\partial u_j \partial u_h} \frac{\partial f_j}{\partial u_l} = \sum_{j=1}^{m} \frac{\partial^2 U}{\partial u_j \partial u_l} \frac{\partial f_j}{\partial u_h}.$$

If we now consider (13.2), take into account the definition of $\mathbf{A}$, and multiply it by $\dfrac{\partial^2 U}{\partial u_j \partial u_h}$, we obtain

$$\sum_{j=1}^{m} \frac{\partial^2 U}{\partial u_j \partial u_h} \partial_t u_j + \sum_{j,l=1}^{m} \frac{\partial^2 U}{\partial u_j \partial u_h} \frac{\partial f_j}{\partial u_l} \partial_x u_l = 0, \quad h = 1, \ldots, m.$$

The new system is equivalent to (13.2), since $\dfrac{\partial^2 U}{\partial u_j \partial u_h}$ is positive definite. We also remark that

$$b_{jh} \overset{\text{def}}{=} \frac{\partial^2 U}{\partial u_j \partial u_h}, \qquad a_{lh} \overset{\text{def}}{=} \frac{\partial^2 U}{\partial u_j \partial u_h} \frac{\partial f_j}{\partial u_l}$$

are symmetric. Moreover, we assume that all the eigenvalues of $\dfrac{\partial f_j}{\partial u_l}$ are not only real but also distinct. As we have seen in Sections 3–5, strictly hyperbolic, symmetric, first-order systems enjoy a unique existence of solutions. Hence, we can conclude that for strictly hyperbolic systems (13.2), we always have local existence and uniqueness of solutions, provided that a convex entropy exists. This was originally observed in Friedrichs and Lax [87].

We want to link what we have done so far, with a different approach to systems of conservation laws, which closely resembles what we discussed in Chapter 7. Indeed, in Section 7 we saw that a way of characterizing physically meaningful solutions to equations is to build them as limit of approximating solutions, namely

$$\partial_t u + a(u) \partial_x u = \epsilon\, \partial_{xx}^2 u,$$

as $\epsilon \to 0$. Let us repeat the same approach here. This leads to

$$\partial_t \mathbf{u} + \mathbf{A}(\mathbf{u}) \partial_x \mathbf{u} = \epsilon\, \partial_{xx}^2 \mathbf{u}, \qquad \epsilon > 0. \tag{13.6}$$

Let us assume that a bounded sequence of solution $\mathbf{u}_\epsilon$ of (13.6) converges to a weak solution $\mathbf{u}$ of (13.2) in L^∞ on compact subsets of $\mathbb{R} \times (0, T)$ as $\epsilon \to 0$. Multiplying (13.6) by $\frac{\partial U}{\partial u_j}$, summing over j, and taking into account (13.4) yields

$$\partial_t U + \partial_x F = \epsilon \sum_{j=1}^{m} \frac{\partial U}{\partial u_j}\, \partial_{xx}^2 u_j = \epsilon\, \nabla U \cdot \partial_{xx}^2 \mathbf{u}.$$

By straightforward computation, we have

$$\partial_{xx}^2 U = \nabla U \cdot \partial_{xx}^2 \mathbf{u} + \sum_{j,l=1}^{m} \frac{\partial^2 U}{\partial u_j \partial u_l} \frac{\partial u_j}{\partial x} \frac{\partial u_l}{\partial x},$$

and assuming U to be strictly convex yields

$$\sum_{j,l=1}^{m} \frac{\partial^2 U}{\partial u_j \partial u_l} \frac{\partial u_j}{\partial x} \frac{\partial u_l}{\partial x} > 0;$$

therefore, we conclude that

$$\partial_{xx}^2 U \geq \nabla U \cdot \partial_{xx}^2 \mathbf{u},$$

and also

$$\partial_t U + \partial_x F \leq \epsilon \, \partial_{xx}^2 U.$$

If we let $\epsilon \to 0$, the right-hand side tends to zero in the sense of distributions, and gives

$$\partial_t U + \partial_x F \leq 0.$$

We summarize the result.

Theorem 13.1. [160] *Let* (13.1) *be a hyperbolic system of conservation laws, which admits the extra conservation law* (13.3) *for a proper pair* (U, F). *Assume that* U *is strictly convex, and that* $\mathbf{u}$ *is a weak solution of* (13.1), *which as* $\epsilon \to 0$ *is the limit in* L^∞ *on compact subsets of* S_T *of a bounded sequence* $\{u_\epsilon\}$ *of solutions of the approximating viscous system* (13.6). *Then,* $\mathbf{u}$ *satisfies*

$$\partial_t U(\mathbf{u}) + \partial_x F(\mathbf{u}) \leq 0 \tag{13.7}$$

in the sense of distributions, that is,

$$\iint_{\mathbb{R}\times[0,T]} [U(\mathbf{u})\partial_t\varphi + F(\mathbf{u})\partial_x\varphi] \, dxdt \geq 0$$

for all $\varphi \in C_o^\infty(\mathbb{R} \times [0, T])$, *with* $\varphi \geq 0$.

Remark 13.1 Suppose $\mathbf{u}$ satisfies (13.7) for a proper pair (U, F) and has compact support. Then, if we integrate with respect to x we obtain

$$\int_{\mathbb{R}} \partial_t U(\mathbf{u}) \, dx \leq 0 \Rightarrow \int_{\mathbb{R}} U(\mathbf{u}(t)) \, dx \text{ is monotone decreasing with respect to } t.$$

Following Bressan [20], we introduce a new concept.

Definition 13.2. *Let* $\mathbf{u} : S_T \to \mathbb{R}^m$. *We say that* $\mathbf{u}$ *is* piecewise Lipschitz continuous *if it satisfies the following assumptions:*

1. $\mathbf{u}$ *is measurable and bounded;*

2. *There exist a finite number of points $P_i(x_i, t_i)$ and finitely many disjoint Lipschitz-continuous curves $\gamma_i : (a_j, b_j) \to \mathbb{R}$ such that*
   - *Every point $P \in S_T$ not coinciding with some P_i and lying outside the curves γ_j has a neighborhood V such that $\mathbf{u}\big|_V$ is Lipschitz continuous in V;*
   - *Every point $Q(\gamma_j(t), t)$ on each curve γ_j has a neighborhood W such that the restrictions of $\mathbf{u}$ to the subsets*

$$W^+ = W \cap \{x > \gamma_j(t)\}, \qquad W^- = W \cap \{x < \gamma_j(t)\}$$

are Lipschitz continuous both in W^+ and in W^-.

Proposition 13.1 *Suppose that $\mathbf{u}$ is a piecewise Lipschitz continuous, weak solution of* (13.1) *in S_T, which satisfies the additional conservation law* (13.3) *for a proper pair (U, F). Then, $\mathbf{u}$ satisfies* (13.7) *if and only if at a point of discontinuity we have*

$$s\,[U] \le [F]\,, \tag{13.8}$$

where s is the speed of the discontinuity, and, as usual, $[\cdot]$ denotes the width of the jump.

Proof. For every non-negative $\varphi \in C^1$ with compact support, it is not hard to prove that

$$\begin{aligned}
&\iint_{S_T} [U(\mathbf{u})\partial_t\varphi + F(\mathbf{u})\partial_x\varphi]\,dxdt \\
&\quad = -\iint_{S_T} [\partial_t U(\mathbf{u}) + \partial_x F(\mathbf{u})]\varphi\,dxdt \\
&\qquad + \sum_j \int_{a_j}^{b_j} \left\{\gamma_j'[U(\mathbf{u}_r) - U(\mathbf{u}_l)] - [F(\mathbf{u}_r) - F(\mathbf{u}_l)]\right\} \varphi(\gamma_j(t), t)\,dt
\end{aligned}$$

where $\gamma_j = \gamma_j(t)$, $t \in [a_j, b_j]$ are the discontinuity curves.

By assumption, outside the discontinuity curves

$$\begin{aligned}
\partial_t U(\mathbf{u}) + \partial_x F(\mathbf{u}) &= \nabla U \cdot \partial_t\mathbf{u} + \nabla F \cdot \partial_x\mathbf{u} \\
&= \nabla U \cdot \partial_t\mathbf{u} + \nabla U \cdot \mathbf{A} \cdot \partial_x\mathbf{u} \\
&= \nabla U \cdot (\partial_t\mathbf{u} + \mathbf{A}\partial_x\mathbf{u}) = 0.
\end{aligned}$$

Therefore, $\partial_t U(\mathbf{u}) + \partial_x F(\mathbf{u}) \le 0$ will hold, if and only if

$$\begin{aligned}
&\iint_{S_T} [\partial_t U(\mathbf{u}) + \partial_x F(\mathbf{u})]\varphi\,dxdt \\
&= \sum_j \int_{a_j}^{b_j} \left\{\gamma_j'[U(\mathbf{u}_r) - U(\mathbf{u}_l)] - [F(\mathbf{u}_r) - F(\mathbf{u}_l)]\right\} \varphi(\gamma_j(t), t)\,dt \ge 0
\end{aligned}$$

for every non-negative $\varphi \in C^1$ with compact support, which is just saying

$$s[U] - [F] \leq 0.$$

■

Conditions (13.7)–(13.8) are usually referred to as *entropy conditions.* Since we have previously discussed the Lax shock conditions in Section 8, it is quite natural to discuss the connection between (8.1)–(8.2) and (13.8).

Theorem 13.3. [160, 161] *Suppose that the system* (13.1) *is strictly hyperbolic and genuinely nonlinear. Suppose that there exists a strictly convex function* U*, which satisfies the additional conservation law* (13.3)*. Let* $\mathbf{u}$ *be a solution of* (13.1)*, which has a discontinuity propagating with speed* s*. Finally, assume that the values* $\mathbf{u}_l$ *and* $\mathbf{u}_r$ *are close. Then, the Lax shock conditions* (8.1)–(8.2) *are satisfied if and only if the strict entropy condition*

$$s\left[U(\mathbf{u}_l) - U(\mathbf{u}_r)\right] - F(\mathbf{u}_l) + F(\mathbf{u}_r) < 0$$

holds.

Proof. It was shown in Section 9 that for a strictly hyperbolic, genuinely nonlinear system, all the states $\mathbf{u}_r$ which can be connected to a given state $\mathbf{u}_l$ through a k-shock, form a one-parameter family of states $\mathbf{u}_r = \mathbf{u}_r(\epsilon)$ with $\epsilon \in (-\epsilon_o, 0)$ for a proper ϵ_o.

We rely on this fact, and we give a third-order Taylor expansion of the quantity

$$G(\epsilon) = s\left[U(\mathbf{u}_l) - U(\mathbf{u}_r)\right] - F(\mathbf{u}_l) + F(\mathbf{u}_r),$$

for $\epsilon \in (-\epsilon_o, 0]$, that is,

$$G(\epsilon) = G(0) + G'(0)\epsilon + \frac{1}{2}G''(0)\epsilon^2 + \frac{1}{6}G'''(0)\epsilon^3 + o(\epsilon^3).$$

If we consider the Rankine–Hugoniot condition (7.2), differentiate it with respect to ϵ, and take into account that $\nabla \mathbf{f} = \mathbf{A}$, we have

$$\frac{ds}{d\epsilon}[\mathbf{u}_l - \mathbf{u}_r] - s\frac{d\mathbf{u}_r}{d\epsilon} = -\mathbf{A}(\mathbf{u}_r)\frac{d\mathbf{u}_r}{d\epsilon}; \qquad (13.9)$$

without loss of generality, by a proper choice of the parametrization, in the following we can assume that $\frac{ds}{d\epsilon} > 0$. On the other hand, if we differentiate G with respect to ϵ, we have

$$G'(\epsilon) = \frac{ds}{d\epsilon}\left[U(\mathbf{u}_l) - U(\mathbf{u}_r)\right] - s\nabla U\frac{d\mathbf{u}_r}{d\epsilon} + \nabla F(\mathbf{u}_r)\frac{d\mathbf{u}_r}{d\epsilon}. \qquad (13.10)$$

Condition $\nabla U \cdot \mathbf{A} = \nabla F$ and (13.9) yield

$$\nabla F(\mathbf{u}_r)\frac{d\mathbf{u}_r}{d\epsilon} = \frac{ds}{d\epsilon}\nabla U[\mathbf{u}_r - \mathbf{u}_l] + s\nabla U\frac{d\mathbf{u}_r}{d\epsilon}. \tag{13.11}$$

If we substitute (13.11) into (13.10), we have

$$G'(\epsilon) = \frac{ds}{d\epsilon}\left[U(\mathbf{u}_l) - U(\mathbf{u}_r)\right] + \frac{ds}{d\epsilon}\nabla U[\mathbf{u}_r - \mathbf{u}_l] \Rightarrow G'(0) = 0.$$

If we differentiate G once more with respect to ϵ, we have

$$G''(\epsilon) = \frac{d^2 s}{d\epsilon^2}\left[U(\mathbf{u}_l) - U(\mathbf{u}_r)\right] + \frac{d^2 s}{d\epsilon^2}\nabla U[\mathbf{u}_r - \mathbf{u}_l] + \frac{ds}{d\epsilon}\left(\frac{d}{d\epsilon}\nabla U\right)[\mathbf{u}_r - \mathbf{u}_l],$$

which easily implies that $G''(0) = 0$. Finally, if we differentiate G one more time, we obtain

$$\begin{aligned} G'''(\epsilon) =& \frac{d^3 s}{d\epsilon^3}\left[U(\mathbf{u}_l) - U(\mathbf{u}_r)\right] + \frac{d^3 s}{d\epsilon^3}\nabla U[\mathbf{u}_r - \mathbf{u}_l] \\ &+ 2\frac{d^2 s}{d\epsilon^2}\left(\frac{d}{d\epsilon}\nabla U\right)[\mathbf{u}_r - \mathbf{u}_l] + \frac{ds}{d\epsilon}\left(\frac{d^2}{d\epsilon^2}\nabla U\right)[\mathbf{u}_r - \mathbf{u}_l] \\ &+ \frac{ds}{d\epsilon}\left(\frac{d}{d\epsilon}\nabla U\right)\frac{d\mathbf{u}_r}{d\epsilon}, \end{aligned}$$

which yields

$$G'''(0) = \frac{ds}{d\epsilon}\left(\frac{d}{d\epsilon}\nabla U\right)\frac{d\mathbf{u}_r}{d\epsilon}\Big|_{\epsilon=0} = \frac{ds}{d\epsilon}\,\nabla^2 U\left(\frac{d\mathbf{u}_r}{d\epsilon}\right)^2\Big|_{\epsilon=0}.$$

Since $\frac{ds}{d\epsilon} > 0$ by assumption, and $\nabla^2 U > 0$ by the strict convexity of U, we conclude that necessarily $G'''(0) > 0$. By the Taylor expansion, possibly reducing the interval where ϵ takes value, taking into account that $\epsilon < 0$, we conclude that $G(\epsilon) < 0$ and by the definition of G we finish. ■

Remark 13.2 The condition on the profiles being close is quantified by the ϵ that appears in the previous expansion. In some sense, it remains a sort of qualitative statement.

We have seen before that symmetric systems admit an additional conservation law. In Chapter 7 we discussed the case of the single equation. The natural question that arises is whether there are other physically interesting examples of systems, that satisfy (13.3).

In Lax [160], § 3, the author deals quite extensively with 2 × 2 systems, which admit a further conservation law. A very interesting study is developed by Luc Tartar [254], which is further expanded in his 2009 unpublished lecture notes [256].

If we refer to (13.4), and take $m = 2$, we have two equations in terms of $\dfrac{\partial F}{\partial u_1}, \dfrac{\partial F}{\partial u_2}$ and one compatibility condition, so that all entropies U are a solution of a linear second-order hyperbolic equation, that has infinitely many solutions.

If $m \geq 3$, there are more compatibility conditions, and U must be a solution of an overdetermined system, which could have only trivial entropies, namely $U_j = u_j$, $j = 1, \ldots, m$ as solutions. However, particular physical examples might nevertheless allow for the existence of nontrivial entropies. These are dealt with in Section 13c of the Complements.

13.1 Examples of Entropies for 2 × 2 Systems

Here, we follow Tartar [256], to which we refer for other very interesting physical considerations. If we consider the p-system of Section 7.1, it is a matter of straightforward computations to check that the entropies are given by

$$\frac{\partial^2 U}{\partial v^2} = -p'(v) \frac{\partial^2 U}{\partial w^2}, \qquad \text{(recall that } p'(v) < 0).$$

If we let

$$P(v) = \int^v p(z)\, dz,$$

one finds

$$U(v, w) = vw, \qquad F(v, w) = -\frac{w^2}{2} - vp(v) + P(v),$$

and

$$U(v, w) = \frac{w^2}{2} + P(v) \text{ (total energy)}, \qquad F(v, w) = -wp(v).$$

Notice that in the second case, (13.3) can be interpreted as the conservation of the total energy. Hence, the total energy is a mathematical entropy for the p-system.

If we switch to Eulerian coordinates, the 2 × 2 system for the isentropic gas dynamics becomes

$$\begin{cases} \partial_t \rho + \partial_x (\rho u) = 0, \\ \partial_t (\rho u) + \partial_x (\rho u^2 + p(\rho)) = 0, \end{cases}$$

where $t \geq 0$, $x \in \mathbb{R}$, $p(\rho) = \kappa \rho^\gamma$, with $\gamma > 1$ and $\kappa = \dfrac{(\gamma - 1)^2}{4\gamma}$. The unknowns of the system are the density ρ and the momentum $q = \rho u$. By obvious physical motivations, we must have $\rho \geq 0$.

It is a matter of straightforward computations to check that we have the additional conservation law (13.3) with $U = U(\rho, u)$ and $F = F(\rho, u)$, if and only if

$$\begin{cases} \dfrac{\partial F}{\partial \rho} = u \dfrac{\partial U}{\partial \rho} + \dfrac{p'(\rho)}{\rho} \dfrac{\partial U}{\partial u}, \\ \dfrac{\partial F}{\partial u} = \rho \dfrac{\partial U}{\partial \rho} + u \dfrac{\partial U}{\partial u}, \end{cases} \tag{13.12}$$

or, equivalently,

$$\frac{\partial^2 U}{\partial \rho^2} = \frac{p'(\rho)}{\rho^2}\frac{\partial^2 U}{\partial u^2}.$$

In Lions et al. [178], to which we refer for the details of the proofs, the authors consider so-called *weak entropies*, namely functions U, which satisfy (13.12) and also the further conditions

$$U(0,u) = 0, \qquad U_\rho(0,u) = g(u),$$

where g is a function to be chosen. The general solution is then given by

$$U(\rho,u) = \int_{\mathbb{R}} g(\xi)\chi(\rho,\xi-u)\,d\xi,$$

where $\chi(\rho,w) = (\rho^{\gamma-1} - w^2)_+^\lambda$ with $\lambda = \frac{3-\gamma}{2(\gamma-1)}$. It is not hard to check that U is convex in ρ and ρu if and only if g is convex. Moreover,

$$F(\rho,u) = \int_{\mathbb{R}} g(\xi)[\theta\xi + (1-\theta)u]\chi(\rho,\xi-u)\,d\xi, \qquad \theta = \frac{\gamma-1}{2}.$$

In particular, if we take $g(\xi) = \frac{1}{2}\xi^2$, the corresponding entropy is

$$U = \frac{1}{2}\rho u^2 + \frac{\kappa}{\gamma-1}\rho^\gamma = \frac{1}{2}\frac{q^2}{\rho} + \frac{\kappa}{\gamma-1}\rho^\gamma.$$

14 The Glimm Existence Result

We can finally come to the fundamental existence result attributed to Glimm. Consider the Cauchy problem

$$\begin{cases} \partial_t \mathbf{u} - \partial_x \mathbf{f}(\mathbf{u}) = 0, & x \in \mathbb{R},\ t > 0, \\ u(x,0) = u_o(x), & x \in \mathbb{R}. \end{cases} \tag{14.1}$$

We have the following.

Theorem 14.1. *Assume that the first of* (14.1) *is strictly hyperbolic with* $\mathbf{f}$ *smooth and defined for* $\mathbf{u}$ *in an open set* $\Omega \subset \mathbb{R}^m$. *Moreover, suppose that each characteristic field is either linearly degenerate or genuinely nonlinear. Then there exists a constant* $\delta_o > 0$ *such that for every initial condition* $\mathbf{u}_o \in L^1(\mathbb{R};\mathbb{R}^m)$ *with*

$$TV(\mathbf{u}_o) \le \delta_o,$$

the Cauchy problem (14.1) *has a weak solution* $\mathbf{u}(x,t)$ *defined for all* $t \ge 0$.

We cannot go into details of the proof of this key result. It suffices to say that in Glimm [103], first an approximate solution of (14.1) is built by patching together solutions of several Riemann problems, using a restarting procedure based on random sampling. The fundamental step in the proof is then an a priori estimate on the total variation of the approximate solutions, obtained by introducing a proper potential. Finally, the control of the total variation yields the compactness of the family of approximate solutions, and hence the existence of a strongly convergent subsequence.

In the following we study the connection of Theorem 14.1 with the theory developed in the previous section.

As we have just explained above, Glimm builds solutions of systems of conservation laws as limits of approximate solutions. These solutions are piecewise Lipschitz continuous, weak solutions in each strip $k\Delta t < t < (k+1)\Delta t$, and all their discontinuities are admissible shocks. Hence, outside of the jumps, they are regular and we can apply Proposition 13.1 and Theorem 13.3 and conclude that

$$\partial_t U(\mathbf{u}) + \partial_x F(\mathbf{u}) \leq 0$$

for each approximate solution in each strip.

Take a smooth, positive test function φ with compact support, multiply the previous relation by φ and integrate over each strip. If we now integrate by parts with respect to t over each strip, and sum over all strips, we obtain

$$\sum_{k=1}^{\infty} \int_{\mathbb{R}} \varphi(x, k\Delta t)\, [U(x,(k+1)\Delta t) - U(x, k\Delta t)]\; dx \\ + \int_0^T \int_{\mathbb{R}} [-\partial_t \varphi\, U + \varphi\, \partial_x F]\; dxdt \leq 0.$$

Lemma 5.1 of [104] shows that

$$\sum_{k=1}^{\infty} \int_{\mathbb{R}} \varphi(x, k\Delta t)\, [U(x,(k+1)\Delta t) - U(x, \Delta t)]\; dx \to 0$$

for a suitable selected subsequence of mesh length Δt. This leaves us with

$$\int_0^T \int_{\mathbb{R}} [-\partial_t \varphi\, U + \varphi\, \partial_x F]\; dxdt \leq 0$$

for all positive test functions φ supported in $t > 0$. Integrating by parts with respect to t once more yields

$$\int_0^T \int_{\mathbb{R}} [\partial_t U + \partial_x F]\, \varphi\, dxdt \leq 0$$

for all such φ. This implies the entropy condition in the whole strip. Thus, we have proved the following

Theorem 14.2. *Suppose that the first of* (14.1) *is strictly hyperbolic, and that there is a convex function U with the corresponding entropy production function F, which satisfies the additional conservation law* (13.3)*. Then, all weak solutions $\mathbf{u}$ built with Glimm's method satisfy the entropy inequality* (13.7)*.*

15 Some Final Comments

In the previous sections we have dealt with the so-called *geometric theory* of systems of conservation laws, which mainly deals with problems concerning the existence and qualitative behavior of solutions, assuming a bound in BV. Moreover, we have confined ourselves to the one-dimensional case, namely $x \in \mathbb{R}$. Finally, the theory we have presented is mainly limited to the fundamental papers of Lax and Glimm, as presented in Lax [161].

In the last 40 years, the theory has seen tremendous developments concerning uniqueness, stability, and qualitative properties of solutions in the one-dimensional case. Although limited to the literature at the end of the 1990s, the book by Bressan [20] gives a clear and thorough picture.

Huge steps forward have also been made in the multidimensional framework. To have at least a very first introduction to the study in several space variables, the interested reader can refer to Majda [180].

However, there exists a different approach to systems of conservation laws, the so-called *functional analytic theory*, based on the notion of compensated compactness, originally introduced by François Murat and Luc Tartar, and later developed by Ronald DiPerna. It is a completely different approach, based not on BV bounds but only on the uniform L^∞ norm of solutions. Unfortunately, we cannot go into detail about this very interesting theory here. To have just a glimpse of these two different approaches and a comparison of them, the reader can refer to DiPerna [57] and the references therein, updated to 1984. For a much more thorough introduction to the topic, there are extremely useful works by Tartar [253, 255].

Finally, coming back to the origin of the modern theory of shock waves and hyperbolic systems of conservation laws, a very interesting reading is represented by von Neumann [269] and by the comments thereupon [231]. As Denis Serre remarks at the end of his paper [231], "Sixty years after (1949), we read these records as if they had been written a decade ago. In a few pages, all the important questions are raised and examples are given, which still serve as paradigms."

Problems and Complements

2c Some Examples

Since the very beginning of the existence of the petroleum industry, engineers have injected water in order to recover oil from its fields to maintain pressure. The so-called *Buckley–Leverett model* describes the flow of two immiscible fluids in a porous medium and solves the corresponding problem in one spatial dimension [22, 170]. It was later discovered that injecting natural gas and water leads to a better displacement of the oil. Of particular interest is the case in which three immiscible fluids, such as water, gas, and oil, are used in any proportion to displace a single fluid, such as oil.

Here we limit ourselves to a simple presentation of the model, and we refer to Azevedo et al. and Glimm et al. [13, 105] and the references therein for more details on the results.

Immiscible three-phase flow in porous media is governed by a system of two conservation laws, and in its simplest case, this system fails to be strictly hyperbolic at a point interior to its domain.

Indeed, consider the flow of a mixture of three fluid phases (which, for concreteness, are called water, oil, and gas) in a thin, horizontal cylinder of porous rock. Let $s_w(x,t)$, $s_g(x,t)$, and $s_o(x,t)$ denote the respective saturations at distance x along the cylinder, at time t: we have

$$s_w + s_g + s_o = 1, \quad \text{and} \quad 0 \le s_w,\, s_o,\, s_g \le 1. \tag{2.1c}$$

The porous rock cylinder is initially assumed to contain pure oil, for simplicity, and a specific mixture of the so-called water, gas, and oil is injected into the left end of the cylinder. In *Corey's model with quadratic relative permeabilities* the three-phase flow is governed by the system

$$\begin{cases} \dfrac{\partial s_w}{\partial t} + \dfrac{\partial f_w}{\partial x} = 0, \\ \dfrac{\partial s_g}{\partial t} + \dfrac{\partial f_g}{\partial x} = 0, \end{cases}$$

where

$$f_w = \frac{s_w^2}{r_w \mathcal{D}}, \quad f_g = \frac{s_g^2}{r_g \mathcal{D}}, \quad \text{and} \quad \mathcal{D} = \frac{s_w^2}{r_w} + \frac{s_g^2}{r_g} + s_o^2,$$

and we take into account, as mentioned in (2.1c), that $s_w + s_g + s_o = 1$. The parameters r_w and r_g are the viscosity ratios water/oil and gas/oil. Since the viscosity of gas is definitely lower than the other ones, r_w and r_g satisfy

$$0 < r_g < 1, \qquad r_g < r_w.$$

We also define the total and net water/gas viscosity ratios

$$r_{tot} = r_w + r_g + 1, \qquad r_{wg} = r_w + r_g.$$

If J denotes the Jacobian matrix of the vector $\mathbf{f} = [f_w, f_g]_T$, we have

$$\operatorname{tr} J = \frac{2R_1}{r_w r_g \mathcal{D}^2}, \qquad \det J = \frac{4s_w s_g(1 - s_w - s_g)}{r_w r_g \mathcal{D}^3},$$

where

$$R_1 \overset{\text{def}}{=} s_w^2 s_g + s_w s_g^2 + (1 - s_w - s_g)[r_g s_w(1 - s_g) + r_w s_g(1 - s_w)] \geq 0;$$

R_1 is strictly positive, except when $s_w = 1$ (all water, no gas, no oil), $s_g = 1$ (all gas, no water, no oil), $s_w = s_g = 0$ (all oil, no water, no gas).

It is a matter of straightforward calculation to check that the eigenvalues of J are

$$\lambda_1 = \frac{R_1 - \sqrt{R_2}}{r_w r_g \mathcal{D}^2}, \qquad \lambda_2 = \frac{R_1 + \sqrt{R_2}}{r_w r_g \mathcal{D}^2},$$

where

$$R_2 \overset{\text{def}}{=} R_1^2 - 4s_w s_g(1 - s_w - s_g)(r_g s_w^2 + r_w s_g^2 + r_w r_g(1 - s_g - s_w)^2).$$

If $s_w = \frac{r_w}{r_{tot}}$ and $s_g = \frac{r_g}{r_{tot}}$, it is easy to check that $R_2 = 0$, so that $\lambda_1 = \lambda_2$ and the system is weakly hyperbolic.

5c Existence of Solutions: The Nonlinear Theory

An interesting historical perspective of the development of the theory of hyperbolic equations is given in Gårding [96]. The existence result of the Cauchy problem for strongly hyperbolic linear and quasi-linear systems was originally given in Petrowsky [202], a paper that is rather hard to understand. Things were much clarified in Leray [166] and carried further in Dionne [59].

6c Proof of Theorem 6.1

By a linear change of variables, which involves only $x_1, \dots, x_k$, we may assume that $P(t, \xi_1, \dots, \xi_k) = 0$ has the root $t = \tau$ when $\xi_1 = 1$, $\xi_2 = \cdots = \xi_k = 0$. Therefore, we can limit ourselves to $k = 1$. Moreover, it suffices to prove the theorem for the operator $\left(\frac{\partial}{\partial t} - \tau\frac{\partial}{\partial x}\right)^r$, that is, it is enough to find u and an operator Q of degree ≤ 1, both vanishing for $t \leq 0$, such that

$$\left(\frac{\partial}{\partial t} - \tau\frac{\partial}{\partial x}\right)^r u + Q(D)u = 0.$$

Finally, a further straightforward change of variable, which is admissible because τ is real, transforms the operator $\left(\frac{\partial}{\partial t} - \tau\frac{\partial}{\partial x}\right)^r$ into $\left(\frac{\partial}{\partial t'}\right)^r$. Therefore, if we relabel t' with t, we are eventually led to work with

$$\frac{\partial^r u}{\partial t^r} + Q(D)u = 0, \tag{6.1c}$$

and we may assume $\tau = 0$. In the following, we equivalently use $P(D)u$ and $\dfrac{\partial^r u}{\partial t^r}$. Before giving the actual proof, we need some introductory material. The first is a Real Analysis Lemma, of which we omit the easy proof.

Lemma 6.1c *Let $I = [\alpha, \beta]$ be an interval of length h, and $C > 0$. Then there exists a function $f : \mathbb{R} \to \mathbb{R}$, $f \in C^\infty(\mathbb{R})$, such that*

$$\begin{cases} f(t) = 0 & \forall\, t \geq \beta, \\ f(t) = C & \forall\, t \leq \alpha, \\ |f^{(s)}(t)| \leq KCh^{-s} & \forall\, s \in \mathbb{N}, \forall\, t \in \mathbb{R}, \end{cases}$$

where $K > 0$ is a constant, which may depend on s.

Now we build a sequence $\{t_m\}_{m\geq 1} \subset (0, +\infty)$ with the following properties, which are repeatedly needed in the proof:

- $t_m \to 0$ as $m \to +\infty$;
- In the interval $I_m \overset{\text{def}}{=} [t_{m+1}, t_m]$ we choose the numbers $t_{m+1} < t_m^4 < t_m^3 < t_m^2 < t_m^1 < t_m$, such that the ratios $\dfrac{t_m - t_m^i}{t_m - t_{m+1}}$, $i = 1, \ldots, 4$ are all independent of m;
- $t_m^1 - t_m^3 = 4(t_m^1 - t_m^2)$.

It is straightforward to see that all these requirements are satisfied if we let

$$t_m = \frac{C}{m}, \quad t_m^i = \frac{C}{m}\left(1 - \frac{H_i}{m+1}\right), \; i = 1, \ldots, 4,$$

with $C > 0$ to be determined, $H_i \in (0,1)$ such that $0 < H_1 < H_2 < H_3 < H_4 < 1$ and $H_3 = 4H_2 - 3H_1$. Notice that quantities H_i can be chosen independently of C. We let $\Delta t_m = t_m - t_{m+1}$ and

$$\begin{aligned} &I_m^1 \overset{\text{def}}{=} [t_m^1, t_m], \;\; I_m^2 \overset{\text{def}}{=} [t_m^2, t_m^1], \\ &I_m^3 \overset{\text{def}}{=} [t_m^3, t_m^2], \;\; I_m^4 \overset{\text{def}}{=} [t_m^4, t_m^3], \;\; I_m^5 \overset{\text{def}}{=} [t_{m+1}, t_m^4]. \end{aligned}$$

The solution u is defined by induction on m in the intervals I_m for $m \geq 1$. Moreover, we set

$$\lambda_m = 2^m, \qquad \beta_m = 2^{\sqrt{m}}, \qquad \gamma_m = \zeta\beta_m \left(\frac{\lambda_{m+1}}{\lambda_m}\right)^{\frac{1}{r}} = \zeta\beta_m 2^{\frac{1}{r}},$$

where ζ is the r^{th} root of the unity with negative real part. Whereas with the parameters H_i no further constraints are needed, as far as C is concerned we require that it is large enough to satisfy

$$\beta_m(t_m^1 - t_m^2) \geq \ln 2.$$

Since

$$\begin{aligned}\beta_m(t_m^1 - t_m^2) &= 2^{\sqrt{m}} \frac{C(H_2 - H_1)}{m(m+1)} \\ &\geq 2^{\sqrt{m}} \frac{C(H_2 - H_1)}{2m^2} \geq \frac{2^{\frac{4}{\ln 2}}}{32} (\ln 2)^2 \, C(H_2 - H_1),\end{aligned}$$

it suffices to choose

$$C \geq \frac{32}{2^{\frac{4}{\ln 2}} \ln 2} \left(\frac{1}{H_2 - H_1} \right).$$

Finally, we let

$$\alpha_m \overset{\text{def}}{=} \left| e^{\frac{1}{2}(\beta_m - \gamma_m)(t_m^3 - t_m^1)} \right|. \tag{6.2c}$$

Since $\mathrm{Re}\,\gamma \leq 0$, we conclude that $\alpha_m \leq e^{\frac{1}{2}\beta_m(t_m^3 - t_m^1)}$.

The gist of the proof is first in *defining* u and then in *computing* a perturbation $Q(D)$, such that u satisfies (6.1c). The function u is built in such a way that as t approaches 0 from above, u involves *frequencies* $e^{i\lambda x}$ with increasing λ. Consequently, for t decreasing, the function u decreases ever more rapidly, so that it identically vanishes for $t \leq 0$.

In the following, the constant K might change value from line to line; whenever estimating derivatives it may depend on the order of derivation, but it never depends on the index m.

Finally, for t in the neighborhood of t_m, the function u is always of the form $u = C_m \, e^{i\lambda_m x}$, where $C_m > 0$ is a proper quantity that we are going to define as the construction evolves; for $t \geq t_1$ we put

$$u \overset{\text{def}}{=} e^{i\lambda_1 x},$$

so that $C_1 = 1$.

We now proceed with building u and $Q(D)$ in the intervals I_m^i, $i = 1, \ldots, 5$.

First step: $t \in I_m^1 = [t_m^1, t_m]$. Let $f_m = f_m(t)$ be a $C^\infty(\mathbb{R})$ function that equals zero in a neighborhood of t_m, is one in a neighborhood of t_m^1, and for every $s \geq 0$ satisfies $|f_m^{(s)}(t)| \leq K(\Delta t_m)^{-s}$. Such a function certainly exists owing to Lemma 6.1c. In I_m^1 we let

$$u \overset{\text{def}}{=} C_m e^{i\lambda_m x} + \alpha_m C_m f_m(t) e^{i\lambda_{m+1} x} \overset{\text{def}}{=} u_1 + u_2,$$

where $C_m > 0$ is still to be defined, as mentioned above. It is as if we were perturbing u_1 with a *small* term u_2. The operator Q is merely defined as the product by $a(x,t)$, where

$$a(x,t) = -\frac{\frac{\partial^r u}{\partial t^r}}{u}, \tag{6.3c}$$

which clearly yields $\dfrac{\partial^r u}{\partial t^r} + Q(D)u = 0$. In order for a to be well-defined, we need u not to vanish, which amounts to requiring that

$$1 + \alpha_m f_m(t) e^{2ix} \neq 0.$$

Since $f_m \in [0,1]$, the previous condition is definitely satisfied if $0 < \alpha_m < \frac{1}{2}$, which is surely satisfied by (6.2c).

Owing to the requirements on Q, we are interested in the size of the function a and of all its derivatives. In I_m^1 we have

$$\frac{\partial^r u}{\partial t^r} = \frac{\partial^r u_2}{\partial t^r} = \alpha_m C_m f_m^{(r)}(t) e^{i\lambda_{m+1}x}.$$

Since $\alpha_m \in (0, \frac{1}{2})$, it is straightforward to see that $|u| \geq \frac{1}{2} C_m$. As far as the numerator of (6.3c) is concerned, owing to the properties of f_m, any one of its derivatives is bounded by $\alpha_m C_m K (\mathit{\Delta} t_m)^{-N_1} \lambda_m^{N_1}$ for N_1 sufficiently large.

As for the denominator, $K C_m (\mathit{\Delta} t_m)^{-N_2} \lambda_m^{N_2}$ similarly bounds all its derivatives, again for N_2 sufficiently large.

Therefore, taking into account the previous lower bound for u and the rules for the derivative of a quotient, we conclude that any derivative of a is bounded by $\alpha_m K (\mathit{\Delta} t_m)^{-N} \lambda_m^N$ for N large enough. Because of the previous condition we have set, such a quantity tends to zero as $m \to \infty$.

Second step: $t \in I_m^2 = [t_m^2, t_m^1]$. Let $\varphi_1 = \varphi_1(t)$ be a $C^\infty(\mathbb{R})$ function that equals zero in a neighborhood of t_m^1, is β_m in a neighborhood of t_m^2, and for every $s \geq 0$ satisfies $|\varphi_1^{(s)}(t)| \leq K \beta_m (\mathit{\Delta} t_m)^{-s}$. Analogously, let $\varphi_2 = \varphi_2(t)$ be a $C^\infty(\mathbb{R})$ function that equals zero in a neighborhood of t_m^1, is γ_m in a neighborhood of t_m^2, and for every $s \geq 0$ satisfies $|\varphi_2^{(s)}(t)| \leq K |\gamma_m| (\mathit{\Delta} t_m)^{-s} \leq K \beta_m (\mathit{\Delta} t_m)^{-s}$. Such functions certainly exist because of Lemma 6.1c. In I_m^2 we let

$$u \overset{\text{def}}{=} C_m e^{\varphi_1(t)(t - t_m^1)} e^{i\lambda_m x} + \alpha_m C_m e^{\varphi_2(t)(t - t_m^1)} e^{i\lambda_{m+1}x} \overset{\text{def}}{=} u_1 + u_2.$$

For t in a neighborhood of t_m^2, we have

$$\frac{\partial^r u_1}{\partial t^r} = \beta_m^r u_1 = \beta_m^r (i\lambda_m)^{-1} \frac{\partial u_1}{\partial x}$$

and

$$\frac{\partial^r u_2}{\partial t^r} = \gamma_m^r u_2 = 2\beta_m^r u_2 = \beta_m^r (i\lambda_m)^{-1} \frac{\partial u_2}{\partial x},$$

so that in a neighborhood of t_m^2, u satisfies

$$\frac{\partial^r u}{\partial t^r} - \beta_m^r (i\lambda_m)^{-1} \frac{\partial u}{\partial x} = 0. \tag{6.4c}$$

Moreover, by its definition, for any $t \in I_m^2$,

$$\frac{\partial^r u_1}{\partial t^r} = p(t)u_1,$$

where $p(t)$ is a sum of products of no more than r derivatives of φ_1 and possibly a term $(t - t_m^1)$. By the definition of φ_1 we have

$$|p(t)| \leq K\beta_m^r(\varDelta t_m)^{-N} \quad \text{and} \quad |p^{(s)}(t)| \leq K\beta_m^r(\varDelta t_m)^{-N} \tag{6.5c}$$

for some proper N. Precisely in the same manner, we can conclude that

$$\frac{\partial^r u_2}{\partial t^r} = q(t)u_2, \quad \text{where} \quad |q^{(s)}(t)| \leq K\beta_m^r(\varDelta t_m)^{-N}. \tag{6.6c}$$

We now define

$$Q_1 \overset{\text{def}}{=} -\frac{p(t)}{i\lambda_m}\frac{\partial}{\partial x} \quad \Rightarrow \quad \frac{\partial^r u_1}{\partial t^r} + Q_1 u_1 = 0.$$

Owing to the conditions we imposed before, we have that $\beta_m^r\lambda_m^{-1}(\varDelta t_m)^{-N} \to 0$, which in turn yields that the coefficients of Q_1 and all its derivatives tend to zero. Next, we define Q_2 by

$$Q_2 \overset{\text{def}}{=} -\frac{Pu + Q_1 u}{u} = -\frac{Pu_2 + Q_1 u_2}{u}.$$

From (6.4c) it is immediate to conclude that Q_2 vanishes in the neighborhood of t_m^2. Moreover, for $t \in I_m^2$, we have

$$|u_1| \geq C_m e^{\beta_m(t_m^2 - t_m^1)}, \quad |u_2| \leq \alpha_m C_m \left|e^{\gamma_m(t_m^2 - t_m^1)}\right|,$$

and this yields

$$\begin{aligned}\frac{|u_1|}{|u_2|} &\geq \frac{1}{\alpha_m}\left|e^{(\beta_m - \gamma_m)(t_m^2 - t_m^1)}\right| \\ &\geq \left|e^{(\beta_m - \gamma_m)[(t_m^2 - t_m^1) - \frac{1}{2}(t_m^3 - t_m^1)]}\right| \geq \left|e^{-(\beta_m - \gamma_m)(t_m^2 - t_m^1)}\right|,\end{aligned} \tag{6.7c}$$

since we have required $t_m^1 - t_m^3 = 4(t_m^1 - t_m^2)$. Moreover, since we have also imposed that $\beta_m(t_m^1 - t_m^2) \geq \ln 2$, we have $|u_1| \geq 2|u_2|$ for any $t \in I_m^2$. Now we have all we need to estimate Q_2. From its definition, the coefficient of Q_2 is a quotient of two quantities. If we factor out u_1 from the top and the bottom, we obtain $Q_2 = A/B$, where

$$\begin{aligned}A &= \alpha_m e^{(\varphi_2(t) - \varphi_1(t))(t - t_m^1)}(-q(t) + 2p(t))e^{i\lambda_{m+1}x}, \\ B &= 1 + \alpha_m e^{(\varphi_2(t) - \varphi_1(t))(t - t_m^1)}e^{i\lambda_{m+1}x}, \quad |B| \geq \frac{1}{2},\end{aligned}$$

and we relied on $0 < \alpha_m < \frac{1}{2}$ for the lower bound on $|B|$. From (6.5c)–(6.6c), the rule for the differentiation of a quotient, and (6.7c) it follows that any derivative of Q_2 is majorized by

$$\begin{aligned}&1+K\lambda_m^N(\Delta t_m)^{-N}\alpha_m e^{(\varphi_2(t)-\varphi_1(t))(t_m^2-t_m^1)}\\ <&1+K\lambda_m^N(\Delta t_m)^{-N}\left|\frac{u_2(t_m^2)}{u_1(t_m^2)}\right|\\ <&K\lambda_m^N(\Delta t_m)^{-N}e^{\beta_m(t_m^2-t_m^1)}.\end{aligned}$$

Owing to the previous conditions we have set, we have

$$\lambda_m^N(\Delta t_m)^{-N}e^{\beta_m(t_m^2-t_m^1)}\to 0$$

as $m\to\infty$ for all N.

Third step: $t\in I_m^3=[t_m^3,t_m^2]$. In I_m^3 we define u as

$$u\overset{\text{def}}{=}C_m e^{\beta_m(t-t_m^1)}e^{i\lambda_m x}+\alpha_m C_m e^{\gamma_m(t-t_m^1)}e^{i\lambda_{m+1}x}\overset{\text{def}}{=}u_1+u_2,$$

and Q as $Q\overset{\text{def}}{=}-\dfrac{\beta_m^r}{i\lambda_m}\dfrac{\partial}{\partial x}$ so that $\dfrac{\partial^r u}{\partial t^r}+Qu=0$. Hence, Q is constant and it has the same expression as Q_1 in I_m^2 for t in a neighborhood of t_m^2. When $t=t_m^3$, we require

$$|u_1(t_m^3)|=\alpha_m|u_2(t_m^3)|,$$

which is the inverse of what we required for u_1 and u_2 when $t=t_m^1$, and this justifies the previous definition given in (6.2c).

Fourth step: $t\in I_m^4=[t_m^4,t_m^3]$. In I_m^4, u is defined in such a way as to cancel u_1. As we have $|u_1(t_m^3)|=\alpha_m|u_2(t_m^3)|$, we now let

$$u\overset{\text{def}}{=}g_m(t)C_m e^{\beta_m(t-t_m^1)}e^{i\lambda_m x}+\alpha_m C_m e^{\gamma_m(t-t_m^1)}e^{i\lambda_{m+1}x}\overset{\text{def}}{=}u_1+u_2,$$

where g_m is a C^∞ function, such that $g_m(t_m^3)=1$, $g_m(t_m^4)=0$, and $\forall\, s\geq 0$ $|g_m^{(s)}(t)|\leq K(\Delta t_m)^{-s}$. We observe that u_2 satisfies

$$\frac{\partial^r u_2}{\partial t^r}+Q_1u_2=0,$$

where Q_1 is the same operator as Q in the interval I_m^3. Therefore, we set

$$Q_2\overset{\text{def}}{=}-\frac{Pu+Q_1u}{u}=-\frac{Pu_1+Q_1u_1}{u},$$

so that $Pu+Q_1u+Q_2u=0$.

Since u_1 without the factor g_m would be decreasing as t decreases in I_m^4, and u_2 is increasing as t decreases, it is apparent that $\left|\dfrac{u_1}{g_m u_2}\right|\leq\alpha_m$. Therefore, exactly as in I_m^2, we conclude that any derivative of Q_2 is bounded above by $K\alpha_m\lambda_m^N(\Delta t_m)^{-N}$ for N properly large, and all the derivatives of $Q=Q_1+Q_2$ tend to zero in I_m^4.

Fifth step: $t\in I_m^5=[t_{m+1},t_m^4]$. We finally come to the last interval. We let $\varphi_3=\varphi_3(t)$ be a C^∞ function, such that $\varphi_3=\gamma_m$ in a neighborhood of t_m^4, vanishes in a neighborhood of t_{m+1}, and $\forall\, s \geq 0$ $|\varphi_3^{(s)}| \leq K|\gamma_m|(\Delta t_m)^{-s} \leq K\beta_m(\Delta t_m)^{-s}$. We define

$$u \overset{\text{def}}{=} \alpha_m C_m e^{\varphi_3(t)(t-t_m^1)} e^{i\lambda_{m+1}x}.$$

Moreover, by its definition, for any $t \in I_m^5$,

$$\frac{\partial^r u}{\partial t^r} = p_1(t)u,$$

where $p_1(t)$ is a sum of products of no more than r derivatives of φ_3 and possibly a term $(t - t_m^1)$. By the definition of φ_3 we have

$$|p_1^{(s)}(t)| \leq K\beta_m^r(\Delta t_m)^{-N}$$

for some proper N. Thus, we define

$$Q(D) \overset{\text{def}}{=} -\frac{p_1(t)}{i\lambda_{m+1}}\frac{\partial}{\partial x},$$

and the analysis of the behavior of the coefficient of $Q(D)$ is precisely the same as for $Q_1(D)$ in I_m^2, so that we can avoid repeating it. We eventually set

$$C_{m+1} = \alpha_m C_m,$$

so that we satisfy the requirement that u is of the form $C_{m+1}e^{i\lambda_{m+1}x}$ in the neighborhood of $t = t_{m+1}$. This completes the construction of u and $Q(D)$ for $t \geq 0$. As for $t < 0$ we let $u = Q = 0$.

By the definition of α_m, it follows that

$$C_{m+1} \leq C_m e^{-K\frac{2^{\sqrt{m}}}{m^2}}$$

for some sufficiently small constant K. By iteration, we easily conclude that

$$C_m \leq e^{-K\sum_{k=1}^m \frac{2^{\sqrt{k}}}{k^2}},$$

once we recall that $C_1 = 1$. Finally, as far as the derivatives of u are concerned, aside from factors such as λ_m^N and $(\Delta t_m)^{-N}$, we also have a third kind of factor, namely $e^{\beta_m \Delta t_m}$, owing to the presence of the term $e^{\varphi_2(t)(t-t_m^1)}$. Therefore, any derivative of u is bounded above by

$$K\lambda_m^N(\Delta t_m)^{-N}C_m e^{\beta_m\Delta t_m} \leq K2^{mN}m^{2N}e^{K_1\frac{2^{\sqrt{m}}}{m^2} - K_2\sum_{k=1}^m \frac{2^{\sqrt{k}}}{k^2}};$$

it is a matter of straightforward computations to check that the function $f(x) = 2^{\sqrt{x}}/x^2$ is monotone increasing and convex for $x \geq 34$. Moreover, since we are interested in the limit as $m \to \infty$, we may write

$$\sum_{k=1}^{m} \frac{2^{\sqrt{k}}}{k^2} = \sum_{k=1}^{34} \frac{2^{\sqrt{k}}}{k^2} + \sum_{k=35}^{m} \frac{2^{\sqrt{k}}}{k^2},$$

and we may estimate

$$\begin{aligned}\sum_{k=35}^{m} \frac{2^{\sqrt{k}}}{k^2} &\geq \int_{34}^{m} \frac{2^{\sqrt{x}}}{x^2}\,dx \geq \frac{1}{m^2}\int_{34}^{m} 2^{\sqrt{x}}\,dx \\ &= \frac{2}{m^2 \ln 2}\left[\sqrt{x}2^{\sqrt{x}} - \frac{1}{\ln 2}2^{\sqrt{x}}\right]_{34}^{m}.\end{aligned}$$

Therefore, further straightforward computations allow us to conclude that

$$K 2^{mN} m^{2N} e^{K_1 \frac{2^{\sqrt{m}}}{m^2} - K_2 \sum_{k=1}^{m} \frac{2^{\sqrt{k}}}{k^2}} \leq K_3 2^{mN} m^{2N} e^{-\frac{2^{\sqrt{m}}}{m^2}(K_4\sqrt{m} - K_5)},$$

and such a quantity tends to zero as $m \to \infty$. This finishes the proof. ■

7c Back to Quasi-Linear First-Order Strictly Hyperbolic Systems

We prove Proposition 7.1. Assume that $\mathbf{u}$ is a weak solution of (7.1), let P be a point on the discontinuity curve Γ, and let B be a small ball centered around P.

Without loss of generality, we may assume that $\Gamma \cap B$ is the only curve of discontinuity of $\mathbf{u}$ in B. We denote the two open components of B on each side of Γ as B_+ and B_-. Let $\boldsymbol{\varphi} \in C^1_o(B)$ and $\mathbf{n} = (n_t, n_x)$ the unit normal to Γ. We have

$$\begin{aligned} 0 &= \iint_B [\mathbf{u}\cdot\partial_t\boldsymbol{\varphi} + \mathbf{f}\cdot\partial_x\boldsymbol{\varphi}]\,dxdt \\ &= \iint_{B_+} [\mathbf{u}\cdot\partial_t\boldsymbol{\varphi} + \mathbf{f}\cdot\partial_x\boldsymbol{\varphi}]\,dxdt + \iint_{B_-} [\mathbf{u}\cdot\partial_t\boldsymbol{\varphi} + \mathbf{f}\cdot\partial_x\boldsymbol{\varphi}]\,dxdt \\ &= -\int_{\Gamma\cap B} [n_t\mathbf{u}_r + n_x\mathbf{f}(\mathbf{u}_r)]\cdot\boldsymbol{\varphi}\,d\sigma - \iint_{B_+} [\partial_t\mathbf{u} + \partial_x\mathbf{f}(\mathbf{u})]\cdot\boldsymbol{\varphi}\,dxdt \\ &\quad + \int_{\Gamma\cap B} [n_t\mathbf{u}_l + n_x\mathbf{f}(\mathbf{u}_l)]\cdot\boldsymbol{\varphi}\,d\sigma - \iint_{B_-} [\partial_t\mathbf{u} + \partial_x\mathbf{f}(\mathbf{u})]\cdot\boldsymbol{\varphi}\,dxdt. \end{aligned}$$

Since $\mathbf{u}$ is a classical solution away from Γ, the second and fourth integrals vanish and the result is

$$\int_{\Gamma\cap B} [n_t(\mathbf{u}_r - \mathbf{u}_l) + n_x(\mathbf{f}(\mathbf{u}_r) - \mathbf{f}(\mathbf{u}_l))]\cdot\boldsymbol{\varphi}\,d\sigma = 0;$$

as this needs to be satisfied for every $\boldsymbol{\varphi}$, we obtain (7.2).

On the other hand, using the same derivation, without assuming that the equation vanishes, we see that if a piecewise C^1 function $\mathbf{u}$ is a solution in any set where it is of class C^1 and satisfies (7.2), then the equation must vanish, and $\mathbf{u}$ is a weak solution.

12c The Riemann Problem

We deal with the model for isentropic gas dynamics in Lagrangian coordinates, which we have already considered in Section 7.1. Here we use an explicit expression for p. We have

$$\begin{bmatrix} v \\ w \end{bmatrix}_t + \begin{bmatrix} 0 & -1 \\ p'(v) & 0 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix}_x = 0, \quad t > 0,\, x \in \mathbb{R},$$

where

$$p(v) = \kappa v^{-\gamma}, \quad \gamma > 1, \quad \kappa = \frac{(\gamma-1)^2}{4\gamma}.$$

An analogous study with a general p is given in Bressan and Smoller [20, 239]. The initial condition for the Riemann problem is

$$\begin{bmatrix} v \\ w \end{bmatrix}(x,0) = \begin{cases} \begin{bmatrix} v^- \\ w^- \end{bmatrix} & \text{if } x < 0, \\ \begin{bmatrix} v^+ \\ w^+ \end{bmatrix} & \text{if } x > 0. \end{cases}$$

The eigenvalues are

$$\lambda_1 = -\sqrt{-p'(v)} = -\sqrt{\frac{\kappa\gamma}{v^{\gamma+1}}}, \qquad \lambda_2 = \sqrt{-p'(v)} = \sqrt{\frac{\kappa\gamma}{v^{\gamma+1}}},$$

and for the corresponding eigenvectors we can take

$$\mathbf{r}_1 = \begin{bmatrix} 1 \\ \sqrt{\frac{\kappa\gamma}{v^{\gamma+1}}} \end{bmatrix}, \qquad \mathbf{r}_2 = \begin{bmatrix} -1 \\ \sqrt{\frac{\kappa\gamma}{v^{\gamma+1}}} \end{bmatrix}.$$

We have already seen that both characteristic fields are genuinely nonlinear.

The 1-rarefaction wave is given by the solution of the Cauchy problem

$$\begin{cases} \dfrac{dv}{d\epsilon} = 1, \\ \dfrac{dw}{d\epsilon} = \sqrt{\dfrac{\kappa\gamma}{v^{\gamma+1}}}, \\ v(0) = v^-, \\ w(0) = w^-, \end{cases}$$

that is,

$$R_1: \begin{cases} v = v^- + \epsilon, \\ w = -(v^- + \epsilon)^{-\frac{\gamma-1}{2}} + w^- + (v^-)^{-\frac{\gamma-1}{2}}, \end{cases} \qquad \epsilon \geq 0.$$

In the same way, the 2-rarefaction wave is given by the solution of the Cauchy problem

$$\begin{cases} \dfrac{dv}{d\epsilon} = -1, \\ \dfrac{dw}{d\epsilon} = \sqrt{\dfrac{\kappa\gamma}{v^{\gamma+1}}}, \\ v(0) = v^-, \\ w(0) = w^-, \end{cases}$$

which yields

$$R_2 : \begin{cases} v = v^- - \epsilon, \\ w = (v^- - \epsilon)^{-\frac{\gamma-1}{2}} + w^- - (v^-)^{-\frac{\gamma-1}{2}}, \end{cases} \qquad \epsilon \geq 0.$$

Since $\epsilon \geq 0$, the Riemann problem admits a solution in the form of a 1-rarefaction wave if $v^+ > v^-$, and a solution in the form of a 2-rarefaction wave if $v^+ < v^-$.

If we now move to shocks, we rely on the Rankine–Hugoniot conditions, which in this case read

$$\begin{cases} \lambda(v - v^-) = -(w - w^-), \\ \lambda(w - w^-) = p(v) - p(v^-), \end{cases} \qquad p(v) = \kappa v^{-\gamma}.$$

We need to distinguish whether $\lambda = -\dfrac{w - w^-}{v - v^-}$ is positive or negative. Hence, we have two families of curves.

1. If $\lambda < 0$, either $v > v^-$ and $w > w^-$, or $v < v^-$ and $w < w^-$, which implies

$$S_- : \begin{cases} w = w^- + \sqrt{-(v - v^-)(p(v) - p(v^-))} & \text{if } v > v^-, \\ w = w^- - \sqrt{-(v - v^-)(p(v) - p(v^-))} & \text{if } v < v^-. \end{cases}$$

2. Otherwise, if $\lambda > 0$, either $v > v^-$ and $w < w^-$, or $v < v^-$ and $w > w^-$, which implies

$$S_+ : \begin{cases} w = w^- - \sqrt{-(v - v^-)(p(v) - p(v^-))} & \text{if } v > v^-, \\ w = w^- + \sqrt{-(v - v^-)(p(v) - p(v^-))} & \text{if } v < v^-. \end{cases}$$

We need to check which of these are admissible, taking into account Lax shock conditions (8.1).

1. For λ_1 this amounts to

$$\lambda < \lambda_1(v^-), \qquad \lambda_1(v) < \lambda < \lambda_2(v). \tag{12.1c}$$

2. For λ_2 this amounts to

$$\lambda_1(v^-) < \lambda < \lambda_2(v^-), \qquad \lambda_2(v) < \lambda. \tag{12.2c}$$

Consider (12.1c). Since $\lambda_1(v^-) < 0$, we have that $\lambda < 0$ too, and we need to work with S_-. On the other hand, we must also have $\lambda_1(v) < \lambda < \lambda_1(v^-)$, which yields $v < v^-$. Hence, we conclude that for $\lambda < 0$ we work with

$$S_1: \qquad w = w^- - \sqrt{-(v - v^-)(p(v) - p(v^-))} \quad \text{if } v < v^-.$$

Consider (12.2c). Since $\lambda_2(v) > 0$, we have that $\lambda > 0$ too, and we need to work with S_+. On the other hand, we must also have $\lambda_2(v) < \lambda < \lambda_2(v^-)$, which yields $v > v^-$. Hence, we conclude that for $\lambda > 0$ we work with

$$S_2: \qquad w = w^- - \sqrt{-(v - v^-)(p(v) - p(v^-))} \quad \text{if } v > v^-.$$

Altogether, if we take into account the lines originating from (v^-, w^-), which we have obtained so far, we have R_1 and R_2, which corresponds to $\epsilon > 0$, and S_1 and S_2, which can be parametrized with $\epsilon < 0$. Correspondingly, we have four regions in the half plane $v > 0$ (see Figure 12.1c). Notice that, as shown in Proposition 10.1, we have a second-order contact at $\mathbf{u}_l$ between R_1 and S_1, and between R_2 and S_2.

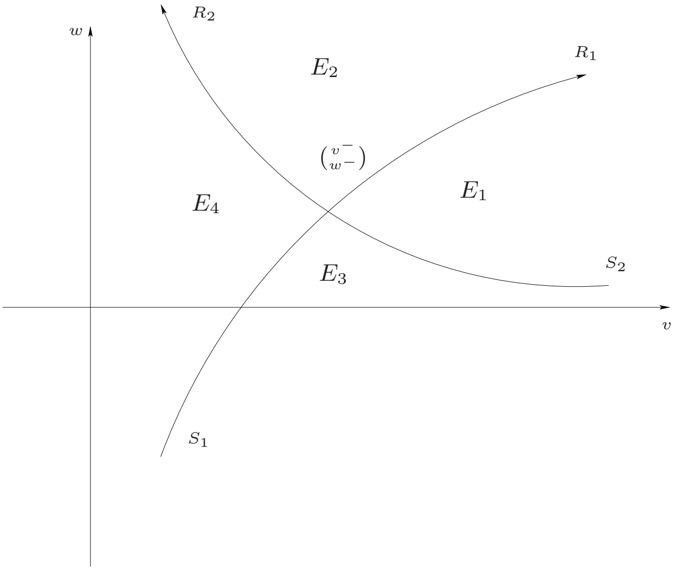

Fig. 12.1c

Assuming that $\mathbf{u}_r = (v^+, w^+)$ is sufficiently close to $\mathbf{u}_l = (v^-, w^-)$, the structure of the general solution of the Riemann problem corresponds to the region in which $\mathbf{u}_r$ is located.

Suppose first that $\mathbf{u}_r \in S_1$; in such a case the situation is described in Figure 12.2c, where the speed is given by $s = -\frac{w^+ - w^-}{v^+ - v^-} < 0$. A completely analogous situation occurs if $\mathbf{u}_r \in S_2$.

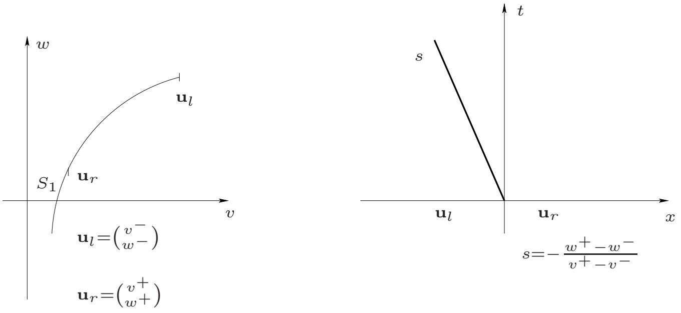
w
$\mathbf{u}_l$
S_1
$\mathbf{u}_r$
v
$\mathbf{u}_l = \binom{v^-}{w^-}$
$\mathbf{u}_r = \binom{v^+}{w^+}$
t
s
$\mathbf{u}_l$
$\mathbf{u}_r$
x
$s = -\frac{w^+ - w^-}{v^+ - v^-}$

Fig. 12.2c

On the other hand, if $\mathbf{u}_r \in R_1$, the solution is described in Figure 12.3c, where $\lambda_1^- = \lambda_1(\mathbf{u}_l)$, $\lambda_1^+ = \lambda_1(\mathbf{u}_r)$, and the shaded region represents the rarefaction wave. Again, the situation is perfectly analogous if $\mathbf{u}_r \in R_2$.

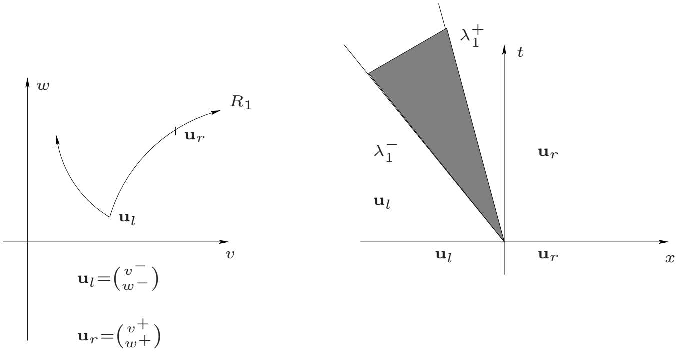

Fig. 12.3c

Suppose now that $\mathbf{u}_r \in E_2$: in such a case, it is not hard to show that there exists a unique $\mathbf{u}' \in R_1$ such that $\mathbf{u}_r$ belongs to the corresponding line R_2' through $\mathbf{u}'$. Following the discussion of Section 12, the solution is given in Figure 12.4c, where $\lambda_1^- = \lambda_1(\mathbf{u}_l)$, $\lambda_1^+ = \lambda_1(\mathbf{u}')$, $\lambda_2^- = \lambda_2(\mathbf{u}')$, $\lambda_2^+ = \lambda_2(\mathbf{u}_r)$, and the two shaded regions correspond to the two rarefaction waves connecting $\mathbf{u}_l$ and $\mathbf{u}'$, and $\mathbf{u}'$ and $\mathbf{u}_r$ respectively.

If $\mathbf{u}_r \in E_3$, again it is not hard to see that we have a unique $\mathbf{u}'$ such that $\mathbf{u}_r$ belongs to the corresponding line S_2' through $\mathbf{u}'$. According to the study of Section 12, the solution is described in in Figure 12.5c, where $s_1 = -\frac{w'-w^-}{v'-v^-}$ and $s_2 = -\frac{w^+-w'}{v^+-v'}$.

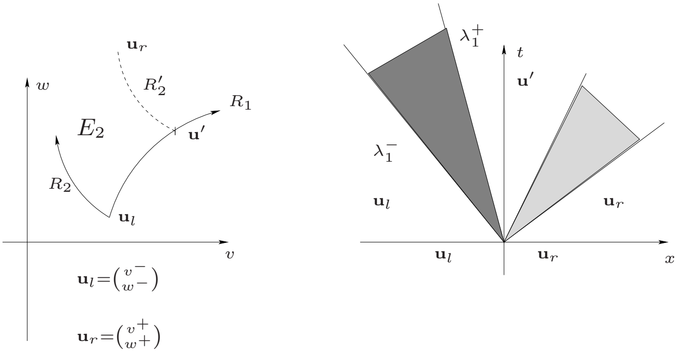

Fig. 12.4c

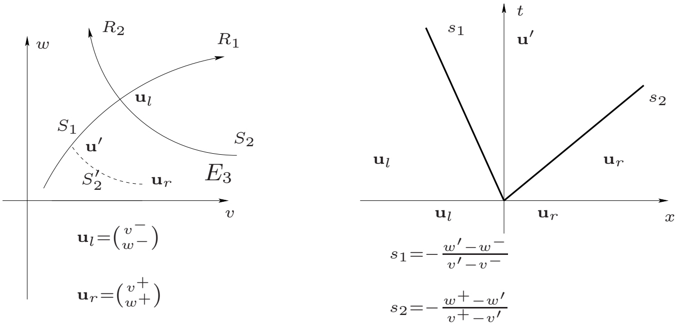

Fig. 12.5c

Summarizing, this is what we have according to the four different regions that the half plane $v > 0$ is divided into:

- $\mathbf{u}_r \in E_1$: one 1-rarefaction wave and one 2-shock;
- $\mathbf{u}_r \in E_2$: two rarefaction waves;
- $\mathbf{u}_r \in E_4$: one 1-shock and one 2-rarefaction wave;
- $\mathbf{u}_r \in E_3$: two shocks.

13c Convex Entropies

We get back to the full system of gas dynamics in Lagrangian coordinates, which we introduced in (12.1). We rewrite it as

$$\begin{cases} \partial_t \tau - \partial_x u = 0, & \\ \partial_t u + \partial_x p = 0, & x \in \mathbb{R},\ t > 0, \\ \partial_t E + \partial_x (pu) = 0, & \end{cases}$$

where $\tau > 0$ is the specific volume (i.e., $\tau = \frac{1}{\rho}$), u is the velocity, and $E > 0$ is the specific energy (i.e., total energy per unit mass). We discuss the results of § 3.3 of Coquel and LeFloch [36], to which we refer for all the details (see also § 2 of Bereux et al. [15]). The unknowns of the problem are τ, u, E, and the pressure p is given by

$$p = \frac{\gamma - 1}{\tau}\left(E - \frac{1}{2}u^2\right), \qquad \gamma > 1,$$

which implies that necessarily $E - \frac{1}{2}u^2 > 0$. The entropies for the system are given by

$$U = U(\tau, u, E) = -G(S), \qquad S = \ln\left((\gamma - 1)\tau^{\gamma-1}\left(E - \frac{1}{2}u^2\right)\right),$$

where $G : \mathbb{R} \to \mathbb{R}$ is an arbitrary smooth function. It turns out that U is a convex function if $G'(S) \geq 0$ and $R \leq \frac{1}{\gamma}$, where $R(S) = \frac{G''(S)}{G'(S)}$.

The Eulerian formulation for the complete gas dynamics was given in Section 7.2. If we rewrite it in terms of ρ, $u\, s$, where s is the physical entropy, and we assume that $p = p(\rho, s)$, we have

$$\begin{cases} \partial_t \rho + u\partial_x \rho + \rho \partial_x u = 0, \\ \partial_t u + \dfrac{1}{\rho}\dfrac{\partial p}{\partial \rho}\partial_x \rho + u\partial_x u + \dfrac{1}{\rho}\dfrac{\partial p}{\partial s}\partial_x s = 0, \qquad x \in \mathbb{R},\ t > 0. \\ \partial_t s + u\partial_x s = 0, \end{cases}$$

From it, one sees that smooth solutions satisfy

$$\partial_t(\rho f(s)) + \partial_x(\rho f(s) u) = 0,$$

for every regular function f, and this gives

$$U(\rho, u, s) = \rho f(s), \qquad F(\rho, u, s) = \rho f(s) u.$$

In Tartar [256] a careful analysis is developed, in order to show that these are the only non trivial entropies in this context.

References

1. *Œuvres complètes de N. H. Abel mathématicien*, M.M. L. Sylow and S. Lie, Eds. 2 Vols. (Oslo 1881).
2. N. Abel, Solution de quelques problèmes à l'aide d'intégrales définies, *Œuvres, #1, 11–27.*
3. N. Abel, Résolution d'un problème de mécanique, *Œuvres, #1, 97–101.*
4. R. Abraham, J.E. Marsden and T. Ratiu, *Manifolds, Tensor Analysis, and Applications*, Applied Mathematical Sciences, 75, II edition, Springer-Verlag, New York, 1988.
5. R.A. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.
6. S. Agmon, *Lectures on Elliptic Boundary Value Problems*, Van Nostrand, Princeton NJ, 1965.
7. S. Aizawa, A Semigroup Treatment of the Hamilton-Jacobi Equations in Several Space Variables, *Hiroshima Math. J., No. 6, (1976), 15–30*
8. A.D. Aleksandrov, Uniqueness conditions and estimates of the solution of Dirichlet's problem, *Vestn. Leningr. Un-ta. Ser. Matematika, Mekhanika, Astronomiya,* **13***(3), (1963), 5–29.*
9. A. Ambrosetti and G. Prodi, *Analisi Non-Lineare*, Quaderni della Scuola Normale Superiore di Pisa, 1973.
10. P. Appell, Sur l'équation $(\partial^2 z/\partial x^2) - (\partial z/\partial y) = 0$ et la théorie de chaleur, *J. Math. Pures Appl.,* **8***, (1892), 187–216.*
11. D.G. Aronson, Bounds for the fundamental solution of a parabolic equation, *Bull. Amer. Math. Soc.,* **73***, (1967), 890–896.*
12. D.G. Aronson and J. Serrin, Local behavior of solutions of quasilinear parabolic equations, *Arch. Rational Mech. Anal.,* **25***, (1967), 81–122.*
13. A.V. Azevedo, A.J. de Souza, F. Furtado and D. Marchesin, Uniqueness of the Riemann Solution for Three-Phase Flow in a Porous Medium, *SIAM J. Appl. Math.,* **74***(6), (2014), 1967–1997.*
14. G. Barenblatt, On some unsteady motions of a liquid or a gas in a porous medium, *Prikl. Mat. Meh.* **16***, (1952), 67–78.*
15. F. Bereux, E. Bonnetier and P.G. LeFloch, Gas Dynamics System: Two Special Cases, *SIAM J. Math. Anal.,* **28***, (1997), 499–515.*
16. D. Bernoulli, Réflexions et éclaircissements sur les nouvelles vibrations des cordes, *Mémoires de l'Academie Royale des Sciences et belles lettres, Berlin, (1755).*

© Springer Nature Switzerland AG 2023

E. DiBenedetto, U. Gianazza, *Partial Differential Equations*,
Cornerstones, https://doi.org/10.1007/978-3-031-46618-2

17. D. Bernoulli, Mémoire sur les vibrations des cordes d'une épaisseur inégale, *Mémoires de l'Academie royale des Sciences et belles lettres, Berlin, (1765).*
18. D. Bernoulli, Commentatio physico-mechanica generalior principii de coexistentia vibrationum simplicium haud perturbaturum in systemate compositio, *Novi Commentarii Academiae Scientiarum Imperialis Petropolitanae,* **19***, (1775), 239.*
19. F. Bowman, *Introduction to Bessel Functions*, Dover, New York, 1958.
20. A. Bressan, *Hyperbolic systems of conservation laws*, Oxford Lecture Series in Mathematics and its Applications, 20, The one-dimensional Cauchy problem, Oxford University Press, Oxford, 2000.
21. L.E.J. Brouwer, Über Abbildungen von Mannigfaltigkeiten, *Math. Ann.,* **71***, (1911), 97–15.*
22. S.E. Buckley and M. Leverett, Mechanism of fluid displacements in sands, *Trans. AIME,* **146***(1), (1942), 107–116.*
23. C. Burch, A Semigroup Treatment of the Hamilton-Jacobi Equations in Several Space Variables, *J. Diff. Eq.,* **23***, (1977), 107–124.*
24. J.M. Burgers, Application of a Model System to Illustrate Some Points of the Statistical Theory of Free Turbulence, *Proc. Acad. Sci. Amsterdam,* **43***, (1940), 2–12.*
25. J. M. Burgers, A Mathematical Model Illustrating the Theory of Turbulence, in *Advances in Applied Mechanics, Ed. R. von Mises and T. von Kármán, Vol. 1, Academic Press, New York, 1948, 171–199.*
26. L. Caffarelli, R. Kohn and L. Nirenberg, Partial regularity of suitable weak solutions of the Navier-Stokes equations, *Comm. Pure Appl. Math.,* **35***(6), (1982), 771–831.*
27. A.P. Calderón and A. Zygmund, On the existence of certain singular integrals, *Acta Math.,* **88***, (1952), 85–139.*
28. S. Campanato, Proprietà di una famiglia di spazi funzionali, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3),* **18***, (1964), 137–160.*
29. C. Carathéodory, *Vorlesungen über reele Funktionen*, Teubner, 1918.
30. H.S. Carlslaw and J.C. Jaeger, *Conduction of Heat in Solids*, Oxford Univ. Press, Oxford, 1959.
31. H. Cartan, *Functions Analytiques d'une Variable Complexe*, Dunod, Paris, 1961.
32. A. Cauchy, Mémoire sur les systèmes d'équations aux derivée partielles d'ordre quelconque, et sur leur réduction à des systèmes d'équations linéaires du premier ordre, *C.R. Acad. Sci. Paris,* **40***, (1842), 131–138.*
33. A. Cauchy, Mémoire sur les intégrales des systèmes d'équations différentielles ou aux derivées partielles, et sur les développements de ces intégrales en séries ordonnés suivant les puissances ascendentes d'un paramètre que renferment les équations proposées, *C.R. Acad. Sci. Paris,* **40***, (1842), 141–146.*
34. P.G. Ciarlet *The Finite Element Method for Elliptic Problems*, SIAM, Classics in Analysis No. 40, Philadelphia, 2002.
35. P. Cohen, The Non-Uniqueness of the Cauchy Problem, *O.N.R. Technical report n. 93, Stanford University (1960).*
36. F. Coquel and P.G. LeFloch, An entropy satisfying MUSCL scheme for systems of conservation laws, *Numer. Math.,* **74***(1), (1996), 1–33.*
37. R. Courant and K.O. Friedrichs, *Supersonic Flow and Shock Waves*, Interscience, New York, 1948.
38. R. Courant and D. Hilbert, *Methods of Mathematical Physics*, Vols. I and II, Interscience, New York, 1953, 1962.

39. R. Courant and P. Lax, Cauchy's problem for non-linear hyperbolic differential equations in two independent variables, *Ann. Mat. Pura Appl.,* **40***(1), (1955), 161–166.*
40. C.M. Dafermos, Hyperbolic Systems of Conservation Laws, Proceedings of *Systems of Nonlinear Partial Differential Equations*, (Oxford, 1982), 25–70, NATO ASI Sci. Inst. Ser. C: Math. Phys. Sci. **111**, Reidel, Dordrecht–Boston, Mass., 1983.
41. G. Darboux, Sur l'existence de l'intégrale dans les équations aux derivées partielles d'ordre quelconque, *C. R. Acad. Sci. Paris,* **80***, (1875), 317–318.*
42. G. Darboux, *Leçons sur la Théorie Générale des Surfaces et les Applications Géometriques du Calcul Infinitesimal*, Gauthiers–Villars, Paris, 1896.
43. P. Davidson, *An Introduction to Magnetohydrodynamics*, Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge, 2001.
44. E.B. Davies, *Heat kernels and spectral theory*, Cambridge Tracts in Mathematics, 92, Cambridge University Press, Cambridge, 1990.
45. E. DeGiorgi, Un esempio di non-unicità della soluzione del problema di Cauchy, relativo ad una equazione differenziale lineare a derivate parziali di tipo parabolico, *Rend. Mat. e Appl. (5),* **14***, (1955), 382–387.*
46. E. DeGiorgi, Un teorema di unicità per il problema di Cauchy, relativo ad equazioni differenziali lineari a derivate parziali di tipo parabolico, *Ann. Mat. Pura Appl. (4),* **40***, (1955), 371–377.*
47. E. DeGiorgi, Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari, *Mem. Acc. Sc. Torino, Cl. Sc. Mat. Fis. Nat.* **3***(3), (1957), 25–43.*
48. E. DiBenedetto, Harnack Estimates in Certain Function Classes, *Atti Sem. Mat. Fis. Univ. Modena,* **XXXVII***, (1989), 173–182.*
49. E. DiBenedetto, *Degenerate Parabolic Equations*, Springer-Verlag, New York, Series *Universitext*, 1993.
50. E. DiBenedetto, *Real Analysis*, Birkhäuser, Boston, 2002.
51. E. DiBenedetto, *Classical Mechanics*, Birkhäuser, Boston, 2010.
52. E. DiBenedetto and A. Friedman, Bubble growth in porous media, *Indiana Univ. Math. J.,* **35***(3) (1986), 573–606.*
53. E. DiBenedetto and N.S. Trudinger, Harnack Inequalities for Quasi-Minima of Variational Integrals, *Ann. Inst. H. Poincaré, Analyse Non-Linéaire,* **1***(4), (1984), 295–308.*
54. E. DiBenedetto, U. Gianazza, V. Vespri, Local Clustering of the Non-Zero Set of Functions in $W^{1,1}(E)$, *Rend. Acc. Naz. Lincei, Mat. Appl.* **17***, (2006), 223–225.*
55. E. DiBenedetto, U. Gianazza and V. Vespri, Harnack Estimates for Quasi-Linear Degenerate Parabolic Equations, *Acta Math.,* **200***, (2008), 181–209.*
56. E. DiBenedetto, U. Gianazza and V. Vespri, Alternative Forms of the Harnack Inequality for Non-Negative Solutions to Certain Degenerate and Singular Parabolic Equations, *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur.* **20***(4), (2009), 369–377.*
57. R.J. DiPerna, Review of "Shock waves and reaction-diffusion equations," *Bull. Amer. Math. Soc.,* **11***(1), (1984), 204–214.*
58. J. Dieudonné, *Treatise on Analysis*, Academic Press, New York, 1976.
59. Ph.-A. Dionne, Sur les problèmes de Cauchy hyperboliques bien posés, *J. Analyse Math.,* **10***, (1962/1963), 1–90.*
60. P. Dive, Attraction d'ellipsoides homogènes et réciproque d'un théorème de Newton, *Bull. Soc. Math. France, # 59 (1931), 128–140.*

61. J.M. Duhamel, Sur les vibrations d'une corde flexible chargée d'un ou plusieurs curseurs, *J. de l'École Polytechnique, No.* **17***, (1843), cahier 29, 1–36.*
62. N. Dunford and J.T. Schwartz, *Linear operators*, New York, Wiley, 1957,
63. A. Einstein, Investigations on the Theory of the Brownian Movement, Dover, New York, 1956.
64. A. Einstein, Über die von der molekularkinetischen Theorie der Wärme geforderte Bewegung von in ruhenden Flüssigkeiten suspendierten Teilchen, *Ann. Phys.* **17***, (1909), 549–560.*
65. L. Escauriaza, G.A. Seregin and V. Sverak, $L^{3,\infty}$-solutions of the Navier-Stokes equations and backward uniqueness, *Uspekhi Mat. Nauk* **58***(2), (2003), 3–44;* English translation *Russ. Math. Surveys* **58***(2), (2003), 211–250.*
66. L.C. Evans, *Partial Differential Equations*, American Mathematical Society, Graduate Studies in Mathematics **19**, Providence R.I., 1998.
67. E.B. Fabes, Singular integrals and partial differential equations of parabolic type, *Studia Mathematica, T. XXVIII, 81–131, (1966).*
68. E.B. Fabes, Gaussian upper bounds on fundamental solutions of parabolic equations; the method of Nash, in *Dirichlet forms, Lecture Notes in Mathematics, 1563, Lectures given at the First C.I.M.E. Session held in Varenna, June 8–19, 1992, Edited by G. Dell'Antonio and U. Mosco, Springer-Verlag, Berlin, 1993, 1–20.*
69. E.B. Fabes and D.W. Stroock, A new proof of Moser's parabolic Harnack inequality using the old ideas of Nash, *Arch. Rational Mech. Anal.,* **96***(4), (1986), 327–338.*
70. E.B. Fabes, B.F. Jones and N.M. Riviere, The initial value problem for the Navier-Stokes equations with data in L^p, *Arch. Rational Mech. Anal.,* **45***, (1972), 222–240.*
71. G. Fichera, *Linear Elliptic Differential Systems and Eigenvalue Problems*, Lecture Notes in Mathematics, No. 8, Springer-Verlag, Berlin, 1965.
72. R. Finn, On steady-state solutions of the Navier-Stokes partial differential equations, *Arch. Rat. Mech. Analysis,* **3***, 381–396, (1959).*
73. R. Finn, On the steady-state solutions of the Navier-Stokes equations. III, *Acta Math.,* **105***(3–4), (1961), 197–244.*
74. R. Finn, Stationary solutions of the Navier-Stokes equations, *Proc. Symp. Appl. Math.* **17***, 1965. Amer. Math. Soc.*
75. R. Finn, Mathematical questions relating to viscous flow in an exterior domain, *Rocky Mt. J. of Math.* **3***, (1973), 107–140.*
76. R. Finn and W. Noll, On the uniqueness and non-existence of Stokes flows, *Arch. Rat. Mech. Analysis,* **1***, 95–106, (1957).*
77. W.H. Fleming and R. Rischel, An integral formula for total gradient variation, *Arch. Math.* **XI***, (1960), 218–222.*
78. J.B. Fourier, *Théorie Analytique de la Chaleur*, Chez Firmin Didot, Paris, 1822.
79. I. Fredholm, Sur une nouvelle méthode pour la résolution du problème de Dirichlet, *Kong. Vetenskaps-Akademiens Fröh. Stockholm, (1900), 39–46.*
80. I. Fredholm, Sur une classe d'équations fonctionnelles, *Acta Math.,* **27***, (1903), 365–390.*
81. A. Friedman, Interior Estimates for Parabolic Systems of Partial Differential equations, *J. Math. Mech.,* **7***, (1958), 393–417.*
82. A. Friedman, A new proof and generalizations of the Cauchy-Kowalewski Theorem, *Trans. Amer. Math. Soc.,* **98***, (1961), 1–20.*

83. A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice Hall, Hoboken NJ, 1964.
84. A. Friedman, *Partial Differential Equations*, Holt, Rinehart and Winston, New York, 1969.
85. A. Friedman, *Variational Principles and Free Boundary Problems*, John Wiley & Sons, New York, 1982.
86. K. Friedrichs, Spektraltheorie halbbeschränkter Operatoren und Anwendung auf die Spektralzerlegung von Differentialoperatoren, *Mathematischen Annalen* **109***, (1934), 465–487.*
87. K.O. Friedrichs and P.D. Lax, Systems of conservation equations with a convex extension, *Proc. Nat. Acad. Sci. U.S.A.,* **68***, (1971), 1686–1688.*
88. Y. Fujie and H. Tanabe, On some parabolic equations of evolution in Hilbert spaces, *Osaka J. Math.,* **10***, (1973), 115–130.*
89. H. Fujita, On the existsence and regularity of the steady-state solutions of the Navier-Stokes equations, *J. Fac. Sci. Univ. of Tokio,* **9***, (1961), 59–102.*
90. E. Gagliardo, Proprietà di Alcune Funzioni in n Variabili, *Ricerche Mat.* **7***, (1958), 102–137.*
91. G.P. Galdi, *An introduction to the mathematical theory of the Navier-Stokes equations. Vol. I*, Springer Tracts in Natural Philosophy, 38, Linearized steady problems, Springer-Verlag, New York, 1994.
92. G.P. Galdi, *An introduction to the mathematical theory of the Navier-Stokes equations. Vol. II*, Springer Tracts in Natural Philosophy, 39, Nonlinear steady problems, Springer-Verlag, New York, 1994.
93. G.P. Galdi, An Introduction to the Navier-Stokes Initial-Boundary Value Problem, in G.P. Galdi, J.G. Heywood and R. Rannacher eds. *Fundamental Directions in Mathematical Fluid Mechanics, Birkhäuser Basel, Basel, (2000), 1–70.*
94. P.R. Garabedian, *Partial Differential Equations*, John Wiley & Sons, New York, 1964.
95. L. Gårding, Hörmander's work on linear differential operators, *Proc. Internat. Congress Math. 1962, Institut Mittag-Leffler, Djursholm, Sweden, (1963), 44–47.*
96. L. Gårding, Hyperbolic equations in the twentieth century, in *Matériaux pour l'histoire des mathématiques au XX*$^{\text{e}}$ *siècle (Nice, 1996), Sémin. Congr.,* **3***, 37–68.*
97. M. Gevrey, Sur les équations aux dérivées partielles du type parabolique, *J. Math. Pures et Appl.* **9***(VI), (1913), 305–471*; ibidem **10***(IV), (1914), 105–148.*
98. M. Giaquinta and E. Giusti, Quasi-Minima, *Ann. Inst. Poincaré, Analyse non Lineaire,* **1***, (1984), 79–107.*
99. Y. Giga, Solutions for semilinear parabolic equations in L^p and regularity of weak solutions of the Navier-Stokes system, *J. Differential Equations,* **62***, (1986), 186–212.*
100. D. Gilbarg and J. Serrin, On isolated singularities of solutions of second order elliptic differential equations, *J. d'Analyse Math.* **4***, (1955), 309–340.*
101. D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, (2$^{\text{nd}}$ ed.), Die Grundlehren der Mathematischen Wissenschaften, No. 224, Springer-Verlag, Berlin, 1983.
102. E. Giusti, *Functions of Bounded Variation*, Birkhäuser, Basel, 1983.
103. J. Glimm, Solutions in the large for nonlinear hyperbolic systems of equations, *Comm. Pure Appl. Math.,* **18***, (1965), 697–715.*

104. J. Glimm and P.D. Lax, *Decay of Solutions of Systems of Non-Linear Hyperbolic Conservation Laws, Mem. Amer. Math. Soc.* **10***(101), (1970).*
105. J. Glimm, D. Marchesin and B. Plohr, The Theory of Shock Waves: From Riemann through Today, in *The legacy of Bernhard Riemann after one hundred and fifty years. Vol. I, Adv. Lect. Math. (ALM), 35, 251–273, Int. Press, Somerville, MA, 2016.*
106. G. Green, *An Essay on the Applications of Mathematical Analysis to the Theories of Electricity and Magnetism*, Nottingham, 1828.
107. K.K. Golovkin, Embedding Theorems, *Doklady Akad. Nauk, USSR,* **134***(1), (1960), 19–22.*
108. E. Goursat, *Cours d'Analyse Mathématique*, Gauthiers–Villars, Paris, 1913.
109. P. Grisvard, Commutativité de deux foncteurs d'interpolation et applications, II, *J. Math. Pures Appl.* **45** *(1966), 207–290.*
110. J. Hadamard, *Le problème de Cauchy et les Équations aux Deriveées Partielles Linéaires Hyperboliques*, Hermann et Cie, Paris, 1932.
111. J. Hadamard, *Lectures on Cauchy's Problem in Linear Partial Differential Equations*, Dover, New York, 1952.
112. J. Hadamard, Extension à l'équation de la chaleur d'un théorème de A. Harnack, *Rend. Circ. Mat. di Palermo, Ser. 2* **3***, (1954), 37–346.*
113. J.K. Hale, *Ordinary differential equations*, Pure and Applied Mathematics, Vol. XXI, Wiley-Interscience [John Wiley & Sons], New York-London-Sydney, 1969.
114. A. Hammerstein, Nichtlineare Integralgleichungen nebst Anwendungen, *Acta Math.,* **54***, (1930), 117–176.*
115. G.H. Hardy, Note on a theorem of Hilbert, *Math. Z.,* **6***, (1920), 314–317.*
116. G.H. Hardy, J.E. Littlewood, G. Pólya, *Inequalities*, Cambridge Univ. Press, 1963.
117. A. Harnack, *Grundlagen der Theorie des Logarithmischen Potentials*, Leipzig, 1887.
118. A. Harnack, Existenzbeweise zur Theorie des Potentiales in der Ebene und in Raume, *Math. Ann.,* **35***(1–2), (1889), 19–40.*
119. D. Hilbert, *Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen*, Leipzig, 1912.
120. E.W. Hobson, *The Theory of Spherical and Ellipsoidal Harmonics*, Chelsea Pub. Co., New York, 1965.
121. E. Holmgren, Ueber Systeme von linearen partiellen Differentialgleichungen, *Öfversigt af Kongl. Vetenskaps Akamemiens Förhandlingar,* **58***, (1901), 91–103.*
122. E. Hopf, The Partial Differential Equation $u_t + uu_x = \mu u_{xx}$, *Comm. Pure Appl. Math.,* **3***, (1950), 201–230.*
123. E. Hopf, Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen, *Math. Nachrichten,* **4***, (1950-1951), 213–231.*
124. E. Hopf, Generalized Solutions of Non Linear Equations of First Order, *J. Math. Mech.* **14***, (1965), 951–974.*
125. E. Hopf, On the Right Weak Solution of the Cauchy Problem for a Quasi Linear Equation of First Order, *J. Mech. Math.,* **19***, (1969/1970), 483–487.*
126. L. Hörmander, Uniqueness theorems and estimates for normally hyperbolic partial differential equations of the second order, *Comptes rendus du Douzième Congrès des Mathematiciens Scandinaves, Lund, (1953), 105–115.*
127. L. Hörmander, *Linear Partial Differential Operators*, Springer-Verlag, Berlin–Heidelberg–New York, 1963.

128. L. Hörmander, *An Introduction to Complex Analysis in Several Variables*, D. Van Nostrand Co. Inc., Princeton, New Jersey, 1966.
129. H. Hugoniot, Sur la Propagation du Mouvement dans les Corps et Spécialement dans les Gaz Parfaits, *Journ. de l'École Polytechnique,* **58***, (1889), 1–125.*
130. F. John, *Plane Waves and Spherical Means Applied to Partial Differential Equations*, Interscience Publishers (1955).
131. F. John, *Partial Differential Equations*, Springer-Verlag, New York, 1986.
132. C. Kahane, On the spatial analyticity of solutions of the Navier-Stokes equations, *Arch. Rational Mech. Anal.,* **33***, (1969), 386–405.*
133. A.S. Kalashnikov, Construction of Generalized Solutions of Quasi-Linear Equations of First Order Without Convexity Conditions as Limits of Solutions of Parabolic Equations With a Small Parameter, *Dokl. Akad. Nauk SSSR,* **127***, (1959), 27–30.*
134. M. Kassmann, Harnack Inequalities: An Introduction, *Boundary Value Problem, Volume 2007, Article ID 81415.*
135. J.L. Kelley, *General Topology*, Van Nostrand, New York, 1961.
136. O.D. Kellogg, *Foundations of Potential Theory*, Springer-Verlag, Berlin, 1929, reprinted 1967.
137. D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities and Their Applications*, Academic Press, New York, 1980.
138. A.A. Kiselev and O. A. Ladyzhenskaya, On the existence and uniqueness of the solution of the nonstationary problem for a viscous, incompressible fluid, *Izv. Akad. Nauk SSSR Ser. Mat.,* **21***(5), (1957), 655–680.*
139. V.I. Kondrachov, Sur certaines propriétés des fonctions dans les espaces L^p, *C.R. (Dokl.) Acad. Sci. USSR (N.S.),* **48***, (1945), 535–539.*
140. S. Kowalewski, Zur Theorie der Partiellen Differentialgleichungen, *J. Reine Angew. Math.* **80***, (1875), 1–32.*
141. S.N. Kruzhkov, First Order Quasilinear Equations in Several Independent Variables, *Mat. USSR Sbornik* **10***(2), (1970), 217–243.*
142. S.N. Kruzhkov, Generalized Solutions of the Hamilton-Jacobi Equations of Eikonal Type-I, *Soviet Mat. Dokl.,* **16***, (1975), 1344–1348*;
143. S.N. Kruzhkov, Generalized Solutions of the Hamilton-Jacobi Equations of Eikonal Type-II, *Soviet Mat. Dokl.,* **27***, (1975), 405–446*;
144. S.N. Kruzhkov, Nonlocal Theory for Hamilton-Jacobi Equations, *Proceeding of a meeting in Partial Differential Equations* Edited by P. Alexandrov and O. Oleinik, Moscow Univ. 1978.
145. N.V. Krylov, Sequences of convex functions and estimates of the maximum of the solution of a parabolic equation, *Sib. Math. J.* **17***, (1976), 226–236.*
146. N.V. Krylov, *Controlled Diffusion Processes*, Stochastic Modelling and Applied Probability, Springer Verlag, Berlin, Heidelberg, 2009.
147. N.V. Krylov and M.V. Safonov, A certain property of solutions of parabolic equations with measurable coefficients, *Math. USSR Izv.* **16***(1), (1981), 151–164.*
148. O.A. Ladyzhenskaya, New equations for the description of motion of viscous incompressible fluids and solvability in the large of boundary value problems for them, *Proc. Steklov Inst. Math.,* **102***, (1967), 95–118 (transl. Trudy Math. Inst. Steklov,* **102***, (1967), 85–104).*
149. O.A. Ladyzhenskaya, *The Mathematical Theory of Viscous Incompressible Flow*, Second English edition, revised and enlarged. Translated from the Russian

by Richard A. Silverman and John Chu. Mathematics and its Applications, Vol. 2, Gordon and Breach, Science Publishers, New York-London-Paris, 1969.

150. O.A. Ladyzhenskaya, N.N. Ural'tzeva, *Linear and Quasilinear Elliptic Equations*, Academic Press, London–New York, 1968.
151. O.A. Ladyzhenskaya, N.A. Solonnikov, N.N. Ural'tzeva, *Linear and Quasilinear Equations of Parabolic Type*, Transl. Math. Monographs, **23**, AMS, Providence R.I., 1968.
152. L.D. Landau and E. M. Lifshitz, *Electrodynamic of Continuous Media*, Course in theoretical physics (vol. 8), Pergamon Press, Oxford, 1960.
153. E.M. Landis, *Second Order Equations of Elliptic and Parabolic Type*, Transl. Math. Monographs, **171**, AMS Providence R.I., 1997 (Nauka, Moscow, 1971).
154. N.S. Landkof, *Foundations of Modern Potential Theory*, Springer-Verlag, Berlin, 1972.
155. M.M. Lavrentiev, *Some Improperly Posed Problems of Mathematical Physics*, Springer-Verlag, Tracts in Nat. Philosophy, **11**, Berlin–Heidelberg–New York, 1967.
156. M.M. Lavrentiev, V.G. Romanov and S.P. Sisatskij, *Problemi Non Ben Posti in Fisica Matematica e Analisi*, C.N.R. Istit. Analisi Globale **12**, Firenze 1983. Italian transl. of *Nekorrektnye zadachi Matematicheskoi Fisiki i Analisa* Akad. Nauka Moscou, 1980.
157. P.D. Lax, Non-Linear Hyperbolic Equations, *Comm. Pure Appl. Math.,* **4**, *(1953), 231–258.*
158. P.D. Lax, On the Cauchy Problem for Partial Differential Equations With Multiple Characteristics, *Comm. Pure Appl. Math.,* **9**, *(1956), 135–169.*
159. P.D. Lax, Hyperbolic systems of conservation laws. II, *Comm. Pure Appl. Math.,* **10**, *(1957), 537–566.*
160. P.D. Lax, Shock waves and entropy, in *Contributions to nonlinear functional analysis (Proc. Sympos., Math. Res. Center, Univ. Wisconsin, Madison, Wis., 1971), (1971), 603–634.*
161. P.D. Lax, *Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves*, SIAM, Philadelphia, 1973.
162. P.D. Lax and A.M. Milgram, *Parabolic Equations*, Contrib. to the Theory of P.D.E.'s, Princeton Univ. Press, 1954, 167–190.
163. H.L. Lebesgue, Sur des cas d'impossibilité du problème de Dirichlet, *Comptes Rendus Soc. Math. de France, 17, (1913).*
164. J. Leray, *Étude de diverses équations intégrales non linéaires et de quelques problèmes que pose l'hydrodynamique*, Thesis, 1933.
165. J. Leray, Sur le mouvement d'un liquide visqueux emplissant l'espace, *Acta Math.,* **63**, *(1934), 193–248.*
166. J. Leray, *Hyperbolic differential equations*, Institute for Advanced Study (IAS), Princeton, N.J., 1953.
167. J. Leray, Équations hyperboliques non-strictes: Contre-exemples, du type De Giorgi, aux théorèmes d'existence et d'unicité, *Math. Ann.,* **162**, *(1965/1966), 228–236.*
168. J. Leray and Y. Ohya, *Systèmes linéaires, hyperboliques non stricts*, Deuxième Colloq. l'Anal. Fonct, 105–144, Centre Belge Recherches Math., Librairie Universitaire, Louvain, 1964.
169. J. Leray and J. Schauder, Topologie et équations fonctionelles, *Ann. Sci. Ècole Norm. Sup.,* **13**, *(1934), 45–78.*

170. M.C. Leverett, Capillary Behaviour in Porous Solids, *Trans. AIME,* **142**, *(1941), 152–169.*
171. N. Liao, A unified approach to the Hölder regularity of solutions to degenerate and singular parabolic equations, *J. Differential Equations,* **268***(10), 5704–5750, (2020).*
172. L. Lichenstein, Eine elementare Bemerkung zur reellen Analysis, *Math. Z.,* **30**, *(1929), 794–795.*
173. F. Lin, A new proof of the Caffarelli-Kohn-Nirenberg theorem, *Comm. Pure Appl. Math.,* **51**, *(1998), 241–257.*
174. J.-L. Lions, *Équations différentielles opérationnelles et problèmes aux limites*, Die Grundlehren der mathematischen Wissenschaften, Band 111, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1961.
175. J.-L. Lions and E. Magenes, *Problèmes aux limites non homogènes et applications. Vol. 1*, Travaux et Recherches Mathématiques, No. 17 Dunod, Paris, 1968
176. J.-L. Lions and E. Magenes, *Problèmes aux limites non homogènes et applications. Vol. 2*, Travaux et Recherches Mathématiques, No. 18 Dunod, Paris, 1968
177. J.-L. Lions and G. Prodi, Un théorème d'existence et unicité dans les équations de Navier-Stokes en dimension 2, *Comptes Rendus Acad. Sci. Paris,* **248**, *(1959), 3519–3521.*
178. P.L. Lions, B. Perthame and E. Tadmor, Kinetic formulation of the isentropic gas dynamics and p-systems, *Comm. Math. Phys.* **163**, *(1994), 415–431.*
179. A. Majda, A qualitative model for dynamic combustion, *SIAM J. Appl. Math.,* **41***(1), (1981), 70–93.*
180. A. Majda, *Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables*, Appl. Math. **53**, Springer-Verlag, New York, 1984.
181. V.G. Mazja, *Sobolev Spaces*, Springer-Verlag, New York, 1985.
182. N. Meyers and J. Serrin, $H = W$, *Proc. Nat. Acad. Sci.* **51**, *(1964), 1055–1056.*
183. S.G. Mikhlin, *Integral Equations*, Pergamon Press **4**, Oxford, 1964.
184. C.B. Morrey, On the Solutions of Quasilinear Elliptic Partial Differential Equations, *Trans. AMS* **43**, *(1938), 126–166.*
185. C.B. Morrey, *Multiple Integrals in the Calculus of Variations*, Springer-Verlag, New York, 1966.
186. J. Moser, On Harnack's Theorem for Elliptic Differential Equations, *Comm. Pure Appl. Math.,* **14**, *(1961), 577–591.*
187. J. Moser, A Harnack inequality for parabolic differential equations, *Comm. Pure and Appl. Math.,* **17**, *(1964), 101–134.*
188. J. Moser, Correction to: "A Harnack inequality for parabolic differential equations", *Comm. Pure Appl. Math.,* **20**, *(1967), 231–236.*
189. J. Moser, On a pointwise estimate for parabolic differential equations, *Comm. Pure Appl. Math.,* **24**, *(1971), 727–740.*
190. J. Nash, Continuity of solutions of parabolic and elliptic equations, *Amer. J. Math.,* **80**, *(1958), 931–954.*
191. A.I. Nazarov and N.N. Ural'tzeva, Convex-monotone hulls and an estimate of the maximum of the solution of a parabolic equation, *J. Math. Sci.* **37**, *(1987), 851–859.*
192. C.G. Neumann, *Untersuchungen über das logarithmische und Newtonsche Potential*, Leipzig, 1877.

193. L. Nirenberg, On Elliptic Partial Differential Equations, *Ann. Scuola Norm. Sup. Pisa,* **3***(13), (1959), 115–162.*
194. M. O'Leary, Integrability and Boundedness of Local Solutions to Singular and Degenerate Quasilinear Parabolic Equations, *Differential Integral Equations,* **12***(3), (1999), 435–452.*
195. Y. Ohya, Le problème de Cauchy pour les équations hyperboliques à caractéristique multiple, *J. Math. Soc. Japan,* **16***, (1964), 268–286.*
196. T. Ohyama, Interior regularity of weak solutions of the time-dependent Navier-Stokes equation, *Proc. Japan Acad.,* **36***, (1960), 273–277.*
197. O.A. Oleinik, Discontinuous Solutions of Non-Linear Differential Equations, *Uspekhi Mat. Nauk (N.S.),* **12***(3), (1957), 3–73*; (*Amer. Math. Soc. Transl.,* **2***(26), 95–172*).
198. O.A. Oleinik, Uniqueness and Stability of the Generalized Solution of the Cauchy Problem for a Quasi-Linear Equation, *Uspekhi Mat. Nauk (N.S.),* **14***(6), (1959), 165–170.*
199. R.E. Pattle, Diffusion from an instantaneous point source with a concentration-dependent coefficient, *Quarterly J. of Appl. Math.* **12***, (1959), 407–409.*
200. L.E. Payne, *Improperly Posed Problems in Partial Differential Equations*, SIAM, Vol. 22, Philadelphia, 1975.
201. O. Perron, Eine neue Behandlung der Randewertaufgabe für $\Delta u = 0$. *Math. Z., No. 18, (1923), 42–54.*
202. I. Petrowsky, Über das Cauchysche Problem für Systeme von partiellen Differentialgleichungen, *Rec. Math. [Mat. Sbornik] N.S.,* **2(44)***(5), (1937), 815–870.*
203. B. Pini, Sulla soluzione generalizzata di Wiener per il primo problema di valori al contorno nel caso parabolico, *Rend. Sem. Mat. Univ. Padova,* **23***, (1954) 422–434.*
204. A. Pliś, The problem of uniqueness for the solution of a system of partial differential equations, *Bull. Acad. Polon. Sci. Cl. III.,* **2***, (1954), 55–57.*
205. S.D. Poisson, Mémoire sur la théorie du son, *J. de l'École Polytechnique, Tome VII, Cahier XIV$^{\text{ème}}$, (1808), 319–392.*
206. S.D. Poisson, in Mouvement d'une corde vibrante composée de deux parties de matières differentes, *J. de l'École Polytechnique, Tome XI, Cahier XVIII$^{\text{ème}}$, (1820), 442–476.*
207. G. Prodi, Qualche risultato riguardo alle equazioni di Navier-Stokes nel caso bidimensionale, *Rendiconti del Seminario Matematico della Università di Padova,* **30***, (1960), 1–15.*
208. M.H. Protter and H.F. Weinberger, *Maximum Principles in Differential Equations*, Prentice Hall, Hoboken NJ, 1967.
209. T. Radó, *Subharmonic Functions*, Chelsea Pub. Co., New York, 1940.
210. J. Radon, Über lineare Funktionaltransformationen und Funktionalgleichungen, *Sitzsber. Akad. Wiss. Wien,* **128***, (1919), 1083–1121.*
211. W.J.M. Rankine, On the Thermodynamic Theory of Waves of Finite Longitudinal Disturbance, *Trans. Royal Soc. of London,* **160***, (1870), 277–288.*
212. H. Rhee R. Aris and N.R. Amundson, On the theory of multicomponent chromatography, *Philos. Trans. Roy. Soc. London Ser. A* **267***, (1970), 419–455.*
213. F. Riesz, Über lineare Funktionalgleichungen, *Acta Math.,* **41***, (1918), 71–98.*
214. F. Riesz, Sur les fonctions subharmoniques et leur rapport à la théorie du potentiel, part I and II, *Acta Math.,* **48***, (1926), 329–343*, and ibidem **54***, (1930), 321–360.*

215. F. Rellich, Ein Satz über mittlere Konvergenz, *Nach. Akad. Wiss. Göttingen Math. Phys. Kl., (1930), 30–35*
216. F. Rellich, Spektraltheorie in nicht-separablen Räumen, *Math. Annalen, Band 110, (1934), 342–356.*
217. S.J. Reye, *Fully Non-Linear Parabolic Differential Equations of Second Order*, Thesis (Ph.D.)–The Australian National University (Australia), ProQuest LLC, Ann Arbor, MI, 1985.
218. B. Riemann, Über die Fortpflanzung ebener Luftwellen von endlicher Schwingungsweite, *Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen, Band 8, (1860), 43–66.*
219. F. Riesz and B. Nagy, *Functional Analysis*, Dover, New York, 1990.
220. P.C. Rosenbloom and D.V. Widder, A temperature function which vanishes initially, *Amer. Math. Monthly,* **65**, *(1958), 607–609.*
221. H.L. Royden, *Real Analysis*, 3^{rd} ed., Macmillan, New York, 1988.
222. J.P. Schauder, Über lineare elliptische Differentialgleichungen zweiter Ordnung, *Math. Z.,* **38**, *(1934), 257–282.*
223. J.P. Schauder, Numerische Abschaetzungen in elliptischen linearen Differentialgleichungen, *Studia Math.* **5**, *(1935), 34–42.*
224. J.P. Schauder, Das Anfangswertproblem einer quasilinear hyperbolischen Differentialgleichung zweiter Ordnung, *Fund. Math.* **24**, *(1935), 213–246.*
225. J.P. Schauder, Cauchy'sches Problem für partielle Differentialgleichungen erster Ordnung. Anwendung einiger sich auf die Absolutbetrage der Lösungen beziehenden Abschätzungen, *Comment. Math. Helv.* **2**, *(1937), 263–283.*
226. A.E. Scheidegger, *The Physics of Flow Through Porous Media*, Univ. of Toronto Press, Toronto, 1974.
227. E. Schmidt, Auflösung der allgemeinen linearen Integralgleichungen, *Math. Annalen, Band 64, (1907), 161–174.*
228. L. Schwartz, *Théorie des Distributions*, Herman et Cie, Paris, 1966.
229. Yu.A. Semenov, On perturbation theory for linear elliptic and parabolic operators; the method of Nash, *Applied analysis (Baton Rouge, LA, 1996), Contemp. Math.,* **221**, *217–284, Amer. Math. Soc., Providence, RI, 1999.*
230. G. Seregin, Local regularity theory of the Navier-Stokes equations, *Handbook of mathematical fluid dynamics. Vol. IV, 159–200, Elsevier/North-Holland, Amsterdam, 2007.*
231. D. Serre, Von Neumann's comments about existence and uniqueness for the initial-boundary value problem in gas dynamics, *Bull. Amer. Math. Soc.,* **47**, *(2010), 139–144.*
232. J. Serrin, On the Phragmen-Lindelöf principle for elliptic differential equations, *J. Rat. Mech. Anal.* **3**, *(1954), 395–413.*
233. J. Serrin, On the interior regularity of weak solutions of the Navier-Stokes equations, *Arch. Rat. Mech. Analysis,* **9**, *187–195, (1962).*
234. J. Serrin, The initial value problem for the Navier-Stokes equations, *Nonlinear Problems, R.E. Langer Ed., Madison WI, the Univ. of Wisconsin Press, 1963.*
235. J. Serrin, Local Behavior of Solutions of Quasilinear Elliptic Equations, *Acta Math.,* **111**, *(1964), 101–134.*
236. J. Serrin, *Mathematical Principles of Classical Fluid Mechanics,* Handbuch der Physik, vol VIII **1**, Springer, Berlin-Göttingen-Heidelberg, 1959.
237. H. Shahgolian, On the Newtonian potential of a heterogeneous ellipsoid, *SIAM J. Math. Anal.,* **22**(5), *(1991), 1246–1255.*

238. M. Shimbrot and R.E. Welland, The Cauchy-Kowalewski Theorem, *J. Math. Anal. and Appl.,* **25***(3), (1976), 757–772.*
239. J. Smoller, *Shock Waves and Reaction-Diffusion Equations*, Grundlehren der mathematischen Wissenschaften 258, Springer-Verlag, New York, 1994.
240. M. Smoluchowski, Drei Vorträge über Diffusion, Brownsche Molekularbewegung and Koagulation von Koloidteilchen, *Phys. Zeitschrift,* **17** *557–571, and ibidem 587–599.*
241. S.L. Sobolev, On a Theorem of Functional Analysis, *Math. Sbornik* **46***, (1938), 471–496.*
242. S.L. Sobolev and S.M. Nikol'skii, Embedding Theorems, *Izdat. Akad. Nauk SSSR, Leningrad, (1963), 227–242.*
243. H. Sohr, Zur Regularitätstheorie der Instationären Gleichungen von Navier-Stokes, *Math. Z.,* **184***(3), (1983), 359–375.*
244. H. Sohr and W. von Wahl, On the singular set and the uniqueness of weak solutions of the Navier-Stokes equations, *Manuscripta Math.* **49***(1), (1984), 27–59.*
245. H. Sohr and W. von Wahl, On the regularity of the pressure of weak solutions of Navier-Stokes equations, *Arch. Math,* **46***(5), 1986, 428–439.*
246. E.M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, 1970.
247. W. Strauss, *Non Linear Wave Equations*, Conference Board of the Mathematical Sciences Regional Conf. Series in Math. **73**, AMS, Providence R.I., 1989.
248. M. Struwe, On partial regularity results for the Navier-Stokes equations, *Comm. Pure Appl. Math.,* **41***, (1988), 437–458.*
249. S. Tacklind, Sur les classes quasianalytiques des solutions des équations aux deriveées partielles du type parabolique, *Acta Reg. Sc. Uppsaliensis, (Ser. 4),* **10***, (1936), 3–55.*
250. G. Talenti, Sulle equazioni integrali di Wiener–Hopf, *Boll. Un. Mat. Ital.,* **7***(4), Suppl. fasc. 1, (1973), 18–118.*
251. G. Talenti, Best Constants in Sobolev Inequalities, *Ann. Mat. Pura Appl.* **110***, (1976), 353–372.*
252. G. Talenti, Sui problemi mal posti, *Boll. U.M.I. (5),* **15***, (1978), 1–29.*
253. L. Tartar, The compensated compactness method applied to systems of conservation laws, in *Systems of nonlinear partial differential equations (Oxford, 1982), NATO Adv. Sci. Inst. Ser. C: Math. Phys. Sci., 111, Reidel, Dordrecht, (1983), 263–285.*
254. L. Tartar, *Une introduction à la Théorie Mathématique des Systèmes Hyperboliques des Lois de Conservation*, Pubblic. **682**, Ist. Analisi Numerica, C.N.R. Pavia 1989.
255. L. Tartar, *From hyperbolic systems to kinetic theory*, Lecture Notes of the Unione Matematica Italiana, 6, Springer-Verlag, Berlin; UMI, Bologna, 2008.
256. L. Tartar, *A Short Introduction to Hyperbolic Systems of Conservation Laws*, unpublished lecture notes, (2009), 1–84.
257. W. Thomson Kelvin, Extraits de deux lettres adressées à M. Liouville, *J. de Mathématiques Pures et Appliquées,* **12***, (1847), 256.*
258. M. Torrilhon, Uniqueness conditions for Riemann problems of ideal magnetohydrodynamics, *J. Plasma Phys.,* **69***, (2003), 253–276.*
259. F.G. Tricomi, Sulle equazioni lineari alle derivate parziali di tipo misto, *Atti Accad. Naz. Lincei,* **14***, (1923), 218–270.*

260. F. Tricomi, *Integral Equations*, Dover, New York, 1957.
261. N.S. Trudinger, On Harnack Type Inequalities and their Application to Quasi-Linear Elliptic Partial Differential Equations, *Comm. Pure Appl. Math.,* **20**, *(1967), 721–747.*
262. N.S. Trudinger, Pointwise Estimates and Quasilinear Parabolic Equations, *Comm. Pure Appl. Math.,* **21**, *(1968), 205–226.*
263. A.N. Tychonov, Théorèmes d'unicité pour l'équation de la chaleur, *Math. Sbornik,* **42**, *(1935), 199–216.*
264. A.N. Tychonov and V.Y. Arsenin, *Solutions of Ill–Posed Problems*, Winston/Wiley, 1977.
265. A.I. Vol'pert, The Spaces BV and Quasi-Linear Equations, *Math. USSR Sbornik,* **2**, *(1967), 225–267.*
266. V. Volterra, Sulla inversione degli integrali definiti, *Rend. Accad. Lincei, Ser. 5 (1896), 177–185.*
267. V. Volterra, Sopra alcune questioni di inversione di integrali definiti, *Ann. di Mat.* **25***(2), (1897), 139–178.*
268. R. Von Mises and K.O. Friedrichs, *Fluid Dynamics*, Appl. Math. Sc., **5**, Springer-Verlag, New York, 1971.
269. J. von Neumann, Discussion on the existence and uniqueness or multiplicity of solutions of the aerodynamical equations, *Bull. Amer. Math. Soc.,* **47**, *(2010), 145–154.*
270. W. von Wahl, Regularity of Weak Solutions of the Navier-Stokes Equations, *Proc. Symp. Pure Appl. Math.,* **45**, *Part 2, (1986), 497–503.*
271. N.A. Watson, *Introduction to heat potential theory*, Mathematical Surveys and Monographs, 182, American Mathematical Society, Providence, RI, 2012.
272. P. Weidemaier, The trace theorem $W_p^{2,1}(\Omega_T) \ni f \mapsto \nabla_x f \in W^{1-1/p,1/2-1/2p}(\partial\Omega_T)$ revisited, *Comment. Math. Univ. Carolin.* **32***(2), (1991), 307–314.*
273. P. Weidemaier, Maximal regularity for parabolic equations with inhomogeneous boundary conditions in Sobolev spaces with mixed L_p-norm, *Electron. Res. Announc. Amer. Math. Soc.,* **8**, *(2002), 47–51.*
274. H. Weyl, The Method of Orthogonal Projection in Potential Theory, *Duke Math. J., 7, (1940), 411–444.*
275. H. Weyl, Shock Waves in Arbitrary Fluids, *Comm. Pure Appl. Math.,* **2–3**, *(1949), 103–122.*
276. D.V. Widder, Positive temperatures in an infinite rod, *Trans. Amer. Math. Soc.,* **55**, *(1944), 85–95.*
277. N. Wiener, Certain notions in potential theory, *J. Math. Phys. Mass. Inst. Tech.* **III**, *(1924), 24–51.*
278. N. Wiener, Une condition nécessaire et suffisante de possibilité pour le problème de Dirichlet, *Comptes Rendus Acad. Sci. Paris,* **178**, *(1924), 1050–1054.*
279. N. Wiener and E. Hopf, Über eine Klasse singulärer Integralgleichungen, *Sitzungsber. Preuss. Akad. der Wiss., (1931), 696–706.*
280. W. Wieser, Parabolic Q-Minima and minimal solutions to variational flow, *Manuscripta Math.,* **59**, *(1987), 63–108.*
281. K. Yoshida, *Functional Analysis*, Springer-Verlag, New York, 1974.

Index

$L^{q,r}$ spaces, 389, 616
$V^{m,p}$ spaces, 390, 392
k-shock, 684, 691
p-Laplacean equation(s), 351

Abel integral equation(s), 139
  of the first kind, 138
adjoint
  homogeneous equation(s), 121, 122, 124–128
  operator(s), 2, 119, 221
    heat, 145, 147
Aleksandrov, Aleksandr Danilovich, 529
analytic, 30, 90
  at a point, 30
  data, 30, 44
  in an open set, 30, 31
  locally, 84, 159, 160
    for the heat equation, 168
  solution(s), 30, 31, 38, 44, 159
anisotropic media, 5
Appel transformation, 184
Aronson, Donald G., 424, 508
Ascoli–Arzelà theorem, 83, 110, 112, 444, 611
axiom of countability, first, 351

backward
  Cauchy problem
    for the heat equation, 150, 157, 160
  characteristic cone, 271
  Dirichlet problem
    for the heat equation, 168
Banach space, 314
Barenblatt, Grigory, 182
barrier, 93
  at a point, 68
  at the origin, 69
  postulate, 68, 80
Bernoulli law, 14
Bernoulli, Daniel, 14, 202
Bessel function(s), 232, 240
bilinear form, 349
blow-up
  for super-linear heat equation, 188
bounded variation, 2
Brownian Motion, 16
Buckley–Leverett model, 707
Burgers equation, 246, 248, 253
  explicit solutions, 275
  invariant transformations, 276

Calderón–Zygmund estimates, 96, 656
Cantor diagonalization, 611
capacity
  in potential theory, 92
  of a compact set, 92
Carathéodory, Constantin, 443
Cauchy
  data, 21, 23, 25, 26, 28, 29, 31
  inequality, 88, 608, 610
  problem(s), 21, 26, 29–31, 49, 147, 197, 198, 202, 204, 205, 207, 208, 227, 229, 230, 232, 246, 248, 285, 286, 288–291, 297, 689, 704, 716
   by similarity solutions, 146
   characteristic, 211

© Springer Nature Switzerland AG 2023

E. DiBenedetto, U. Gianazza, *Partial Differential Equations*,
Cornerstones, https://doi.org/10.1007/978-3-031-46618-2

    for systems of conservation laws, 680
    for the backward heat equation, 150, 157, 160
    for the heat equation, 146, 149, 157, 159, 162, 163, 169, 177, 184
    for the wave equation, 220
    homogeneous, 224
    in terms of Lagrangian, 292
    inhomogeneous, 198, 223, 224
    noncharacteristic, 220, 222
Cauchy, Augustin, 32
Cauchy–Dirichlet problem
  for linear parabolic equations
    $L^\infty(E)$ estimates, 448
    Hölder continuity, 423
    homogeneous, 394, 403
    inhomogeneous, 405, 445
  for nondivergence form equations, 519, 524
  for nonlinear operators, 552
  for quasi-linear parabolic equations, 451, 473, 474
  homogeneous
    by Galerkin methods, 397
Cauchy–Kowalewski theorem, 30–32, 37, 38, 44, 49, 146, 159
Cauchy–Neumann problem
  for linear parabolic equations, 409
    $L^\infty(E)$ estimates, 416
    Hölder continuity, 423
  for nondivergence form equations, 522, 527
  for quasi-linear parabolic equations, 448, 451, 479
caustic(s), 207, 290
centered k-rarefaction wave, 689, 691
chain rule in $W^{1,p}(E)$, 342
characteristic(s), 22–24, 26, 29, 30, 38, 196, 200, 217, 241, 242, 250, 252, 283, 284, 290
  Cauchy problem, 211
  cone, backward, 271
  cone, truncated, 211, 271
  cone, truncated backward, 211, 213
  curves, 198, 283, 292
  form, 29
  Goursat problem(s), 217
  line(s), 244, 245, 251
  parallelogram, 196, 197, 217
  projection(s), 242–245, 248, 250
  strips, 284–286, 289
  surface(s), 26, 27, 29, 30, 313, 393
coercive, 293
Cohen, Paul, 677
compact
  embedding(s) of $W^{1,p}(E)$, 316, 344, 651
  kernel(s), 110, 112, 114, 120, 126, 127, 129, 135, 142
  mapping(s), 110, 112, 114
  operator(s), 110, 112, 114, 120, 131, 134
  subset(s) of $L^p(E)$, 110, 344
compatibility condition(s), 25, 26, 29, 30, 38, 200, 213, 326, 327
complete orthonormal system, 135, 136, 142
  in $L^2(E)$, 135, 332
  in $W_o^{1,2}(E)$, 322, 332, 398
  in $W^{1,2}(E)$, 407
  of eigenfunctions, 135
completeness criterion, 134, 135
concave hull, 529
  increasing, 531, 554
condition
  kinematic, 592
  Ladyzhenskaya–Prodi–Serrin, 616, 618, 624, 653
  Navier boundary, 592
  no-slip, 592, 607
cone property, 315, 323, 343, 355
conservation
  law(s), 5, 249, 697
    in one space dimension, 249
    system, 699, 706, 707
  of entropy, 13
  of mass, 4
contact discontinuity, 684, 691
continuity equation, 3, 4, 14, 15
continuous dependence, 198
convex functional(s), 319, 320, 329, 350
  strictly, 319, 327–329
convex hull
  increasing, 568
convexity condition, 351
  of Legendre, 351
Corey's model, 707

counterexample
  of Cohen, 677, 708
  of DeGiorgi, 676, 679
  of Lebesgue, 69
  of Pliś, 674
  of Tychonov, 155
cutoff function(s), 169, 178, 190
cycloid as tautochrone, 140

d'Alembert formula, 197, 199, 202, 223, 231
d'Alembert, Jean le Ronde, 202
d'Alembertian, 203
Darboux formula, 204, 205, 231
Darboux, Gaston, 31, 32, 204, 217
Darboux–Goursat problem, 217
Darcy's law, 5
Davies, Edward Brian, 424
DeGiorgi classes, 342, 363–365
  boundary
    Dirichlet data, 375
    Neumann data, 377, 379
  boundary local continuity
    Dirichlet data, 376
    Neumann data, 379
  homogeneous, 365, 366, 369, 375
  local boundedness, 366
  local Hölder continuity, 369
DeGiorgi, Ennio, 342, 423, 674
derivative(s)
  directional, 321
  Fréchet, 321
  Gâteaux, 321, 352
descent, method of, 204, 209, 230–232
differential operator(s), 2, 87
DiPerna, Ronald, 706
Dirichlet data, 28, 49, 53
  boundary DeGiorgi classes, 375, 376
  parabolic boundary DeGiorgi classes, 472, 476
Dirichlet kernel, 105
Dirichlet problem, 48, 49, 53, 54, 60, 65, 70, 80, 85, 87
  $L^\infty(E)$ estimates, 333, 359, 410
  by integral equations, 104, 115
  for a ball, 53, 56, 85
  for a disc, rectangle, annulus, 85, 86
  for quasi-linear elliptic equations, 357, 363, 375
  for the exterior ball, 93
  for the heat equation, 145
    backward, 168
    homogeneous, 163–167, 180
  Hölder continuity, 341
  homogeneous, 318, 320, 349
    by Galerkin methods, 321
    by variational methods, 319, 350
  inhomogeneous, 325, 357
  sub(super)-solution, 334
  uniqueness, 59, 68
distribution
  of dipoles, 50
  of electrical charges, 50, 83
  of masses, 50, 83
divergence theorem, 1, 2
domain of dependence, 198, 207, 209, 211
double-layer potential, 50, 84, 97–99, 103, 114
  jump condition, 101, 103
Duhamel's principle, 199, 210, 223, 224

eigenfunction(s), 129
  complete system of, 135, 136, 142
  expansion in terms of, 134, 137
  first, 137
  for integral equation(s), 109, 114, 117
  for the Laplacean, 136
  orthogonal, 329, 331
  orthonormal, 129–131, 134, 135, 137, 329, 331
  real-valued, 129–131, 329
  sequence of, 133, 134, 331
eigenspace, 133
eigenvalue problem(s), 129, 133, 328
  for the Laplacean, 109, 114, 136
  homogeneous Neumann data, 117
eigenvalue(s), 124, 125, 127, 129
  countably many, 130
  first, 131, 332
  for integral equation(s), 109, 114, 117
  for the Laplacean, 136
    first, 164
  of an elliptic matrix, 313, 393
  real, 129, 137, 329
  sequence of, 132–134, 137, 142, 331
  simple, 131, 332
eikonal equation, 289, 292

Einstein, Albert, 2, 16
elliptic
  second-order, 313
elliptic equation(s), 22, 23, 27, 29, 313
  linear, 313
  quasi-linear, 353, 363
    Dirichlet problem, 357, 363, 375
    Neumann problem, 358, 363, 377
ellipticity condition, 313, 349, 350, 393, 512, 517, 519
embedding(s)
  compact, 316, 344
  limiting, 315
  multiplicative, 316, 327, 345
    of $W^{1,p}(E)$, 316, 317
  of $W_o^{1,p}(E)$, 316, 345
  of $W^{1,p}(E)$, 315, 343, 635
  theorem(s), 315, 598
energy equality
  for Navier–Stokes equations, 618
energy identity
  for the heat equation, 167
energy inequality
  for Navier–Stokes equations, 608
entropy, 697, 698, 703, 704, 721
  condition, 265, 267, 268, 277, 701, 705
  production, 697
  solution(s), 254, 255, 265–267
    global in time, 272
    maximum principle for, 272
    stability in $L^1(\mathbb{R}^N)$, 272
epigraph, 351, 352
equation(s)
  p-Laplacean, 351
  adjoint homogeneous, 121, 122, 124–128
  Bellman–Dirichlet, 516, 585
  Burgers, 246, 248, 253
  eikonal, 289, 292
  elliptic, 22, 23, 29, 313
  Euler, 13, 321, 329, 352
  Hamilton–Jacobi, 290–292, 298, 299
  heat, 5, 23, 44
  hyperbolic, 22, 23
    in two variables, 195, 208, 217, 239
  in divergence form, 6, 246
  in nondivergence form, 6
  integral, 105, 109, 121, 124, 125, 138
    Wiener–Hopf, 141
  Lagrange, 293, 294
  Laplace, 5, 23, 44
  Navier–Stokes, 13
  of continuity, 3, 4, 14, 15
  of state, 13, 14
  of steady incompressible fluid, 22
  of the porous media, 181
  parabolic, 22, 23, 393
  Poisson, 78, 79, 82
  quasi-linear, 22, 28, 313, 393
  telegraph, 232, 239
  Tricomi, 22
error function, 184
Euler's equation(s), 13, 321, 329, 352
expansion of Green's function, 137
expansion of positivity, 487, 565
expansion of the kernel
  in terms of eigenfunctions, 134
exterior sphere condition, 65, 68, 69, 92
extremal problem(s), 131, 132

fast geometric convergence, 335
first
  axiom of countability, 351
  eigenfunction, 137
  eigenvalue, 131, 137, 332
    for the Laplacean, 164
flattening the boundary, 354, 378, 379, 481, 484
flow(s)
  isentropic, 15, 681, 696, 703, 716
  potential, 15, 47
  sonic, 23
  sub-sonic, 23
  super-sonic, 23
focal curve, 290
focussing effect, 207
Fourier
  coefficients, 201
  law, 5
  series, 135, 136
    convergence of, 135, 136
  transform, 185, 229
    heat kernel, 184, 185
    inversion formula, 186
Fourier, Jean Baptiste Joseph, 5, 202
Fréchet derivative(s), 321
Fredholm
  alternative, 124

  integral equation(s), 121
Friedman, Avner, 32, 189
Friedrichs, Kurt Otto, 612
Fubini's theorem, 77
function(s)
  Green's, 53, 80–82, 85, 109
    for a ball, 55, 58, 80
  Nash, 438
  rapidly decreasing, 185
  Riemann, 220, 222
    symmetry of, 222
functional(s)
  convex, 319, 320, 350
  linear, 319, 595, 599
  lower semi-continuous, 351
  minimum of a, 320, 321, 350
  nonlinear, 326, 328
  stationary point(s) of a, 321
  strictly convex, 329
fundamental solution, 91
  Gaussian bounds, 423
  Lower bound, 436
  of the heat equation, 147, 240, 423, 424, 625
  of the Laplacean, 52, 53, 70, 79, 97, 347, 627
  pole of, 52, 70
  Upper bound, 425

Gâteaux derivative(s), 321, 352
Gårding, Lars, 32
Gagliardo, Emilio, 316
Gagliardo–Nirenberg theorem, 316, 616
Galdi, Giovanni P., 623
Galerkin
  approximation, 637
  approximation(s), 321, 397, 609
  method, 321, 407, 410
genuinely nonlinear, 681, 688
geometric convergence, fast, 335
geometric measure theory, 2
geometrical optics, 289, 292
Gevrey class, 679
Gevrey, Maurice, 168, 679
Giga, Yoshikazu, 633
Glimm, James Gilbert, 695, 704, 706
Goursat problem(s), 220, 239
  characteristic, 217, 220, 223
Goursat, Edouard Jean Baptiste, 217
Green's function, 53, 80–82, 85
  for a ball, 55, 58, 80
  for a half-ball in $\mathbb{R}^N$, 85
  for a quadrant in $\mathbb{R}^2$, 85
  for the half-space, 85
  for the Laplacean, 53, 109
  for the Neumann problem, 107, 108, 116, 117
  kernel generated by, 110
Green's identity, 51, 53, 57, 81, 84
Green's theorem, 1, 2, 10, 12, 597
Green, Gabriel, 53

Hölder
  continuity, 425, 476, 485, 495, 498
    local, 369, 459, 472, 580, 583
  continuous, 332, 341, 342, 364, 423, 452
    locally, 366, 456
    versus Harnack inequality, 383, 386, 494
  inequality, 334, 346, 355, 595, 616, 631, 647, 650
  norm(s), 341, 423
Hörmander, Lars Valter, 677
Hadamard, Jacques, 32, 49, 90, 174, 217
Hamilton–Jacobi equation(s), 290–292, 298, 299
Hamiltonian, 292, 293
  strictly convex, 306, 307
  system, 292
Hammerstein integral equations, 141
Hardy's inequality, 95
harmonic
  extension, 61, 70, 71, 87, 91, 92, 357
  function(s), 47, 49, 50, 52–55, 59, 60, 62, 63, 65, 66, 81, 84, 87, 90–92, 94
    analyticity of, 63, 64
    Taylor's series of, 89
  polynomial(s), 84
Harnack inequality, 62, 66, 366, 380
  for the heat equation, 173
  parabolic, 425, 456, 472, 485, 552, 569, 576
  versus Hölder continuity, 383, 386, 494
Harnack, Axel, 62
heat equation, 5, 18, 23, 44, 147, 495

  backward Cauchy problem, 150
  Cauchy problem, 146, 149, 157, 159, 184
    homogeneous, 163
    inhomogeneous, 162, 163, 169
    positive solutions, 177
  infinite speed of propagation, 149
  strong solution(s) of, 165
  weak solution(s) of, 165, 166
heat kernel, 147, 181
  by Fourier transform, 184, 185
heat operator, 145, 147, 151
  adjoint, 145, 147
Heaviside function, 181, 190, 278
Helmholtz–Weyl decomposition, 601, 646, 649
Hilbert–Schmidt theorem, 134, 136
  for the Green's function, 136
Holmgren's Theorem, 38
Holmgren, Erik Albert, 40
Hopf
  first variational formula, 294
  second variational formula, 294
  variational solution(s), 293, 295, 297
    regularity of, 296
    semigroup property, 295
    weakly semi-concave, 306, 308, 309
Hopf solution of Burgers equation, 253
Hopf solution of Navier–Stokes equations, 608, 615, 617, 624, 625, 649
Hopf, Eberhard, 253, 608
Huygens principle, 209, 210
hyperbolic equation(s), 22, 23, 27, 195, 208, 217, 239
  in two variables, 195, 208, 217, 239

ill posed problem(s), 49, 90, 150, 151, 168
inequality(ies)
  Cauchy, 88, 190, 608, 610
  continuous Minkowski, 655
  Gronwall, 622
  Hölder, 334, 346, 355, 595, 616, 631, 647, 650
  Hardy, 95
  Harnack, 62, 66, 366, 380
  Jensen's, 60
  Nash, 426, 449
  parabolic Harnack, 425, 456, 472, 485, 498, 507, 552, 569, 576
  Poincaré, 650
  trace(s), 327, 328
  Young's for the convolution, 654
infinite speed of propagation
  for the heat equation, 149
inhomogeneous
  Cauchy problem(s), 223, 224
  Dirichlet problem, 325, 357, 405, 445
  problem(s), 210, 223, 231
initial data in the sense of $L^1_{\text{loc}}(\mathbb{R}^N)$, 158, 159, 249, 263
initial value problem(s), 291, 292
inner product(s)
  equivalent, 317, 319, 322, 350
integral equation(s), 105, 109, 121, 124, 125
  Abel, 138, 139
  Dirichlet problem by, 104, 115
  eigenfunction(s) of, 109, 114, 117
  eigenvalue(s) of, 109, 114, 117
  Fredholm of the second kind, 121
  Hammerstein, 141
  in $L^2(E)$, 121
  Neumann problem by, 105, 106, 109, 116
  of the first kind, 138
  Volterra, 140
  Wiener–Hopf, 141
integral surface(s), 241, 242, 248–250, 281, 284–286
  Monge's construction, 281
inversion formula of the Fourier transform, 186, 230

Jensen's inequality, 60, 267, 295
jump condition, 24, 129
  of a double-layer potential, 101, 103, 129

Kelvin transform, 55, 94
Kelvin, William Thomson, 55, 94
kernel(s)
  almost separable, 126, 127, 129
  compact, 110, 112, 114, 120, 126, 127, 129, 135, 142
  degenerate, 120, 122, 141
  Dirichlet, 105

  expansion in terms of eigenfunctions, 134, 137
  generated by Green's function, 110, 114, 120, 135, 142
  Hammerstein, 141
  in $L^2(E)$, 119, 120
  Neumann, 106
  of finite rank, 120, 122, 141
  on ∂E, 120
  Poisson, 71
    for the half-space, 93
  potential, 127
  separable, 120, 122, 141
  symmetric, 120
  Volterra, 139
  Wiener–Hopf, 141
kinetic momentum(a), 292
Kirchoff formulae, 207
Kiselev, Aleksandr Alekseevich, 624
Kiselev–Ladyzhenskaya solution of Navier–Stokes equations, 625
Kondrachov, Vladimir Iosifovich, 316
Kowalewskaja, Sonja, 31, 32
Kruzhkov uniqueness theorem, 269
Kruzhkov, Stanislav Nikolaevich, 265
Krylov, Nikolay Vladimirovich, XXIX, 529, 558, 564

Ladyzhenskaya, Olga Aleksandrovna, 624
Lagrange equation(s), 293, 294
Lagrange, Joseph Louis, Compte de, 47
Lagrangian, 292, 293
  configuration(s), 294
  coordinate(s), 292, 294
  path, 13
Laplace equation, 5, 23, 44
Laplace, Pierre Simon Marquis de, 47
Lax shock condition, 683, 684, 692, 701, 717
Lax variational solution(s), 255, 256, 258
  existence of, 261
  stability of, 261
Lax, Peter, 32, 255, 319, 673, 702, 706
Lax–Milgram theorem, 319, 441
least action principle, 294
Lebesgue counterexample, 69, 357
Lebesgue, Henry Léon, 69
Legendre
  condition, 351
  transform, 292, 293, 298, 299
Leibniz rule, 35, 36, 45
  generalized, 35, 36, 45
Leibniz, Gottfried Wilhelm, 37
Lemma
  Friedrichs, 612, 649
  Zorn, 609
Leray, Jean, 633, 646, 649, 678
light rays, 289, 290
linear functional(s), 319, 595, 599
linearly degenerate, 681, 688
Lions, Jacques-Louis, 441
Liouville theorem, 62, 71
Liouville, Joseph, 62
local solution(s), 78, 79, 341, 353, 422
  of quasi-linear elliptic equation(s), 363
  of quasi-linear parabolic equation(s), 452
  of the Poisson equation, 78, 79
logarithmic convexity, 167
logarithmic potential, 50
lower semi-continuous
  functional(s), 351
  weakly, 320, 351

Mach number, 15
Mach, Ernst, 15
majorant method, 32
maximization problem(s), 131, 132
maximizing sequence, 131
maximum principle, 59–61, 81, 92, 149, 254, 272
  Aleksandrov, 525, 529, 579, 589
  for entropy solutions, 272
  for general parabolic equations, 189, 254
  for parabolic equations in nondivergence form, 519
    local, 553
  for the heat equation, 150
    bounded domains, 150, 187
    in $\mathbb{R}^N$, 152, 189
  weak form, 87, 150, 334
mean value property, 59, 62, 86
mechanical system(s), 292–294
method of descent, 204, 209, 230, 231

method of majorant, 32
Meyers–Serrin's theorem, 314
Milgram, Arthur Norton, 319
minimizer(s), 257, 258, 320
minimizing sequence(s), 257, 320, 330
minimum of a functional, 320, 321, 350
Monge's cone(s), 281–283, 289
  symmetric equation of, 282
Monge's construction
  of integral surfaces, 281
Monge, Gaspard, 281
Moser, Jürgen Kurt, 174, 507
Murat, François, 706

Nash's function, 438
Nash's inequality, 426, 449
Nash, John Forbes, 425, 500
Navier, Claude-Louis, 13
Navier–Stokes equations, 13
Neumann data, 49, 106, 326, 377, 407, 480
  boundary DeGiorgi classes, 377, 379
  parabolic boundary DeGiorgi classes, 479, 484
Neumann kernel, 106
Neumann problem, 48, 49, 326, 327, 406, 641
  $L^\infty(E)$ estimates, 336
  by integral equations, 105, 106, 109, 116, 128, 142
  conditions of solvabiliy, 326
  for quasi-linear elliptic equations, 358, 363, 377
  for the half-space, 93
  for the heat equation, 146
  Green's function, 107, 108
    for the ball, 116, 117
  Hölder continuity, 342
  sub(super)-solution, 336, 416
  uniqueness, 328
Newton formula, 37
Newtonian potential(s), 72, 83
  of ellipsoids, 83, 84
  Schauder estimates of, 72
Nikol'skii, Sergei Mikhailovich, 315
Nirenberg, Luis, 316
noncharacteristic, 197
  Cauchy problem(s), 220, 222
norm in $W^{1,p}(E)$, 314
norm of an operator, 119
normal mapping, 530, 539
  of a cone, 531

odd extension(s), 202, 214, 215
Ohya, Yujiro, 678
open covering
  finite, 354
  of ∂E, locally finite, 323
  of E, 354
operator
  uniformly parabolic, 553
operator(s), 119
  adjoint(s), 2, 119
  compact, 110, 112, 114, 120, 131, 133, 134
  second-order, 84
    invariant, 84
  symmetric, 133, 134
  translation, 110
order of a PDE, 16, 28
orthonormal
  complete system for $L^2(E)$, 164, 201, 332
  complete system for $W_o^{1,2}(E)$, 321, 322, 332, 398
  eigenfunctions, 129–131, 134, 135, 137, 329, 331
  in the sense of $L^2(E)$, 119

parabolic, 22, 27
  boundary, 150, 153, 188
  second-order, 393
parabolic DeGiorgi classes, 423, 451, 452, 456, 508, 565
  boundary
    Dirichlet data, 472, 476
    Neumann data, 479, 484
  boundary local continuity
    Dirichlet data, 476
    Neumann data, 484
  homogeneous, 456, 457, 459, 460, 476, 494–496, 498
  local boundedness, 456, 457
  local Hölder continuity, 459, 472
parabolic embedding, 389, 458, 462
parabolic equation(s), 22, 23, 393
  linear, 393
  quasi-linear, 451

    - Dirichlet problem, 451, 474
    - Neumann problem, 451, 479
- parabolic Harnack
  - inequality, 498, 507
- parabolic quasi-minima, 446
- Parseval's identity, 165, 609, 610
- partition of unity, 354
- Pattle, Richard, 182
- periodic function(s), 143
- Perron, Otto, 65
- phase space, 292
- Phragmen–Lindelöf-type theorems, 90
- piecewise Lipschitz continuous, 699
- Pini, Bruno, 174
- Pliś, Andrzej, 674
- Poincaré inequality, 650
- Poiseuille, Jean Léonard Marie, 636
- Poisson
  - equation, 78, 79, 82
  - formula, 59, 62, 86, 91, 94, 231
  - integral, 94
    - for the half-space, 70
  - kernel, 71
    - for the half-space, 93
  - representation, 56
- Poisson, Siméon Denis, 12, 56, 202
- porous media equation, 181
- positive geometric density, 375, 474
- positive solutions of the heat equation
  - decay at infinity, 177, 179
  - uniqueness, 177, 179
- postulate of the barrier, 68, 80
- potential(s)
  - double-layer, 50, 84, 97–99, 103, 114
    - jump condition, 101, 103
  - estimates in $L^p(E)$, 75, 95
  - flow(s), 15, 47
  - for the Laplacean, 50
  - generated by
    - charges, masses, dipoles, 50, 83
  - gravitational, 47
  - kernel(s), 127
  - logarithmic, 50
  - Newtonian, 72, 83, 84
    - of ellipsoids, 83
    - Schauder estimates of, 72
  - retarded, 211
  - Riesz, 76, 95, 343
  - single-layer, 50, 84, 105
  - weak derivative of, 76
- principal part of a PDE, 16
- principle(s)
  - Duhamel's, 199, 210, 223, 224
  - Huygens, 209, 210
  - least action, 294
  - reflection, 71, 91
- probability measure(s), 254, 258–260
- Prodi, Giovanni, 618
- propagation of disturbances, 198

- quasi-linear, 16, 21, 22, 25, 28, 241
  - elliptic equation(s), 353, 357, 363
    - Dirichlet problem, 357, 363, 375
    - Neumann problem, 358, 363, 377
    - sub(super)-solution(s), 363, 375, 377
  - Neumann problem
    - sub(super)-solution(s), 480
  - parabolic equation(s), 451
    - Dirichlet problem, 451, 474
    - Neumann problem, 451, 479
    - sub(super)-solution(s), 451, 456, 473, 474, 480
- quasi-minima, 353
  - parabolic, 446

- Rademacher's theorem, 297
- radial solution(s)
  - of Laplace's equation, 50
  - of the wave equation, 227, 228
- radius of convergence
  - for harmonic functions, 89
- Rankine–Hugoniot shock condition, 248, 267, 681, 683, 692, 701, 717
- rapidly decreasing function(s), 185, 229
- reflection
  - even, 204
  - map, 85
  - odd, 202, 215
  - principle, 71, 87, 91
  - technique, 191, 213, 236
- Rellich, Franz, 316
- Rellich–Kondrachov theorem, 316, 596, 606
- removable singularity(ies), 91
- representation formula
  - for the heat equation, 148
- retarded potential(s), 211

Reynolds number, 592, 599, 607
Reynolds, Osborne, 592
Riemann function, 220, 222, 239, 240
  pole of, 240
  symmetry of, 222
Riemann problem(s), 252, 255, 694, 705, 716
  generalized, 276
Riemann, Bernhard, 222, 694
Riesz
  potential(s), 76, 95, 343
  representation theorem, 318, 319, 349, 441, 595, 599
Riesz, Frigyes, 89

Safonov, Mikhail V., XXIX, 558, 564
Schauder estimate(s)
  of Newtonian potentials, 72
  up to the boundary, 94
Schauder, Julius Pavel, 72
Schauder–Leray fixed point theorem, 444, 597, 600
Schwartz class(es), 185, 229
segment property, 323–325, 328, 354, 357, 405
semi-concave, 304, 309
  weakly, 305, 306, 308, 309
semi-continuous
  functional(s), 351
  upper, 89
semigroup property(ies)
  of Hopf variational solution(s), 295
separation of variables, 85, 163, 201, 202, 215, 225, 238
Seregin, Gregory, 653
Serre, Denis, 706
Serrin's theorem, 90
Serrin, James, 314, 508, 623, 624
shock(s)
  condition(s), 248, 251, 252, 267
    Lax, 683, 684, 692, 701, 717
    Rankine–Hugoniot, 248, 267, 681, 683, 692, 701, 717
  line(s), 247, 248, 252
similarity solutions, 146, 182–184
single-layer potential, 50, 84, 105
singularity(ies), 23
  removable, 91
small vibration(s)
  of a membrane, 8, 9
  of a string, 6, 11
  of an elastic ball, 227
Sobolev space(s), 314, 343, 404, 598
Sobolev, Sergei Lvovich, 314, 315
Sohr, Hermann, 633
solid angle, 97, 102, 315, 343, 344, 355
sonic flow, 23
space(s)
  $W^{s,p}(\partial E)$ and traces, 325
  Banach, 314
  Sobolev, 314, 343, 404
speed of sound, 14, 15
spherical mean(s), 204, 207
spherical symmetry of the Laplacean, 47
stationary point(s)
  of a functional, 321
Steklov average(s), 266, 393
Stirling inequality, 156, 171, 172
Stirling, James, 156
Stokes identity, 51, 52, 54, 57, 81, 84, 99
Stokes, George Gabriel, 13, 636
strictly convex, 319, 327, 328
  functional(s), 329
string, vibrating, 6
structure condition(s), 363, 451, 500
Struwe, Michael, 623, 624
sub(super)-solution(s)
  of linear parabolic equations in nondivergence form, 520, 521, 553, 569
  of quasi-linear elliptic equations, 363, 375, 377
  of quasi-linear parabolic equations, 451, 456, 473, 474, 480
  of quasi-linear parabolic equations in nondivergence form, 577
  of the heat equation, 189
sub-harmonic, 57–60, 86–89
sub-sonic flow, 23
super-harmonic, 58, 59, 61, 68
super-sonic flow, 23
surface(s) of discontinuity(ies), 247, 248
symmetry of Green's function, 53
synchronous variation(s), 294

Talenti, Giorgio, 315
Tartar, Luc, 702, 706

tautochrone, 138, 140
Taylor's series, 32
  of a harmonic function, 64, 89
telegraph equation, 232, 239
theorem(s)
  Ascoli–Arzelà, 83, 110, 112, 444, 611
  Cauchy–Kowalewski, 30–32, 37, 38, 44, 49, 146, 159
  Fubini, 77
  Gagliardo–Nirenberg, 316, 616
  Green, 1, 10, 12, 597
  Hilbert–Schmidt, 134, 136
  implicit functions, 246
  Kruzhkov, uniqueness, 269
  Lax–Milgram, 319, 441
  Liouville, 62, 71
  Meyers–Serrin, 314
  of the divergence, 1, 2
  Phragmen–Lindelöf-type, 90
  Rademacher, 297
  Rellich–Kondrachov, 316, 596, 606
  Riesz representation, 318, 319, 349, 441, 595, 599
  Schauder–Leray fixed point, 444, 597, 600
  Serrin, 90
  Weierstrass, 127
total derivative, 13, 15
trace(s), 324
  inequality, 327, 328, 354
  of functions in $W^{1,p}(E)$, 323, 324
    characterization of, 325, 355
traveling waves, 249
Tricomi equation, 22
Tricomi, Francesco Giacomo, 22
Trudinger, Neil Sidney, 508
Tychonov counterexample, 155, 181, 191

ultra-hyperbolic, 27
undistorted waves, 195, 197
unique continuation, 89
upper semi-continuous, 89

variational
  integral(s), 357, 358
  problem(s), 350
variational solution(s)
  existence of, 261
  Hopf's, 293, 306, 308
  Lax, 255, 256, 258, 266
  of Burgers-type equations, 255, 256, 258, 261, 266
  stability of, 261
  uniqueness of Hopf's, 309
vibrating string, 6, 200, 224
vibration(s), small
  of a membrane, 8, 9
  of a string, 6, 11
  of an elastic ball, 227
viscosity, dynamic, kinematic, 636
Volterra
  integral equation(s), 140
  kernel(s), 139
von Wahl, Wolf, 633

Waelbroeck, Lucien, 678
wave equation, 8, 11, 12, 23, 44, 195
  in two variables, 195
wave front(s), 289, 290
waves undistorted , 195, 197
weak
  convergence, 320, 351
  derivative(s), 313, 314, 342, 393
    of a potential, 76
  formulation(s), 313, 329, 334, 342, 375, 393, 412, 474
    of the Neumann problem, 326, 328, 408
  maximum principle, 334
  solution(s), 197, 246–249, 251, 252, 254, 256, 264, 265, 297, 313, 358
    local, 363, 452
    of steady-state flow, 594, 597, 600, 601
    of the Poisson equation, 79
  sub(super)-solution
    of the Dirichlet problem, 334
    of the Neumann problem, 336, 416
weakly
  closed epigraph, 352
  lower semi-continuous, 320, 351
  semi-concave, 305
    Hopf variational solution(s), 306, 308, 309
  sequentially compact, 358
Weierstrass theorem, 127
well posed

  in the sense of Hadamard, 49, 198, 208
Wiener, Norbert, 92
Wiener–Hopf integral equations, 141
Wieser, Wilfried, 446